3264 & All That
Intersection Theory in Algebraic Geometry

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Contents

0.1 Why you should read this book 1
0.2 Why we wrote this book 3
0.3 What's with the title? 3
0.4 What's in this book 4
0.4.1 Overture 4
0.4.2 Second beginning 4
0.4.3 Using the tools 5
0.4.4 Further developments 6
0.4.5 Relation to "Intersection Theory" 7
0.4.6 Keynote Problems 7
0.5 What you should know before you begin 7
0.6 What's in a scheme? 8

1 Overture **11**
1.1 The Chow Ring 12
1.1.1 Cycles 12
1.1.2 Rational equivalence 13
1.1.3 First Chern class of a line bundle 15
1.1.4 First results on the Chow Group 16
1.1.5 Intersection products 16
Multiplicities 19
1.1.6 Functoriality 20
Proper Pushforward 20
Flat and Projective Pullback 22
1.2 Chow Ring Examples 23

1.2.1 Varieties built from Affine Spaces 23
Open Subsets of Affine Space 23
Affine stratifications 24
1.2.2 The Chow ring of $\mathbb{P}^n$ 25
Degrees of Veronese varieties 27
Degree of the dual of a hypersurface 28
1.2.3 Products of projective spaces 28
Degrees of Segre varieties 30
The class of the diagonal 31
The graph of a morphism 32
1.2.4 The blowup of $\mathbb{P}^n$ at a point 33
1.2.5 Loci of singular plane cubics 39
The locus of reducible cubics 41
Triangles 42
Asterisks 43
1.3 Chern Classes 43
1.3.1 Chern Classes as Dependency Loci 44
1.3.2 Whitney's Formula and the Splitting Principle . . . 46
1.3.3 The Chern Character 47
1.4 Chern Class Examples 47
1.4.1 The 27 Lines on a Cubic Surface 47
1.4.2 First Computations 49
Sums of Line Bundles 49
Duals 49
1.4.3 The Cotangent Bundle 49
The Genus of a Curve on a Surface 52
The Self-Intersection of a Curve on a Surface 53
Linked curves in $\mathbb{P}^3$ 53
The topological Euler Characteristic 54
1.5 Exercises 56

2 Introduction to Grassmannians and Lines in $\mathbb{P}^3$ 65
2.1 Introduction to Grassmann Varieties 65
2.1.1 The Plücker embedding 66
2.1.2 Covering by affine spaces; local coordinates 69
2.1.3 Universal sub and quotient bundles 71
2.1.4 The Tangent Bundle of the Grassmann Variety . . . 73
2.2 The Chow Ring of $\mathbb{G}(1,3)$ 76
2.2.1 Schubert Cycles in $G = \mathbb{G}(1,3)$ 77
2.2.2 Equations of the Schubert Cycles 79
2.2.3 Ring Structure 80
Tangent spaces to Schubert cycles 83
2.2.4 A Specialization Argument 84
2.3 Applications 87
2.3.1 How Many Lines Meet Four General Lines? 87

2.3.2 Lines meeting a curve . . . 88
2.3.3 Lines meeting a curve via specialization . . . 90
2.3.4 Chords to a space curve . . . 91
2.3.5 Chords via specialization . . . 93
2.3.6 Lines on a quadric . . . 95
2.3.7 Tangent Lines to a Surface . . . 95
2.4 Exercises . . . 97

3 Grassmannians in General 105
3.1 Schubert Cells and Schubert Cycles . . . 106
3.2 Intersections in complementary dimension . . . 111
3.2.1 Varieties swept out by linear spaces . . . 113
3.3 Grassmannians of Lines . . . 114
3.3.1 Dynamic specialization . . . 117
3.4 How to Count Schubert Cycles with Young Diagrams . . . 119
3.5 Linear spaces on quadrics . . . 121
3.6 Pieri and Giambelli . . . 124
3.6.1 Intersecting with σ_1 . . . 124
3.6.2 Pieri's formula in general . . . 126
3.6.3 Giambelli's formula . . . 126
3.7 Exercises . . . 127

4 Chow Groups 133
Keynote Questions: . . . 133
4.1 Chow groups and basic operations . . . 134
4.1.1 Examples of Rational Equivalences . . . 138
Dynamic projection . . . 139
4.2 Rational Equivalence through Divisors . . . 143
4.3 Proper push-forward . . . 146
4.3.1 The Determinant of a Homomorphism . . . 149
4.4 Flat Pullback . . . 155
4.4.1 Affine Space Bundles . . . 157
4.5 The connection between A_{n-1} and Pic. . . . 158
4.6 Exercises . . . 159

5 Intersection Products and Pullbacks 163
5.1 The Chow Ring . . . 163
5.1.1 Products via the Moving Lemma . . . 164
5.1.2 From Intersections to Pullbacks . . . 168
5.2 Proof of the Moving Lemma . . . 170
5.3 Kleiman's Transversality Theorem: The Case of a Transitive Group Action . . . 181
5.4 Intersections on singular varieties . . . 183
5.5 Intersection multiplicities . . . 185
5.6 Refinements and Other Approaches . . . 188

5.6.1 Refined and Excess Intersection 188
5.6.2 Reduction to the Diagonal and Deformation to the Normal Cone 189
5.7 Proofs of the results on Pullbacks 190
5.8 Exercises 193

6 Interlude: Vector Bundles and Direct Images 195
6.1 Vector Bundles and Locally Free Sheaves 195
6.2 Pullbacks 197
6.3 Flat Families of Sheaves 198
6.4 Direct Images 199
6.5 Cohomology and Base Change 203
6.5.1 Tools From Commutative Algebra 205
6.5.2 Proof Of the Theorem On Cohomology and Base Change 208
6.6 Examples 209
6.6.1 Applications to Line Bundles 209
6.6.2 Double covers 210
6.6.3 Projective bundles 212
6.6.4 Ideals of points in projective space 213
6.7 Applications I: Bundles revisited 214
6.7.1 Symmetric powers of tautological bundles 214
6.8 Exercises 217

7 Vector Bundles and Chern Classes 219
7.1 Chern Classes 220
7.1.1 Chern Classes as Dependency Loci 220
7.2 Basic properties 223
7.2.1 Pullbacks 224
7.2.2 Line bundles 224
7.2.3 Whitney sum 224
7.2.4 The splitting principle 228
7.2.5 Tensor products with line bundles 229
7.2.6 Definition of Chern Classes in General 231
7.2.7 Dual Bundles 233
7.2.8 Determinant of a Bundle 233
7.3 Application: A strong Bertini Theorem 234
7.4 Chern Classes of Some Interesting Bundles 235
7.4.1 Universal bundles on projective space 236
7.4.2 Chern Classes of Tangent Bundles 236
7.4.3 Tangent Bundles of Projective Spaces 237
7.4.4 Tangent bundles to hypersurfaces 238
7.4.5 Bundles on Grassmannians 240
7.5 Generators and Relations for $A^*(G(k,n))$ 241
7.6 Exercises 245

8 Lines on Hypersurfaces 249
8.1 What to Expect . . . 249
8.2 Fano Schemes and Chern Classes . . . 251
8.2.1 Counting lines on cubics . . . 253
8.3 Tangent spaces to Fano schemes . . . 254
8.3.1 Normal Bundles and the Smoothness of the Fano Scheme . . . 255
8.3.2 First-order Deformations . . . 260
8.3.3 Normal bundles of lines on hypersurfaces . . . 263
8.4 Lines on Quintic Threefolds and Beyond . . . 266
8.5 The Universal Fano Scheme and the Geometry of Families of Lines . . . 269
8.5.1 Lines on the quartic surfaces in a pencil . . . 274
8.6 Local Equations for $F_1(X)$ . . . 274
8.6.1 Tangent spaces to Fano schemes via local equations 275
8.6.2 Lines on singular cubics . . . 277
8.7 The Debarre/de Jong Conjecture . . . 279
8.7.1 Further open problems . . . 282
8.8 Exercises . . . 283

9 Singular Elements of Linear Series 289
9.1 General Singular Hypersurfaces and the Universal Singularity 290
9.2 The bundles of principal parts . . . 293
9.3 Singular Elements of a Pencil . . . 296
9.3.1 From Pencils to Degeneracy Loci . . . 296
9.3.2 The Chern Class of a Bundle of Principal Parts . . 297
9.3.3 Triple Points of Plane Curves . . . 301
9.3.4 Higher Dimensions and Other Line Bundles . . . 302
9.4 Geometry Of Nets of Plane Curves . . . 304
9.4.1 Class Of The Universal Singular Point . . . 304
9.4.2 Singular Points and Discriminant in a Net of Plane Curves . . . 305
9.5 Flexes, and inflection points in general . . . 308
9.5.1 Inflection points . . . 308
9.5.2 The Plücker formula . . . 310
9.5.3 Consequences of the Plücker formula . . . 312
9.5.4 Curves with Hyperflexes . . . 314
9.6 The Topological Hurwitz Formula . . . 314
9.6.1 Application to pencils . . . 316
9.6.2 Multiplicities of the discriminant hypersurface . . . 318
9.6.3 Tangent cones . . . 319

10 Compactifying Parameter Spaces 323
10.1 Approaches To The Five Conic Problem . . . 324
Blowing Up The Excess Locus . . . 325

Excess Intersection Formulas 326
Changing the Parameter Space 326
10.2 Complete conics 327
10.2.1 Informal Description 327
Degenerating the dual 327
Types of Complete Conics 329
10.2.2 Rigorous Description 329
Duals of Quadrics 329
Equations For the Variety of Complete Conics . . . 331
10.2.3 Solution to the Five Conic Problem 333
Outline 334
Complete Conics Tangent To Five General Conics are Smooth 334
Transversality 335
10.2.4 Chow Ring Of The Space Of Complete Conics . . . 337
The Class of the Divisor of Complete Conics Tangent to C 338
10.3 Parameter spaces of curves 340
10.3.1 Hilbert schemes 340
10.3.2 Examples of Hilbert Schemes 340
Hypersurfaces 340
Twisted Cubics 340
Report Card for the Hilbert Scheme 341
10.3.3 The Kontsevich space 342
10.3.4 Examples of Kontsevich spaces 343
Plane conics 343
Conics in space 343
Plane cubics 344
Twisted cubics 344
Report Card for the Kontsevich Space 345
10.4 Rational plane curves 346

11 Chow Rings of Projective Bundles 351
11.1 Projective bundles and the tautological divisor class 351
11.2 Chow ring of a projective bundle 353
11.2.1 Example: the blow-up of $\mathbb{P}^n$ along a linear space . . 358
11.3 Chow ring of a Grassmannian bundle 358
11.4 Conics in $\mathbb{P}^3$ Meeting Eight Lines 360
11.4.1 The Parameter Space 360
11.4.2 Transversality 363
11.4.3 The Chow Ring 366
11.4.4 The Cycle of Plane Conics Meeting a Line 367

12 Segre Classes and Varieties of Linear Spaces 371
12.1 Segre Classes 372

12.2 Universal hyperplane 375
12.3 Varieties Swept Out by Linear Spaces 376
12.3.1 Lines on hypersurfaces revisited 376
12.4 Secants of Rational Curves 378
12.4.1 Points on Secants and Sums of Powers 378
12.4.2 Degrees of secant varieties to rational curves 382
12.4.3 Chern classes of E 383
12.5 Subbundles of Projective Bundles 386
12.5.1 The Class of a Subbundle 386
12.5.2 Ruled surfaces 387
12.5.3 Self-intersection of the zero section of a vector bundle 388
12.6 Dual varieties and conormal varieties 390
12.6.1 The universal hyperplane as a projective bundle . . 390
12.6.2 Conormal and dual varieties 391
12.6.3 Proof of the duality theorem 393
12.6.4 Degree of the dual 397
12.6.5 Pencils of hypersurfaces 399

13 The Tangent Bundle of a Projective Bundle and Relative Principal Parts 401
13.0.6 Chern Classes of Relative Tangent Bundles 402
13.1 Application: Contact problems 403
13.1.1 Contact bundles 405
13.1.2 Chern classes of contact bundles 406
13.1.3 Counting lines with contact of order 5 407
13.2 The Case of Negative Expected Dimension 409
13.2.1 Predestination versus Free Will 409
13.2.2 Lines on surfaces of degree $d \geq 4$ in $\mathbb{P}^3$ 410
13.2.3 Other configurations with negative expected dimension 412
13.3 Flexes, again 413
13.3.1 The Cartesian view of flexes 413
13.4 Cusps of plane curves 420
13.4.1 Another approach to the cusp problem 423

14 Porteous' Formula 429
14.1 Porteous' formula 429
14.2 The top Chern class of a tensor product 432
14.2.1 Sylvester's determinant 432
14.3 Back to our original problem 436
14.3.1 One last wrinkle 439
14.4 Applications 440
14.4.1 Degrees of determinantal varieties 440
14.4.2 Pinch points of surfaces 443
14.4.3 Quadrisecant lines to a rational space curve 445

Other proofs 449
14.5 Miscellaneous 450
14.5.1 Porteous for symmetric and skew-symmetric bundle maps 450
14.5.2 Positive vector bundles; Statement of Fulton-Lazarsfeld theorem 450
14.5.3 Webs of quadrics 450

15 The Chow Ring of a Blowup **451**
15.1 The Chow Ring of a Blowup 452
15.2 The blow-up of $\mathbb{P}^3$ along a curve 455
15.2.1 The intersection of three surfaces in $\mathbb{P}^3$ containing a curve 456
15.3 Intersections in a Subvariety 459
15.3.1 Deformation to the normal cone 460

16 Excess Intersection Formulas **465**
16.1 Introduction to the Excess Intersection Formula 467
16.1.1 First Examples 469
16.1.2 Intersection of two surfaces in $\mathbb{P}^4$ containing a curve 470
16.2 Justification of the Formula 471
16.3 Avoiding Smoothness Hypotheses 473
16.3.1 Computing the Segre class of a Cone 475
16.3.2 How Many Conics are Tangent to Five Lines? . . . 476
16.4 Pullbacks 478
16.5 The double point formula 480
16.5.1 Apparent Double Points of a Surface 481
16.5.2 Generalizations 484

17 The Grothendieck-Riemann-Roch Theorem **485**
17.1 The Riemann-Roch Formula for Curves and Surfaces . . . 485
17.1.1 Nineteenth Century Riemann-Roch 485
17.2 The Chern character 487
17.2.1 K-theory 490
17.2.2 Chern classes of coherent sheaves 490
17.3 Hirzebruch-Riemann-Roch 491
17.4 Higher Direct Images of sheaves 494
17.5 Grothendieck-Riemann-Roch 497
17.6 Application: Jumping lines 499
17.6.1 Vector bundles with even first Chern class 500
17.6.2 Vector bundles with odd first Chern class 502
17.6.3 Examples and Generalizations 504

18 Brill-Noether **507**
18.1 What maps to projective space "should" a curve have? . . 507

The Jacobian 508
Abel's Theorem 509
Riemann-Roch 510
18.1.1 Behavior on special curves: Clifford and Castelnuovo Theorems 511
18.1.2 Behavior on general curves: naive dimension counts leading to statement of Brill-Noether 512
How you might be led to the Brill-Noether Theorem 513
18.1.3 Description of various methods of proof 514
18.2 Chow rings of Jacobians and Symmetric Products of Curves 516
18.2.1 Poincaré's formula 518
18.2.2 Symmetric products as projective bundles 519
18.2.3 Chern classes of tautological bundles on Jacobians . 521
18.3 Application of Porteous; calculation of class of W_d^r 522
18.3.1 Construction of the Poincaré bundle and the bundle E 522
18.3.2 Calculation of the class of W_d^r 523

19 Families of Curves **527**
19.1 Invariants of families of curves 527
19.1.1 Examples 529
Pencils of quartics in the plane 529
Pencils of plane sections of a quartic surface 531
Families of hyperelliptic curves 532
19.1.2 The Mumford Relation 534
19.1.3 A final word 536
19.2 The moduli space of curves 536
19.2.1 Moduli spaces in general; examples in genus 2 . . . 536
19.2.2 Stacks 537
19.2.3 Stable curves, and the Deligne-Mumford compactification 537
19.2.4 Divisor classes on $\overline{M}_g$ 538
19.2.5 Recasting the Mumford relation 538
19.3 The Kodaira dimension of $\overline{M}_g$ 538
19.3.1 Unirationality in low genus 538
19.3.2 The canonical class of $\overline{M}_g$ 539
The Reid-Tai criterion 539
19.3.3 The Brill-Noether divisor 539

20 Appendix: Intersection Theory on Surfaces **541**
20.1 Introduction 541
20.2 Riemann-Roch and Serre Duality for Curves 542
20.3 Curves and adjunction 542
20.3.1 Genus of a Curve in the Plane or on a Quadric Surface 544
20.3.2 Self-Intersection of a Curve on a Surface 545

20.4 Blow-ups of surfaces 546
20.5 The Riemann-Roch Theorem for Surfaces 547

21 Appendix: Other Cycle Theories 549
21.1 Examples 549
21.2 Algebraic equivalence 551
21.3 Homological equivalence 552
21.4 Cohomology 553
21.5 Numerical equivalence 554
21.6 Comparing the Theories 555
21.7 Chern classes in topology and differential geometry 556
21.7.1 Chern Classes and Obstructions 556
21.7.2 Chern Classes and Curvature 558

22 Appendix: Lefschetz Hyperplane Theorem and Applications 561
22.1 Statement of the Lefschetz Hyperplane Theorem 561
22.2 Cohomology of hypersurfaces 563
22.3 Application: points on the universal hypersurface 567
22.4 Extensions and generalizations 569
22.5 Open problems 570

23 Solutions to Selected Exercises 571
23.1 Chapter 1 571
23.2 Chapter 2 588
23.3 Chapter 3 592
23.4 Chapter 4 595
23.5 Chapter 5 603
23.6 Chapter 6 609
23.7 Chapter 8 610

References 617

Introduction

"Es gibt nach des Verf. Erfarhrung kein besseres Mittel, Geometrie zu lernen, als das Studium des Schubertschen 'Kalküls der abzählenden Geometrie'."
(There is, in the author's experience, no better means of learning geometry than the study of Schubert's "Calculus of Enumerative Geometry.")
–B. L. van der Waerden (in a Zentralblatt review of an introduction to enumerative geometry by Hendrik de Vries).

1066 & All That (Sellar and Yeatman [1992]) is "A memorable history of England, comprising all the parts you can remember, including one hundred and three *good* things, five *bad* kings, and two *genuine* dates. . . . History is not what you thought. *It is what you can remember.*

0.1 Why you should read this book

Algebraic geometry is one of the central subjects of mathematics. All but the most analytic of number theorists speak our language, as do mathematical physicists, homotopy theorists, complex analysts, symplectic geometers, representation theorists. . . How else could you get between such apparently disparate fields as topology and number theory in one hop, except via algebraic geometry?

And intersection theory is at the heart of algebraic geometry. From the very beginnings of the subject, the fact that the number of solutions to a

system of polynomial equations is, in many circumstances, constant as we vary the coefficients of those polynomials has fascinated algebraic geometers. The distant extensions of this idea still drive the field forward.

At the outset of the 19th century, it was to extend "preservation of number" that algebraic geometers made two crucial choices: to work over the complex numbers rather than the real numbers, and to work in projective space rather than affine space. (With these choices the two points of intersection of a line and an ellipse have somewhere to go as the ellipse moves away from the real points of the line, and the same vor the point of intersection of two lines as the lines become parallel.) Over the course of the century, geometers refined the art of counting solutions to geometric problems—introducing the central notion of a parameter space; proposing the notions of an equivalence relation on cycles and a product on the equivalence classes and using these in many subtle calculations. These constructions were fundamental to the developing study of algebraic curves and surfaces.

In a different field, it was the search for a mathematically precise way of describing intersections that underlay Poincaré's study of what became algebraic topology; his inability to work with continuous spaces (now called manifolds) led him to develop the idea of a simplicial complex, too. We owe Poincaré duality and a great deal more in algebraic topology directly to this impulse.

Despite the lack of precise foundations, nineteenth century enumerative geometry rose to impressive heights: for example Schubert, whose *Kalkül der abzählenden Geometrie* (Schubert [1979]) represents the summit of intersection theory at the time, calculated the number of twisted cubics tangent to 12 quadrics—and got the right answer (5,819,539,783,680). Imagine landing a jumbo jet blindfolded!

At the outset of the 20th century, Hilbert made finding rigorous foundations for the subject one of his celebrated Problems, and the quest to put intersection theory on a sound footing drove much of algebraic geometry for the following century; the search for a definition of multiplicity fueled the subject of commutative algebra in work of van der Waerden, Zariski, Samuel, Weil and Serre. This progress culminated, towards the end of the century, in the work of Fulton and MacPherson and then in Fulton's landmark book *Intersection Theory* (Fulton [1984]), which both greatly extended the range of intersection theory and put the subject on a precise and rigorous foundation.

The development of intersection theory is far from finished. Today the focus is on things like virtual fundamental cycles, quantum intersection rings, Gromov-Witten theory and the extension of intersection theory from schemes to stacks.

A central part of a central subject of mathematics—of course you want to read this book!

0.2 Why we wrote this book

Given the centrality of the subject, it is not surprising how much of algebraic geometry one encounters in learning enumerative geometry. And that's how this book came to be written, and why: like van der Waerden, we found that intersection theory made for a great "second course" in algebraic geometry, weaving together threads from all over the subject. Moreover, the new ideas one meets have a context in which they're not merely more abstract definitions for the student to memorize, but tools that help answer concrete questions.

0.3 What's with the title?

The number in the title of this book is a reference to the solution of a classic problem in enumerative geometry: the determination, by Chasles, of the number of smooth conic plane curves tangent to five given general conics. The problem is emblematic of the dual nature of the subject. On the one hand, the number itself is of little significance: would you really want to write down all the conics tangent to five given ones? Life would not be materially different if there were more or fewer. But the fact that the problem is well-posed—that there is a Zariski open subset of the space of 5-tuples of conics $(C_1, \ldots, C_5)$ for which the number of conics tangent to all five is constant, and that we can in fact determine that number—is at the heart of algebraic geometry. And the insights developed in the pursuit of a rigorous derivation of the number—the recognition of the need for, and introduction of, a new parameter space for plane conics, and the understanding of why intersection products are well defined for this space—are landmarks in the development of algebraic geometry.

The rest of the title is from "1066 & All That" by W. C. Sellar and R. J. Yeatman, a parody of English history textbooks; in many ways the number 3264 of conics tangent to five general conics is as emblematic of enumerative geometry as the date 1066 of the Battle of Hastings is of English history.

0.4 What's in this book

It is commonly the case that an author finds it hard to know how to end a book and some even include two endings. This book has two beginnings!

0.4.1 Overture

In the first, the "Overture," we very quickly introduce rational equivalence, the Chow ring, and the pull-back and push-forward maps, and some other tools—the "Dogma" of the subject—up to but not including Chern classes. We follow this with a range of simple examples to give the reader a sense of the themes to come: the computation of Chow rings of affine and projective spaces, their products and blowups. To illustrate how intersection theory is used in algebraic geometry, we examine loci of various types of singular cubic plane curves, thought of as subvarieties of the projective space $\mathbb{P}^9$ of plane cubics; and to illustrate how the language of intersection theory has become pervasive in the subject of algebraic geometry, we give a quick run-through of the theory of algebraic surfaces.

We should mention here that there are many other possible cycle theories that allow us to carry out the basic constructions of intersection theory. These, and the relations among them, are discussed in Appendix 21.

0.4.2 Second beginning

— **Eppur, se muove!** (And yet, it moves!—Attributed to Galileo)

After the overture, the book starts again. The first two chapters are devoted to proofs and substantial discussion of the results of the Dogma section. Chapter 4 introduces the foundational ideas of commutative algebra, centered around finite extensions of one-dimensional rings, that are necessary in establishing the properties of the pushforward map.

We have chosen to base our development of the theory on the *Moving Lemma*, a central result that allows us to define intersection products in a relatively intuitive way. Chapter **??** introduces the beautiful geometry, first described by Severi and then refined by Chow and others, of its proof.

This is not the only way to develop intersection products. The "deformation to the normal cone" (explained in Chapter **??**), the foundational idea used in Fulton [1984], has many advantages, and leads ultimately to a sharper and more general theory. The foundations are equivalent where the y both apply, and as the student becomes an expert the transition to the new point of view will be natural. But we felt that, in its more limited sphere, the approach via the Moving Lemma gives a direct intuition for

why the basic results are true that is difficult to gain from the newer approach. One might compare the situation to that encountered in learning basic algebraic geomety: despite the fact that the theory of schemes is a more natural and supple language for the theory, a beginner may be well served by learning the more concrete case of varieties first.

The intersection rings of the Grassmannians are archtypal examples of intersection theory. Chapter 2, is devoted to them and the geometry that underlies them. Here we introduce the Schubert cycles, which form a basis for the Chow ring, and we use them to solve a number of geometric problems.

We then come to a watershed in the subject. In a brief interlude we review the basics of the theory of vector bundles (a.k.a. locally free sheaves) and their direct images (referring for a full treatment of cohomology and base change to Chapter 17). With these notions in hand, Chapter 7 introduces a notion that is at the center of modern intersection theory, and indeed of modern algebraic geometry: Chern classes. As with the development of intersection theory we define these in the classical style, as degeneracy loci of collections of sections (rather than via the Chow ring of projective bundles, an idea explained in Chapter **??**). We feel that this interpretation provides useful intuition and in any case it is basic to many applications of the theory.

0.4.3 Using the tools

The introduction of Chern classes is motivated by a discussions of two classical problem with modern ramifications: Chapter 8 takes up the question of how many lines lie on a hypersurface (for example there are exactly 27 lines on each smooth cubic surface), and Chapter 9 looks at the singular hypersurfaces in a one-dimensional family (what is the degree of the discriminant). Using the basic technique of *linearization*, these problems can be translated into problems of computing Chern classes. These and the next few chapters are organized around the different constructions of vector bundles to which one can apply the technique of Chern classes.

Chapter 10 deals with an area in which intersection theory has had a profound influence on modern algebraic geometry: the notion of *parameter spaces* and their compactifications. This is illustrated with the five-conic problem; there is also a discussion of the more modern example of Kontsevich spaces, and an application of that theory.

0.4.4 Further developments

The remainder of the book introduces a series of increasingly advanced topics. Chapters 11, 12 and 13 deal with a situation ubiquitous in the subject, the intersection theory on projective bundles, and its applications to subjects such as projective duality and the enumerative geometry of contact conditions.

Chern classes are defined in terms of the loci where a collection of sections of a vector bundle become dependent. This can be interpreted as the locus where a map from a trivial vector bundle drops rank. One might think that one would need new invariants to describe the locus where a map between more general vector bundles drops rank, but this is not the case: the Porteous formula, proved and applied in Chapter 14, expresses these in terms of the Chern classes of the two bundles involved.

Next, we come to some of the developments of the modern theory of intersections. In Chapter 15, we introduce the idea of *deformation to the normal cone*, a construction fundamental to the work of Fulton and Macpherson; we use this to derive the famous "key formula" comparing intersections of cycles in a subvariety $Z \subset X$ to the intersections of those cycles in X, and use this in turn to give a description of the Chow ring of a blow-up.

Chapter 16 deals in turn with what may be viewed as a tremendous generalization of the key formula: the notion of "excess" intersections and the *excess intersection formula*, one of the subjects that was particularly mysterious in the nineteenth century. This theory makes it possible to describe the intersection class of two cycles even if their intersection has "too large" dimension.

Chapter 17 contains an account of Riemann-Roch formulas, leading up to a description (but not a proof) of Grothendieck's version. Along the way we define the direct image complex of a sheaf, and prove the Theorem on Cohomology and Base Change. The chapter concludes with a number of examples and applications showing how Grothendieck's formula can be used.

The last Chapter of the book, Chapter **??**, explains a larger-scale application of enumerative geometry to a problem that is central in the study of algebraic curves and their moduli spaces: the existence half of the Brill-Noether theorem on linear series on curves. We give the original proof of this theorem (by Kempf and Kleiman-Laksov), which draws upon many of the ideas and techniques of the book, plus one new one: the use of topological cohomology in the context of intersection theory. This is also a wonderful illustration of the way in which enumerative geometry can be the essential ingredient in the proof of a purely qualitative result.

0.4.5 Relation to "Intersection Theory"

Fulton's book (Fulton [1984]) on intersection theory is a great work, setting up a rigorous framework for intersection theory in great generality, significantly extending what was known before and laying out an enormous number of applications. But because each notion is developed in full generality, many beautiful and elementary things wait in the wings while the heavy artillery—necessary only for the more difficult applications—is prepared. It is a work that can serve as an encyclopedic reference to the subject.

By contrast, the present volume is intended as a textbook, a second course in algebraic geometry in which the classical side of intersection theory is a starting point for exploring many topics in geometry. We introduce the intersection product at the outset, and focus on basic examples. We use concrete problems to motivate the introduction of new tools from all over algebraic geometry. Our book is not a substitute for Fulton's: it has a different aim. We hope that it will, in particular provide the reader with intuition and motivation that will make reading Fulton's book easier.

0.4.6 Keynote Problems

To highlight the sort of problems we'll learn to solve, and to motivate the material we present, we'll begin each chapter with some *Keynote Questions*.

0.5 What you should know before you begin

When it comes to prerequisites, there are two distinct questions: what you need to know to start reading this book; and what you should be prepared to learn along the way.

In the course of developing and applying intersection theory we introduce many key techniques of algebraic geometry, such as deformation theory, specialization methods, characteristic classes, commutative and homological algebra and topological methods, in contexts in which they're used. We sketch the necessary theory, but don't always give a detailed account. That's not to say that you need to know these things going in. Just the opposite, in fact: reading this book should be viewed as an occasion to learn them.

To aid you we try to provide a guide to the relevant literature. In each Chapter, we indicate what new ideas or techniques from algebraic geometry are being used, and give suggestions of where the reader might go

for background or a further development of the subject. For example, you certainly need not have read the first author's *Commutative algebra with a view toward algebraic geometry* (Eisenbud [1995]), but there are many specific references to it.

So what do you need before starting?

(a) An undergraduate course in classical algebraic geometry or its equivalent, comprising the elementary theory of affine and projective varieties. *An Invitation to Algebraic Geometry* (Smith et al. [2000]) contains very nearly just what's required. Other books that cover this material include *Undergraduate Algebraic Geometry* (Reid [1988a]), *Introduction to Algebraic Geometry* (Hassett [2007]) and, at a somewhat more advanced level (with more than we need) *Algebraic Geometry I: Complex Projective Varieties* (Mumford [1976]), *Basic Algebraic Geometry, Volume I* (Shafarevich [1974]) and *Algebraic Geometry: A First Course* (Harris [1992]). The last three include much more than we'll use here.

(b) An acquaintance with the language of schemes. This would be amply covered by the first two or three Chapters *The Geometry of Schemes* (Eisenbud and Harris [2000]). Chapter 2 of *Algebraic Geometry* (Hartshorne [1977]) is more than enough: we won't need all the technical results so much as just a sense of what affine and projective schemes are like.

(c) An acquaintance with coherent sheaves and their cohomology. For this, *Faisceax Algébriques Cohérents* (Serre [1955]) remains an excellent source (it's written in the language of varieties, but applies nearly word for word to projective schemes over a field, the context in which this book is written). Chapter 3 of *Algebraic Geometry* (Hartshorne [1977]) is more than you need to know to get started.

(d) A willingness to look things up or work them out for yourself as you go along.

Of these, the fourth is the most important!

0.6 What's in a scheme?

Throughout this book, a *scheme* X will be a scheme of finite type over a field K, and a *variety* will be a reduced, irreducible scheme. If X is a variety we write $K(X)$ for the field of rational functions on X.

By a *point* we mean a closed point. Recall that a *locally closed* subscheme U of a scheme X is a scheme that is an open subset of a closed subscheme of X, which may of course be taken to be the closure $\overline{U} \subset X$. We generally use the term "subscheme" (without any modifier) to mean a closed subscheme, and similarly for "subvariety".

An important consequence of the finite type hypothesis is that primary decomposition and dimension theory work well: any subscheme of X has a primary decomposition with finitely many components. We distinguish the *isolated* components (maximal as sets) and *embedded* components (the rest). We often use the term *irreducible component* to refer to a reduced isolated irreducible component. We say that a scheme X is *equidimensional* if every irreducible component has the same dimension. When X is equidimensional and Y is a subvariety, we have $\dim Y + \operatorname{codim} Y = \dim X$. Moreover, the *normalization* of X is again of finite type over K.

We write $\mathcal{O}_{X,Y}$ for the local ring of X along Y, and more generally, if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules then we write $\mathcal{F}_Y$ for the corresponding $\mathcal{O}_{X,Y}$-module.

We will often use Krull's Principal Ideal Theorem (Eisenbud [1995] Theorem 10.2), which says that an ideal generated by n elements in a Noetherian ring cannot have codimension greater than n. We will also use the Chinese Remainder Theorem in the form saying that a module of finite length is the direct sum of its localizations at finitely many maximal ideals, and the Jordan-Hölder Theorem that over a local ring such a module has a unique composition series (Eisenbud [1995] Theorem 2.13). Putting these together, we see that a module of finite length has a well-defined *length*—the sum of the lengths of the composition series of its localizations.

1

Overture

Keynote Questions

(a) Let F_0, F_1 and $F_2 \in \mathbb{C}[X, Y, Z]$ be three general homogeneous cubic polynomials in three variables. How many linear combinations $t_0F_0 + t_1F_1 + t_2F_2$ (up to scalars) factor as a product of a linear and a quadratic polynomial?

(b) Now let F_0, F_1, F_2 and $F_3 \in \mathbb{C}[X, Y, Z]$ be four general homogeneous cubic polynomials in three variables. How many linear combinations $t_0F_0 + t_1F_1 + t_2F_2 + t_3F_3$ (again, up to scalars) factor as a product of three linear polynomials?

(c) If A, B, C are general homogeneous quadratic polynomials in 3 variables, for how many triples $t = (t_0, t_1, t_2)$ is $(A(t), B(t), C(t))$ proportional to (t_0, t_1, t_2)?

(d) Let $S \subset \mathbb{P}^3$ be a smooth cubic surface, and $\{H_\lambda \subset \mathbb{P}^3\}_{l \in \mathbb{P}^1}$ a general pencil of planes (that is, the set of planes containing a fixed general line $L \subset \mathbb{P}^3$). For how many values of λ is the intersection $H_\lambda \cap S$ a singular curve?

The machinery necessary for intersection theory is beautiful and subtle, and it will take us quite a few pages to develop. Nevertheless, a few facts suffice to begin using the theory. In this chapter we will explain some of these facts without worrying about how they are proved and give a number of simple examples. Our goal is to give the reader a road-map for reading

the rest of the book, and to provide a box of tools that are used most commonly in applications—the "dogma" of intersection theory.

We begin by describing the construction of the Chow Ring of a smooth projective variety and explaining some of the properties that make it useful. The second section is a series of progressively more complicated examples of computations of Chow rings of some familiar varieties, and easy applications of their structure. The section concludes with an example of a different kind: we use facts about the Chow ring to describe some geometrically interesting loci in the projective space of singular cubic plane curves.

In the third section of this chapter we turn to the other major pillar of the subject, the theory of Chern classes. Again this is followed by a section explaining some of the simplest applications.

Very little is proven in the theory sections of this chapter; but in the sections of examples we have tried to give fairly complete expositions. In the following chapters we start fresh, and build up the theory rigorously.

1.1 The Chow Ring

We start with the Chow groups of a scheme. These were perhaps the first examples of what we now know as homology theories—abelian groups associated to a geometric object that are described as a group of cycles modulo an equivalence relation.

1.1.1 Cycles

Let X be any algebraic variety or scheme. The *group of cycles* on X, denoted $Z(X)$, is the free abelian group generated by the set of subvarieties (reduced irreducible subschemes) of X. The group $Z(X)$ is graded by dimension: we write $Z_k(X)$ for the group of cycles that are formal linear combinations of subvarieties of dimension k—these are called *k-cycles*. A cycle $Z = \sum n_i Y_i$ is *effective* if the coefficients n_i are all non-negative. A *divisor* (sometimes called a *Weil Divisisor*) is an $n-1$-cycle on an n-dimensional variety. It follows from the definition that $Z(X) = Z(X_{\mathrm{red}})$; that is, $Z(X)$ is insensitive to whatever non-reduced structure X may have.

We can associate an effective cycle with any closed subscheme $Y \subset X$ by using an idea from algebra: If A is any commutative ring then we say that an A-module M has finite length if it has a finite *composition series* (a sequence of submodules $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_l = 0$ such that each factor module M_i/M_{i+1} is a simple A-module). The Jordan-Hölder Theorem

(Eisenbud [1995], Theorem 2.13) asserts that the number of times a given simple A-module appears as a factor M_i/M_{i+1} does not depend on the composition series chosen; in particular, the length l, written $\text{length}_A(M)$, depends only on A and M. Since our schemes are Noetherian, the local ring of a generic point of an irreducible component of any scheme has finite length as a module over itself.

Returning to the case of a subscheme $Y \subset X$, we let $Y_1, \ldots, Y_s$ be the (isolated, reduced) irreducible components of Y. Each local ring $\mathcal{O}_{Y,Y_i}$ has finite length, say l_i. We define the cycle Z associated to Y to be the formal combination $\sum l_i Y_i$. When we want to emphasize the distinction we will write $\langle Y \rangle$ for the cycle associated to the subscheme Y.

1.1.2 Rational equivalence

Next we will define *rational equivalence* between cycles, and also the Chow group, which is the group of cycles mod rational equivalence. Informally speaking, we say that two cycles $A_0, A_1 \in Z(X)$ are rationally equivalent if there is a rationally parametrized family of cycles interpolating between them—that is, a cycle on $\mathbb{P}^1 \times X$ whose restrictions to two fibers $\{t_0\} \times X$ and $\{t_1\} \times X$ are A_0 and A_1. Here is the formal definition:

Definition 1.1. Let $Rat(X) \subset Z(X)$ be the subgroup generated by differences of the form

$$\big\langle \Phi \cap (\{t_0\} \times X) \big\rangle - \big\langle \Phi \cap (\{t_1\} \times X) \big\rangle,$$

where t_0 and $t_1 \in \mathbb{P}^1$ and Φ is a subvariety of $\mathbb{P}^1 \times X$not contained in any fiber $\{t\} \times X$. Two cycles are *rationally equivalent* if their difference is in $Rat(X)$, and two subschemes are rationally equivalent if their associated cycles are rationally equivalent. The *Chow group* of X is the quotient $A(X) = Z(X)/Rat(X)$, the *group of rational equivalence classes of cycles on* X. If Y is a subscheme, we write $[Y]$ for the equivalence class corresponding to the cycle $\langle Y \rangle$.

There is another useful way to characterize the group $Rat(X)$ of cycles rationally equivalent to 0. If X is an affine variety and $f \in \mathcal{O}_X$ is a function on X, then by the Krull's Principal Ideal Theorem the isolated components of the subscheme defined by f are all of codimension 1, so the cycle defined by this subscheme is a divisor; we call it $Div(f)$, the *divisor of* f. If Y is any codimension 1 subscheme of X we write $\text{ord}_Y(f)$ for the order of vanishing of f along Y, so we have

$$Div(f) = \sum_Y ord_Y(f)[Y].$$

If $\alpha = f/g$ is a rational function on X, we define the divisor $Div(f/g) = Div(f) - Div(g)$. We will show after Proposition 4.8, that this divisor is independent of the choice of representation f/g of α.

If X is a variety then the field of rational functions on X is the same as the field of rational functions on any open affine subset U of X, so if α is a rational function on X then, restricting α, we get a divisor $Div(\alpha|_U)$ on U. These agree on overlaps, and thus define a divisor $Div(\alpha)$ on X itself. We will see that the association $\alpha \mapsto Div(\alpha)$ is a homomorphism from the multiplicative group of nonzero rational functions to the additive group of divisors on X.

Proposition 1.2. *The group $Rat(X) \subset Z(X)$ is generated by all divisors of rational functions on all subvarieties of X.*

See Corollary 4.17 for the proof.

Since the divisor of a rational function on an m-dimensional subvariety of X is a sum of cycles of dimension $m-1$, the Chow group is graded by dimension,

$$A(X) = \bigoplus_k A_k(X)$$

and $A_k(X)$ is called the *group of rational equivalence classes of k-cycles.* We will use the notation $[Y]$ for the class modulo rational equivalence of a cycle Y.

Example 1.3. It follows from the Proposition that two finite subsets of a curve (=1-dimensional variety) are rationally equivalent if and only if they differ by the divisor of a rational function. Thus two points on a curve can be rationally equivalent if and only if the curve is isomorphic to $\mathbb{P}^1$, the isomorphism being given by that rational function that defines the rational equivalence.

Example 1.4. We will soon see that two subvarieties of $\mathbb{P}^n$ are rationally equivalent if and only if they have the same dimension and degree—in this case rational equivalence is a rather loose relationship.

When X is equidimensional, the *codimension* of a subvariety $Y \subset X$ is $\dim X - \dim Y$. When X is also smooth, we will write $A^c(X)$ for the group $A_{\dim X - c}$, and think of it as the group of codimension c cycles, mod rational equivalence. (It would occasionally be convenient to adopt the same notation when X is singular, but this would conflict with established convention (see the discussion in Section 5.4 below.)

1.1.3 First Chern class of a line bundle

One way that rational equivalence classes of cycles arise is from line bundles (in this book we will use "line bundle" and "locally free sheaf of rank 1" interchangeably). If L is a line bundle on a variety X and σ is a rational section, then on an open affine set U of a covering of X we may write σ in the form f_U/g_U and define $Div(\sigma)|_U = Div(f) - Div(g)$. Again, this patches on overlaps, and thus defines a divisor on X. Moreover, if τ is another rational section of L then $\alpha = \sigma/\tau$ is a well defined rational function, so

$$Div(\sigma) - Div(\tau) = Div(\alpha) \equiv 0 \bmod Rat(X).$$

Thus for any line bundle L on X we may define the *first Chern class*

$$c_1(L) \in A(X)$$

to be the rational equivalence class of the divisor σ for any rational section σ. Note that there is no distinguished cycle in the equivalence class. As a first example, we see that $c_1(\mathcal{O}_{\mathbb{P}^n}(d))$ is the class of any hypersurface of degree d; in the notation of Section 1.2.2 it is $d\zeta$, where ζ is the class of a hyperplane.

Recall that $\mathrm{Pic}(X)$ is by definition the group of isomorphism classes of line bundles L on X, with addition law $[L] + [L'] = [L \otimes L']$.

Proposition 1.5. *If X is a variety of dimension n then c_1 is a group homomorphism*

$$c_1 : \mathrm{Pic}(X) \to A_{n-1}(X),$$

If X is smooth then c_1 is an isomorphism.

In Section 1.3 we will describe Chern classes for vector bundles more generally. They are one of the most powerful tools of intersection theory.

Proof of Proposition 1.5. To see that c_1 is a group homomorphism, suppose that L and L' are line bundles on X. If σ and σ' are rational sections of L and L' respectively, then $\sigma \otimes \sigma'$ is a rational section of $L \otimes L'$ whose divisor is $Div(\sigma) + Div(\sigma')$.

Now assume that X is smooth and projective. Since the local rings of X are unique factorization domains, every codimension 1 subvariety is a Cartier divisor, so to any divisor we can associate a unique line bundle and a rational section. Forgetting the section, we get a line bundle, and thus a map from the group of divisors to $\mathrm{Pic}(X)$. By Proposition 1.2, rationally equivalent divisors differ by the divisor of a rational function and thus correspond to different rational sections of the same bundle. It follows that the map on divisors induces a map on $A^1(X)$, inverse to the map c_1. □

In case X is singular the map $c_1 : \mathrm{Pic}(X) \to A_{n-1}(X)$ is in general neither injective or surjective. For example, if X is an irreducible plane cubic with a node, then $c_1 : Pic(X) \to A_1(X)$ is not a monomorphism (Exercise 1.52). On the other hand, if $X \subset \mathbb{P}^3$ a quadric cone with vertex p then $A_1(X) = \mathbb{Z}$, generated by the class of a line, and the image of $c_1 : Pic(X) \to A_1(X)$ is $2\mathbb{Z}$ (Exercise 1.53).

1.1.4 First results on the Chow Group

Proposition 1.2 makes it obvious that if $Y \subset X$ is a closed subscheme then the identification of the cycles on $\mathbb{P}^1 \times Y$ as cycles on $\mathbb{P}^1 \times X$ induces a map $Rat(Y) \to Rat(X)$, and thus a map $A(Y) \to A(X)$. Further, the intersection of a subvariety of X with the open set $U = X \setminus Y$ is a subvariety of U (possibly empty), so there is restriction homomorphism $Z(X) \to Z(U)$. It turns out that this induces a homomorphism of Chow groups. From this we get the fourth part of the following useful proposition; see Proposition 4.11 for the proof.

Proposition 1.6. *Let X be a scheme.*

(a) $A(X) = A(X_{\mathrm{red}})$.

(b) *If X is irreducible of dimension k, then $A_k(X) \cong \mathbb{Z}$, with generator $[X]$, called the* fundamental class *of X. More generally, if the irreducible components of X are $X_1, \ldots, X_m$ then the classes $[X_i]$ generate a free abelian subgroup of rank m in $A(X)$.*

(c) *(Mayer-Vietoris) If X_1, X_2 are closed subschemes of X, then there is a right exact sequence*

$$A(X_1 \cap X_2) \longrightarrow A(X_1) \oplus A(X_2) \longrightarrow A(X_1 \cup X_2) \longrightarrow 0.$$

(d) *(Excision) If $Y \subset X$ is a closed subscheme and $U = Y \setminus X$ is its complement, then the the inclusion and restriction maps of cycles give a right exact sequence*

$$A(Y) \longrightarrow A(X) \longrightarrow A(U) \longrightarrow 0.$$

For example, if X is the spectrum of an artinian ring, then X has no proper subvarieties except for its irreducible components, so Parts (a) and (b) imply that $A(X) = A_0(X)$ is the free abelian group on the components of X.

1.1.5 Intersection products

One of the earliest results about intersections is the theorem of Bézout (1776): If curves $D, E \subset \mathbb{P}^2$, of degrees d and e, intersect transversely,

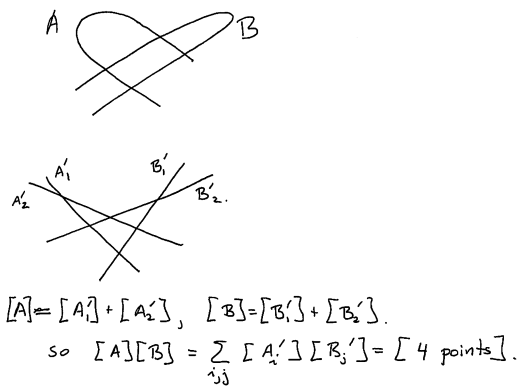


FIGURE 2.01. Two conics meet in four points. [[SILVIO the curves should be conics.]]

then they intersect in exactly *de* points. Two things about this result are striking: first, the cardinality of the intersection does not depend on the choice of curves, beyond knowing their degrees and that they meet transversely. Given this invariance, the theorem follows from the obvious fact that a union of d general lines meets a union of e general lines in de points. Second, the answer, de, is a product, suggesting that some sort of ring structure is present. A great deal of the development of algebraic geometry over the past 200 years—one might say its whole history—is bound up in the attempt to understand, generalize and apply these ideas.

We will show that, indeed, cycles mod rational equivalence on any smooth quasi-projective variety form a ring of this sort. Moreover, the ring structure is unique if we insist on one geometrically natural condition, which we now explain.

We say that subvarieties A, B of X intersect *transversely* at a point p if A, B and X are all smooth at p and the tangent spaces to A and B at p together span the tangent space to X; that is,

$$T_pA + T_pB = T_pX.$$

We say that A and B are *generically transverse*, or that they intersect *generically transversely* if they meet transversely at a general point of each component C of $A \cap B$. If X is smooth, this is equivalent to saying that $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$ and C is generically reduced. Note that if $\operatorname{codim} A + \operatorname{codim} B > \dim X$, then then A and B are generically transverse if and only if they are disjoint.

Theorem 1.7. *If X is a smooth quasiprojective variety then there is a unique product structure on $A(X)$ satisfying the condition:*

(*) If subvarieties A, B of X are generically transverse then

$$[A][B] = [A \cap B]$$

This structure makes $A(X)$ into an associative, commutative ring graded by codimension.

Definition 1.8. The *Chow Ring* of X is the ring described in Theorem 1.7.

The hypothesis of smoothness in Theorem 1.7 cannot be avoided completeley. See Section 5.4 for examples where no intersection product with reasonable properties can be defined in general.

To prove Theorem 1.7 we must somehow compare an arbitrary pair of cycles with a pair of cycles that meet generically transversely. The key to our development will be the following statement:

Theorem 1.9 (Moving Lemma)**.** *Let X be a smooth quasiprojective variety.*

(a) *For every $\alpha, \beta \in A(X)$ there are cycles $A = \sum m_i A_i$ and $B = \sum n_j B_j$ in $Z(X)$ that represent α and β and so that A_i intersects B_j generically transversely for each i and j.*

(b) *The class*

$$\sum_{i,j} m_i n_j [A_i \cap B_j]$$

in $A(X)$ is independent of the choice of such A, B.

The Moving Lemma is easy to prove when a sufficiently large group of automorphisms acts on X: we can simply use an automorphism of X to move one of the cycles we would like to intersect, as in the following special case of a result of Kleiman (see Theorem 5.20):

Theorem 1.10 (Kleiman's Theorem in Characteristic Zero)**.** *Suppose that an affine algebraic group G acts on a variety X and that $A, B \subset X$ are subvarieties. For any $g \in G$ the variety gA is rationally equivalent to A. If the action of G on X is transitive, and the ground field has characteristic zero, then there is an open dense set of $g \in G$ such that gA is generically transverse to B.*

Kleiman showed that the same conclusion holds in positive characteristic under a stronger transitivity hypothesis (Theorem 5.20). Another case when the Moving Lemma is easy is when the class of the cycle to be moved has the form $c_1(L)$ for some line bundle L. In this case we don't need smoothness. We also get a useful formula for the product of any class with $c_1(L)$:

Proposition 1.11. *Suppose that X is a quasiprojective variety, and L is a line bundle on X. If $Y_1, \ldots, Y_n$ are any subvarieties of X then there is a cycle in the class of $c_1(L)$ that is generically transverse to each Y_i. If X is smooth, then*

$$c_1(L)\left[\sum_i m_i Y_i\right] = \sum_i m_i c_1(L|_{Y_i}).$$

We have abused our notation: the class $c_1(L|_{Y_i})$ on the right hand side of the formula is actually a class in $A(Y_i)$, so to be precise we should have written $j_*(c_1(L|_{Y_i})$, where $j : Y_i \to X$ is the inclusion. This imprecision points to an important theoretical fact: the intersection is actually well-defined, not just on X, but as a class on Y_i. This is the beginning of the theory of the "refined intersection product" defined in Fulton [1984].

We imposed the hypothesis of smoothness in Proposition 1.11 because we have only discussed products in this context. In fact, the formula could be used to define an action of a class of the form $c_1(L)$ on $A(X)$ for any scheme X. This is the point of view taken in Fulton [1984].

Sketch of Proof of Proposition 1.11. Since X is quasiprojective there is an ample bundle L' on X. For a sufficiently large integer n both the line bundles $L'^{\otimes n}$ and $L'^{\otimes n} \otimes L$ are very ample, so by Bertini's Theorem there are sections $\sigma \in H^0(L'^{\otimes n})$ and $\tau \in H^0(L'^{\otimes n} \otimes L)$ whose zero loci $Div(\sigma)$ and $Div(\tau)$ are generically transverse to each Y_i. The class $c_1(L)$ is rationally equivalent to the cycle $Div(\sigma) - Div(\tau)$, proving the first assertion. Moreover, $c_1(L)[Y_i] = [Div(\sigma) \cap Y_i] - [Div(\tau) \cap Y_i]$ by definition. Since $Div(\sigma) \cap Y_i = Div(\sigma|_{Y_i})$, and similarly for τ, we are done. □

Multiplicities. We can easily extend the range of cases where intersection products have a simple geometric interpretation: if the intersection of subvarieties A and $B \subset X$ of codimensions a and b on a smooth variety X is *dimensionally proper*—that is, every component Z of $A \cap B$ has codimension $a+b$—then for each component Z of $A \cap B$ there is an *intersection multiplicity* $m_Z(A, B)$, such that the intersection cycle

$$AB = \sum m_Z(A, B) \cdot Z$$

has class $[A][B] \in A^{a+b}(X)$. The key facts about the intersection multiplicity are:

(a) $m_Z(A, B) > 0$ for any component Z of $A \cap B$, with $m_Z(A, B) = 1$ if and only if A and B intersect transversely at a general point of Z;

(b) In case A and B are local complete intersection subvarieties of X (or, more generally, Cohen-Macaulay) at a general point of Z, then $m_Z(A, B)$ is the multiplicity of the component of the scheme $A \cap B$ supported on Z; and

(c) $m_Z(A, B)$ depends only on the local structure of A and B at a general point of Z.

We will give the definition of the intersection multiplicity in Section 5.5, and we will prove part (b) in Section 5.5. The notion of intersection multiplicity has been generalized by Fulton to the notion of a refined intersection class defined in $A^{a+b}(Z)$ even when Z is a component of the intersection that is not dimensionally proper (if Z is a dimensionally proper component, then $A^{a+b}(Z) = \mathbb{Z}$, so giving $m_Z(A, B)$ is equivalent to giving such a class.)

1.1.6 Functoriality

The key to working with Chow groups is to understand how they behave with respect to morphisms between varieties. To know what to expect, think of the analogous situation with homology and cohomology. A smooth projective variety of (complex) dimension n over the complex numbers is a compact oriented $2n$-manifold, so $H_{2m}(X)$ can be identified canonically with $H^{2n-2m}(X)$ (singular homology and cohomology). If we think of $A(X)$ as being analogous to $H_*(X)$, then we should expect $A_m(X)$ to be a covariant functor from smooth projective varieties to groups, via some sort of pushforward maps preserving dimension. If we think of $A(X)$ as analogous to $H^*(X)$, then we should expect $A(X)$ to be a contravariant functor from smooth projective varieties to rings, via some sort of pullback maps preserving codimension. Both these expectations are realized.

Proper Pushforward. If $f : Y \to X$ is a proper map of schemes, then the image of a subvariety $A \subset Y$ is a subvariety $f(A) \subset X$. One might at first guess that the pushforward could be defined by sending the class of A to the class of $f(A)$ but this would not preserve rational equivalence, (an example is pictured in Figure 1.4.) Rather, we must take multiplicities into account.

We say that the map $A \to f(A)$ is *generically finite* if, for a generic point $x \in f(A)$, the preimage $y := f|_A^{-1}(x)$ in A is a finite scheme or, equivalently, if the field of rational functions $K(A)$ is a finite extention of the field $K(f(A))$. The degree $n := [K(A) : K(f(A))]$ of this field extension is then equal to the degree of y over x, and this common value n is called the *degree* of the covering of $f(A)$ by A. Under these circumstances, we must count $f(A)$ with multiplicity n:

Definition 1.12 (Pushforward for Cycles)**.** Let $f : Y \to X$ be a proper map of schemes, and let $A \subset X$ be a subvariety.

(a) If $f(A)$ has strictly lower dimension than A, then we define $f_*\langle A\rangle = 0$.

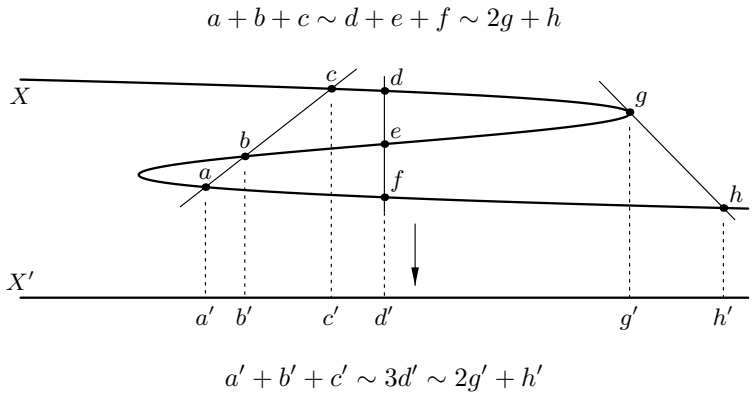


FIGURE 1.2. Pushforwards of equivalent cycles are equivalent

(b) If $\dim f(A) = \dim A$, then the map $f|_A : A \to f(A)$ is generically finite. If $n := [K(A) : K(f(A))]$ is the degree of the extension of fields of rational functions, we say that $f|_A$ is *generically finite of degree* n, and we define $f_*\langle A\rangle = n \cdot \langle f(A)\rangle$.

(c) If $A = \sum n_i A_i \in Z(Y)$ is a formal $\mathbb{Z}$-linear combination of subvarieties, then we define $f_*\langle A\rangle = \sum n_i f_*\langle A_i\rangle$.

With this definition, the pushforward of of cycles preserves rational equivalence (see 4.14):

Theorem 1.13. *If $f : Y \to X$ is a proper map of schemes, then the map $f_* : Z(Y) \to Z(X)$ defined above induces a map of groups $f_* : A_k(Y) \to A_k(X)$ for each k.*

It is often hard to prove that a given class in $A(X)$ is nonzero, but the fact that the pushforward map is well-defined gives us a start:

Proposition 1.14. *Suppose that K is a field. If X is a scheme proper over $\operatorname{Spec} K$, then there is a map $\deg : A_0(X) \to \mathbb{Z}$ taking the class $[p]$ of each closed point $p \in X$ to the degree $(\kappa(p) : K)$ of the extension of K by the residue field $\kappa(p)$ of p.*

We will typically use this proposition together with the intersection product: if A is a k-dimensional subvariety of a smooth projective variety X and B is a k-codimensional subvariety of X such that $A \cap B$ is finite and nonempty, then the map

$$A_k(X) \to \mathbb{Z} : \ [Z] \mapsto \deg\big([Z][B]\big)$$

sends $[A]$ to a nonzero integer. Thus no integer multiple $m[A]$ of the class A could be zero.

Flat and Projective Pullback. We next turn to the pullback. Let $f : Y \to X$ be a morphism, and $A \subset X$ a subvariety of codimension c. A good pullback map f^* on cycles should preserve rational equivalence, and in the nicest case, when $f^{-1}(A)$ is generically reduced of codimension c, it should be geometric, in the sense that $f^*([A]) = [f^{-1}(A)]$. It turns out that when f is a projective map there at most one way to define a map f^* that has these properties. This is a consequence of a refinement of the moving lemma (see Lemma 5.9 and Theorem5.10):

Theorem 1.15. *Let $f : Y \to X$ be a morphism of smooth projective varieties.*

(a) *Given any class $\alpha \in A^c(X)$, there exists a cycle $A = \sum n_i A_i \in Z^c(X)$ representing α such that $f^{-1}(A_i)$ is generically reduced of codimension c in Y for all i; and*

(b) *The class $\sum n_i[f^{-1}(A_i)] \in A^c(Y)$ is independent of the choice of such A.*

In these circumstances, we can define the pullback $f^*(\alpha)$ to be the class $\sum n_i[f^{-1}(A_i)] \in A^c(Y)$:

Theorem 1.16. *Let $f : Y \to X$ be a map of smooth projective varieties.*

(a) *There is a unique way of defining a map of groups $f^* : A^c(X) \to A^c(X)$ such that $f^*([A]) = [f^{-1}(A)]$ whenever $f^{-1}(A)$ is generically reduced of codimension c. The map f^* is a ring homomorphism, and makes A into a contravariant functor from the category of smooth projective varieties to the category of graded rings.*

(b) *(Push-Pull Formula) The map $f_* : A(Y) \to A(X)$ is a map of graded modules over the graded ring $A(X)$. More explicitly, if $\alpha \in A^k(X)$ and $\beta \in A_l(Y)$ then then*

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta \in A_{l-k}(X).$$

The last statement of this Theorem is the result of applying appropriate multiplicities to the set-theoretic equality $f(f^{-1}(A) \cap B) = A \cap f(B)$ (see the proof of Theorem 5.10).

One simple case of a projective morphism is the inclusion map from a closed subvariety $\iota : Y \subset X$. When X and Y are smooth, our definition of intersections and pullbacks makes it clear that if A is any subvariety of X, then $[A][Y]$ is represented by the same cycle as $\iota^*([A])$—except that these are considered as classes in different varieties! More precisely, we can write

$$[A][Y] = \iota_*(\iota^*[A]).$$

In this case the extra content of Theorem 1.16 is that this cycle is well defined as a cycle on Y, not only as a cycle on X. Fulton [1984] Section 8.1 shows that it is even well-defined as a "refined intersection class" on $X \cap Y$ and, more generally, he proves the existence of such a refined version of the pullback under a proper, locally complete intersection morphism (of which a map of smooth projective varieties is an example).

We can also define a good pull-back map for flat morphisms. The flat case is simpler than the projective case for two reasons: first, the preimage of a subvariety of codimension k is always of codimension k; and second, rational functions on the image pull back to rational functions on the source. We will use the flat case as a stepping stone in the proof of the projective case, and also to analyze maps of affine space bundles.

Theorem 1.17. *Let $\pi : Y \to X$ be a flat map of schemes. The map π^* defined on cycles by*

$$\pi^*(\langle A \rangle) := \langle \pi^{-1}(A) \rangle \quad \text{for every subvariety } A \subset X$$

preserves rational equivalence, and thus induces a map of Chow groups preserving the grading by codimension. If X and Y are smooth and quasiprojective, then $\pi^* : A(X) \to A(Y)$ is a ring homomorphism.

1.2 Chow Ring Examples

In this section we will use the techniques described above to compute the Chow groups of a few basic spaces, and to compute the classes of some interesting cycles. We will generally give proofs that are complete modulo the results above.

1.2.1 Varieties built from Affine Spaces

Open Subsets of Affine Space.

Proposition 1.18. *If $U \subset \mathbb{A}^n$ is an open set then $A(U) = A_n(U) = \mathbb{Z} \cdot [U]$.*

Proof. By part (b) of Proposition 1.6, $A_n(U) = \mathbb{Z} \cdot [U]$, so it suffices to show that the class $[Y]$ of any subvariety $Y \subsetneq U$ is zero. By part (d) of 1.6, it is enough to do this in the case $U = \mathbb{A}^n$.

Let $z = z_1, \ldots, z_n$ be coordinates on $\mathbb{A}^n$ and let x_0, x_1 be homogeneous coordinates of $\mathbb{P}^1$. We think of $x_1 = 0$ as the "point at infinity" and $t = x_0/x_1$ as "the coordinate on $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$." If $f(z) = 0$ is a function

vanishing on Y, then $f(z/t) = f(\frac{x_1}{x_0}z)$ is a function vanishing on tY. We let

$$W' = V(\{x_0^{\deg f} f(\frac{x_1}{x_0}z) \mid f(z) \text{ vanishes on } Y\}) \subset \mathbb{A}^n \times \mathbb{P}^1.$$

The fiber of W' over the point $(t, 1) \in \mathbb{P}^1$ is obviously tY, so the restriction of W' to $\mathbb{A}^n \times \mathbb{A}^1 \subset \mathbb{A}^n \times \mathbb{P}^1$, the open set $x_1 \neq 0$, is irreducible. Its closure W dominates $\mathbb{P}^1$and has fiber Y over the point $(1, 1)$.

Since Y is a proper subvariety of $\mathbb{A}^n$ and the ground field K is infinite, there is a K-rational point of $\mathbb{A}^n$ not in Y, and we may assume that it is the origin. It follows that some polynomial $g(z)$ that vanishes on Y has a nonzero constant term. This shows that $G(z, x_0, x_1) = x_0^{\deg g} g(\frac{x_1}{x_0}z)$ evaluates at $x_1 = 0$ to a nonzero multiple of a power of x_0, so the fiber $W \cap \mathbb{A}^n \times \{(1, 0)\}$ is empty. Thus $[Y] = 0$ in $A(\mathbb{A}^n)$. □

See Section 4.1.1 for a more systematic treatment of this idea.

Affine stratifications. In general we will work with very partial knowledge of the the Chow groups of a variety, but when X admits an *affine stratification*—a special kind of decomposition into a union of affine spaces—we can know them completely. This will help us compute the Chow groups of projective space, Grassmannians, and many other interesting rational varieties.

We say that a scheme X is *stratified* by a finite collection of irreducible, locally closed subschemes U_i if X is a disjoint union of the U_i and, in addition, if $\overline{U_i}$ meets U_j, then $\overline{U_i}$ contains U_j. The sets U_i are called the *strata* of the stratification, while the closures $Y_i := \overline{U_i}$ are called the *closed strata*. The stratification can be recovered from the closed strata Y_i: we simply take

$$U_i = Y_i \setminus \bigcup_{Y_j \subset Y_i} Y_j.$$

Definition 1.19. We say that a stratification of X with strata U_i is:

- *affine* if each open stratum is isomorphic to some $\mathbb{A}^k$; and
- *quasi-affine* if each U_i is isomorphic to an open subset of some $\mathbb{A}^k$.

For example, a complete flag of subspaces $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$ is an affine stratification of projective space; the closed strata are just the $\mathbb{P}^i$ and the open strata are affine spaces $U_i \cong \mathbb{A}^i$.

Proposition 1.20. *If a scheme X has a quasi-affine stratification then $A(X)$ is generated by the classes of the closed strata.*

A much stronger assertion holds for affine stratifications: Totaro [to appear] proves that in this case the classes of the closed strata are independent in $A(X)$, so that $A(X)$ is the free abelian group generated by its closed strata. Amazingly, no elementary proof of this simple assertion is known. However, using the intersection product it is easy to prove the result in the cases we will encounter.

Proof of Proposition 1.20. We will do induction on the number of strata U_i. If this number is 1 then the assertion is that of Proposition 1.18.

Let U_0 be a minimal stratum. Since the closure of U_0 is a union of strata, U_0 must already be closed. It follows that $U := X \setminus U_0$ is stratified by the strata other than U_0. By induction, $A(U)$ is generated by the classes of the closures of these strata, and by Proposition 1.18. $A(U_0)$ is generated by $[U_0]$. By part b) of Proposition **??** the sequence

$$\mathbb{Z} \cdot [U_0] = A(U_0) \longrightarrow A(X) \longrightarrow A(X \setminus U_0) \longrightarrow 0.$$

is right exact. Since the classes in $A(U)$ of the closed strata in U come from the classes of (the same) closed strata in X, it follows that $A(X)$ is generated by the classes of the closed strata. □

1.2.2 The Chow ring of $\mathbb{P}^n$

Theorem 1.21. *The Chow ring of $\mathbb{P}^n$ is*

$$A^*(\mathbb{P}^n) = \mathbb{Z}[\zeta]/(\zeta^{n+1}),$$

where ζ is the rational equivalence class of a hyperplane, and, more generally, the class of a variety of codimension k and degree d is $d\zeta^k$.

In particular, the Theorem implies that $A_m(\mathbb{P}^n) \cong \mathbb{Z}$ for $0 \leq m \leq n$, generated by the class of an m-plane. The natural proof, given below, uses the intersection product. A proof that does not use the intersection product is given in Chapter 4 (Theorem 4.3).

We note that since the sections of the *tautological line bundle* $\mathcal{O}_{\mathbb{P}^n}(1))$ correspond to linear forms on $\mathbb{P}^n$, the class ζ could also be described as the first Chern class, $\zeta = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.

Proof. It follows from Proposition 1.20 that $A_k(\mathbb{P}^n)$ is generated by the class of any k-plane $L_k \subset \mathbb{P}^n$. Using Proposition 1.14 we get $A_n(\mathbb{P}^n) = \mathbb{Z}$. Since a general $(n-k)$ plane L_k intersects a general k-plane transversely in one point, multiplication by $[L_k]$ induces a surjective map $A^k(\mathbb{P}^n) \to A^n(\mathbb{P}^n) = \mathbb{Z}$, so $A^k(\mathbb{P}^n) = \mathbb{Z}$ for all k.

A k-plane in $\mathbb{P}^n$ is the transverse intersection of $n-k$ hyperplanes so

$$[L_k] = \zeta^{n-k},$$

where $\zeta = [L_{n-1}] \in A^1(\mathbb{P}^n)$ is the class of a hyperplane. Finally, since a subvariety $X \subset \mathbb{P}^n$ of dimension k and degree d intersects a general $n-k$-plane transversely in d points, we have $\deg([X]\zeta^k) = d$. Since $\deg(\zeta^n) = 1$, we conclude that $[X] = d\zeta^{n-k}$. □

A nice qualitative result follows from Theorem 1.21:

Corollary 1.22. *Any map from $\mathbb{P}^n$ to a quasi-projective variety of dimension $< n$ is constant.*

Proof. Let $\varphi : \mathbb{P}^n \to X$ be the map If φ is not constant, then the preimage of a general hyperplane section of X will be disjoint from the preimage of a general point of X. But if $0 < \dim X < n$, then the preimage of a hyperplane section of X has dimension $n-1$ and the preimage of a point has dimension > 0. Since any two such subvarieties of $\mathbb{P}^n$ must meet, this is a contradiction. □

Theorem 1.21 implies the analogue of Poincaré duality for $A(\mathbb{P}^n)$: $A_k(\mathbb{P}^n)$ is dual to $A^k(\mathbb{P}^n)$ via the intersection product. The reader should be aware that in cases where the Chow groups and the homology groups are different, Poincaré duality generally does *not* hold for the Chow ring; for example, when X is a variety, $A_{\dim X}(X) \cong \mathbb{Z}$, but $A_0(X)$ need not even be finitely generated.

As an immediate consequence of Theorem 1.21, we get a general form of of Bézout's Theorem.

Theorem 1.23 (Bézout's Theorem)**.** *If $Z_1, \ldots, Z_k \subset \mathbb{P}^r$ are subvarieties of codimensions $c_1, \ldots, c_k$ with $\sum c_i \leq r$ and the Z_i intersect generically transversely, then*

$$\deg(Z_1 \cap \cdots \cap Z_k) = \prod \deg(Z_i).$$

In particular, two subvarieties $X, Y \subset \mathbb{P}^r$ having complementary dimension and intersecting transversely will intersect in exactly $\deg(X)\cdot\deg(Y)$ points. □

Using multiplicities we can extend this formula to the case where the varieties intersect properly (that is, the Z_i all have codimension equal to $\sum c_i$); and if we assume as well that the varieties are generically Cohen-Macaulay along their intersections, then the multiplicities are equal to the lengths of the components of the intersection scheme, and thus are already encoded in the intersection cycle. (See Example 5.27 for a case where a more subtle definition of multiplicity is necessary.) These things are satisfied,

for example when two plane curves meet along a finite set of points, the classical case treated by Bézout's Theorem. More generally, we can treat a dimensionally proper intersection of hypersurfaces in any projective space:

Corollary 1.24. *If c hypersurfaces $Z_1, \ldots, Z_c \subset \mathbb{P}^r$ meet in a scheme of codimension c with irreducible (but not necessarily reduced) components $C_1, \ldots, C_t$, then*

$$\sum [C_j] = \prod [Z_i]$$

and in particular

$$\sum \deg[C_j] = \prod \deg[Z_i].$$

Degrees of Veronese varieties. We can use Theorem 1.23 to compute the degrees of a Veronese variety. Let

$$\nu = \nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N \quad \text{where} \quad N = \binom{n+d}{n} - 1$$

be the *Veronese map*

$$[Z_0, \ldots, Z_n] \mapsto [\ldots, Z^I, \ldots]$$

where Z^I ranges over over monomials of degree d in $n+1$ variables.

Alternatively, we can characterize the Veronese map, up to automorphisms of the target $\mathbb{P}^N$, as the map associated to the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$; in other words, by the property that the preimages $\nu^{-1}(H) \subset \mathbb{P}^n$ of hyperplanes $H \subset \mathbb{P}^N$ comprise all hypersurfaces of degree d in $\mathbb{P}^n$.

In characteristic zero there is yet another attractive description: if $\mathbb{P}^n = \mathbb{P}V$ for V an $(n+1)$-dimensional vector space, ν is projectively equivalent to the map taking $\mathbb{P}V \to \mathbb{P}\operatorname{Sym}^d V$ by $[v] \mapsto [v^d]$; for if the coordinates of v are $v_0, \ldots, v_n$ then the coordinates of v^d are

$$\frac{d!}{\prod_i d_i!}(v_0^{d_0} \cdots v_n^{d_n}).$$

and since, if the characteristic is zero, the coefficients are nonzero, we may rescale by an automorphism of $\mathbb{P}^n$ to get the standard Veronese map above.

The image $\Phi = \Phi_{n,d} \subset \mathbb{P}^N$ of the Veronese map $\nu = \nu_{n,d}$ is called the *d-th Veronese variety of* $\mathbb{P}^n$, as is any subvariety of $\mathbb{P}^N$ projectively equivalent to it.

Proposition 1.25. *The degree of $\Phi_{n,d}$ is d^n.*

Proof. The degree of Φ is the cardinality of its intersection with n general hyperplanes $H_1, \ldots, H_n \subset \mathbb{P}^N$; since the map ν is one-to-one, this is in turn the cardinality of the intersection $f^{-1}(H_1) \cap \cdots \cap f^{-1}(H_n) \subset \mathbb{P}^n$. Since the preimages of the hyperplanes H_i are n general hypersurfaces of degree d in $\mathbb{P}^n$, by Bézout's Theorem, this cardinality is d^n. □

Degree of the dual of a hypersurface. The same idea allows us to compute the degree of the the *dual variety* of a smooth hypersurface $X \subset \mathbb{P}^n$, that is, the set of points $X^* \subset \mathbb{P}^{n*}$ corresponding to hyperplanes of $\mathbb{P}^n$ that are tangent to X. In Chapter 12 we will discuss the duals of singular varieties as well.

The set X^* is a variety because it is the image of X under the morphism $\mathcal{G}_X : X \to \mathbb{P}^{n*}$ sending a point $p \in X$ to its tangent hyperplane $\mathbb{T}_p X$; in coordinates, if X is the zero locus of the homogeneous polynomial $F(Z_0, \ldots, Z_n)$ then $\mathcal{G}$ is given by the formula

$$\mathcal{G}_X : p \mapsto [\frac{\partial F}{\partial Z_0}, \ldots, \frac{\partial F}{\partial Z_n}].$$

Since X is smooth, the partials of F have no common zeroes on X, so this is indeed a morphism.

If we assume in addition that the characteristic of the ground field is zero, we can use Bézout's Theorem to compute the degree of X^*. Of course when $d := \deg X = 1$, the dual is a point.

Proposition 1.26. *If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d > 1$ then the dual of X is a hypersurface of degree $d(d-1)^{n-1}$.*

Proof. If $d > 0$, then Corollary 12.29 shows that $\mathcal{G}_X$ is birational onto its image, so the degree of X^*, which is the number of points of intersection of X^* and $n-1$ general hyperplanes, can be computed by pulling back the hyperplanes to X and computing their intersection with X there. Since $\mathcal{G}_X$ is given by forms of degree $d-1$, hyperplanes pull back to the intersection of X with hypersurfaces of degree $d-1$ in $\mathbb{P}^n$, so the result follows by Bézout's Theorem. □

For example, suppose that X is a smooth cubic curve in $\mathbb{P}^2$. By the formula above the degree of X^* is 6. Since a general hyperplane in $\mathbb{P}^{2*}$ corresponds to the set of lines through a general point $p \in \mathbb{P}^2$, there will be exactly six lines in $\mathbb{P}^2$ through p tangent to X, as shown in Figure 1.3.

1.2.3 Products of projective spaces

In general there is no Künneth formula for the Chow rings of products of varieties. Even for a product of two smooth curves C and D of genera $g, h \geq 1$ we have no algorithm for calculating $A^1(C \times D)$, and no idea at all what $A^2(C \times D)$ looks like, beyond the fact that it can't be in any sense finite-dimensional (Mumford [1962]).

However the Chow rings of products of projective space do obey the Künneth formula:

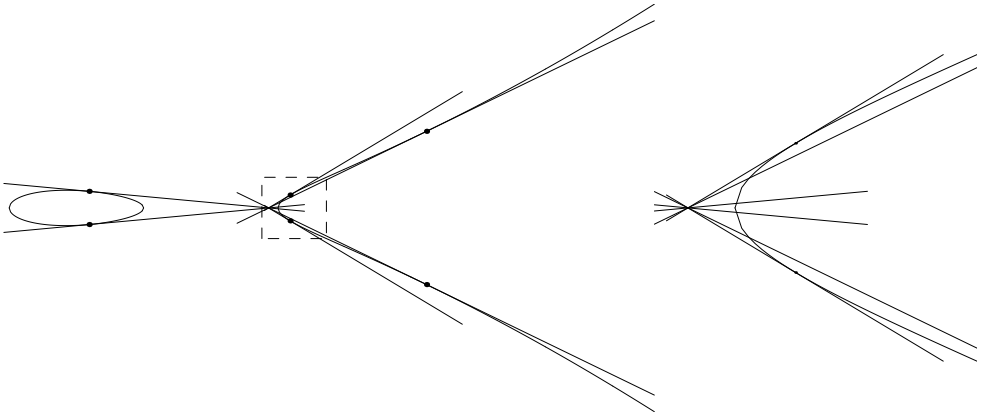


FIGURE 1.3. Six of the lines through a general point are tangent to a smooth plane cubic (but often not all the lines are defined over $\mathbb{R}$). ****show the enlarged "box" around the figure on the right.****

Theorem 1.27. *The Chow ring of $\mathbb{P}^r \times \mathbb{P}^s$ is given by the formula*

$$A(\mathbb{P}^r \times \mathbb{P}^s) \cong A(\mathbb{P}^r) \otimes A(\mathbb{P}^s).$$

Equivalently, if α and $\beta \in A^1(\mathbb{P}^r \times \mathbb{P}^s)$ denote the pullbacks, via the projection maps, of the hyperplane classes on $\mathbb{P}^r$ and $\mathbb{P}^s$ respectively, then

$$A(\mathbb{P}^r \times \mathbb{P}^s) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1}).$$

Moreover, the class of the hypersurface defined by a bihomogeneous form of bidegree (d, e) on $\mathbb{P}^r \times \mathbb{P}^s$ is $d\alpha + e\beta$.

In fact the Künneth formula holds more generally for products of varieties with affine stratifications; see Totaro [to appear] for a proof.

Proof. We can proceed exactly as in Theorem 1.21. We may construct an affine stratification of $\mathbb{P}^r \times \mathbb{P}^s$ by choosing flags of subspaces

$$\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{r-1} \subset \Lambda_r = \mathbb{P}^r$$

and

$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_{s-1} \subset \Gamma_s = \mathbb{P}^s$$

with $\dim \Lambda_i$ and $\dim \Gamma_i = i$, and setting

$$\Xi_{a,b} = \Lambda_{r-a} \times \Gamma_{s-b} \subset \mathbb{P}^r \times \mathbb{P}^s.$$

The open strata

$$\tilde{\Xi}_{a,b} := \Xi_{a,b} \setminus \left(\Xi_{a-1,b} \cup \Xi_{a,b-1}\right)$$

of this stratification are affine spaces. Invoking Proposition 1.20, we conclude that the Chow groups of $\mathbb{P}^r \times \mathbb{P}^s$ are generated by the classes $\varphi_{a,b} =$

$[\Xi_{a,b}] \in A^{a+b}(\mathbb{P}^r \times \mathbb{P}^s)$. Since $\Xi_{a,b}$ is the transverse intersection of the pullbacks of a hyperplanes in $\mathbb{P}^r$ and b hyperplanes in $\mathbb{P}^s$ we have

$$\varphi_{a,b} = \alpha^a \beta^b,$$

and in particular $\alpha^{r+1} = \beta^{s+1} = 0$. This shows that $A(\mathbb{P}^r \times \mathbb{P}^s)$ is a homomorphic image of

$$\mathbb{Z}[\alpha,\beta]/(\alpha^{r+1}, \beta^{s+1}) = \mathbb{Z}[\alpha]/(\alpha^{r+1}) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta]/(\beta^{s+1}).$$

On the other hand, $\Xi_{r,s}$ is a single point, so $\deg \varphi_{r,s} = 1$. The pairing

$$A^{p+q}(\mathbb{P}^r \times \mathbb{P}^s) \times A^{r+s-p-q}(\mathbb{P}^r \times \mathbb{P}^s) \to \mathbb{Z}; \quad ([X],[Y]) \to \deg([X][Y])$$

sends $(\alpha^p\beta^q, \alpha^m\beta^n)$ to 1 if $p+m=r$ and $q+n=s$, because in this case the intersection is transverse and consists of one point, and to 0 otherwise, since then the intersection is empty. This shows that the monomials of bidegree (p,q), for $0 \le p \le r$ and $0 \le q \le s$, are linearly independent over $\mathbb{Z}$, proving the first statement.

If $F(X,Y)$ is a bihomogeneous polynomial of bidegree (d,e), then since $F(X,Y)/X_0^d Y_0^e$ is a rational function on $\mathbb{P}^r \times \mathbb{P}^s$, the class of the hypersurface X defined by $F=0$ is d times the class of the hypersurface $X_0 = 0$ plus e times the class of the hypersurface $Y_0 = 0$. In particular, $[X] = d\alpha + e\beta$. □

Degrees of Segre varieties. The Segre variety $\Sigma_{k,s}$ is by definition the image of the product $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^{(r+1)(s+1)-1}$ under the map

$$\sigma_{r,s} : \big([X_0,\dots,X_r],[Y_0,\dots,Y_s]\big) \mapsto [\dots, X_iY_j, \dots].$$

The map $\sigma_{r,s}$ is an immersion because on each open set where one of the X_i and one of the Y_j are equal to 1 the rest of the coordinates can be recovered from the products.

If V and W are vector spaces of dimensions $r+1$ and $s+1$ we may write $\sigma_{r,s}$ without bases by the formula

$$\begin{aligned} \sigma_{r,s} : \mathbb{P}V \times \mathbb{P}W &\to \mathbb{P}(V \otimes W) \\ (v,w) &\mapsto v \otimes w. \end{aligned}$$

For example, the map $\sigma_{1,1}$ is defined by the four forms $a = X_0Y_0, b = X_0Y_1, c = X_1Y_0, d = X_1Y_1$, and these satisfy the equation $ac - bd = 0$, the Segre variety $\Sigma_{1,1}$ is the nonsingular quadric in $\mathbb{P}^3$.

Proposition 1.28. *The degree of the Segre embedding of* $\mathbb{P}^r \times \mathbb{P}^l$ *is*

$$\deg \Sigma_{r,l} = \binom{r+l}{r}.$$

Proof. The degree of $\Sigma_{r,s}$ is the number of points in which it meets the intersection of $r+s$ hypersurfaces in $\mathbb{P}^{(r+1)(s+1)-1}$. Since $\sigma_{r,s}$ is an embedding, we may compute this number by pulling back these hypersurfaces to $\mathbb{P}^r \times \mathbb{P}^s$ and computing in the Chow ring of $\mathbb{P}^r \times \mathbb{P}^s$. Thus $\deg \Sigma_{r,l} = \deg(\alpha+\beta)^{r+l}$, which gives the desired formula because $(\alpha+\beta)^{r+l} = \binom{r+l}{r}\alpha^r\beta^l$ □

For instance the Segre variety $\mathbb{P}^1 \times \mathbb{P}^r \subset \mathbb{P}^{2r+1}$ has degree $r+1$. These varieties are among those called *rational normal scrolls* (see for example Harris [1992]).

This discussion can be readily generalized to arbitrary products of projective spaces, as in Exercise 1.54.

The class of the diagonal. The next question we want to address is to find the class δ, in the Chow group $A^r(\mathbb{P}^r \times \mathbb{P}^r)$, of the diagonal $\Delta \subset \mathbb{P}^r \times \mathbb{P}^r$, and more generally the class γ_f of the graph of a map $f : \mathbb{P}^r \to \mathbb{P}^s$. Apart from the applications of such a formula, this will introduce a technique for finding the class of a cycle, called the method of *undetermined coefficients*, that we'll use many times in the course of this book. (Another approach to this problem is given in Exercise 1.55.)

By Theorem 1.27 we have

$$A(\mathbb{P}^r \times \mathbb{P}^r) = \mathbb{Z}[\alpha,\beta]/(\alpha^{r+1}, \beta^{r+1}).$$

where α and $\beta \in A^1(\mathbb{P}^r \times \mathbb{P}^r)$ are the pullbacks, via the two projection maps, of the hyperplane class in $A^1(\mathbb{P}^r)$.

To find the class $\delta = [\Delta]$ of the diagonal, we first observe that it is necessarily expressible as a linear combination

$$\delta = c_0\alpha^r + c_1\alpha^{r-1}\beta + c_2\alpha^{r-2}\beta^2 + \cdots + c_r\beta^r$$

for some $c_0, \ldots, c_r \in \mathbb{Z}$. We can determine the coefficients c_i by taking the product of both sides of this expression with various classes of complementary codimension: specifically, if we intersect both sides with the class $\alpha^i\beta^{r-i}$ we have

$$c_i = (\delta \cdot \alpha^i\beta^{r-i}).$$

We can evaluate the product $(\delta \cdot \alpha^i\beta^{r-i})$ directly: if Λ and Γ are general linear subspaces of codimension i and $r-i$ respectively, then $[\Lambda \times \Gamma] = \alpha^i\beta^{r-i}$. Moreover,

$$(\Lambda \times \Gamma) \cap \Delta \cong \Lambda \cap \Gamma$$

is a reduced point, so

$$\begin{aligned} c_i &= (\delta \cdot \alpha^i \beta^{r-i}) \\ &= \#\big(\Delta \cap (\Lambda \times \Gamma)\big) \\ &= \#(\Lambda \cap \Gamma) \\ &= 1. \end{aligned}$$

Thus

$$\delta = \alpha^r + \alpha^{r-1}\beta + \cdots + \alpha\beta^{r-1} + \beta^r.$$

(This formula and its derivation will be familiar to anyone who's had a course in algebraic topology. As loyalists we can't resist pointing out that algebraic geometry had it first!)

The graph of a morphism. Let $f : \mathbb{P}^r \to \mathbb{P}^s$ be a morphism given by $(s+1)$ homogeneous polynomials F_i of degree d that have no common zeros:

$$f : [X_0, \ldots, X_r] \mapsto \big[F_0(X), F_1(X), \ldots, F_s(X)\big],$$

By Corollary 1.22, we must have $s \geq r$. Let $\Gamma_f \subset \mathbb{P}^r \times \mathbb{P}^s$ be the graph of f. What is its class $\gamma_f = [\Gamma_f] \in A^r(\mathbb{P}^r \times \mathbb{P}^s)$?

We will assume that the characteristic of the ground field K is zero so that we can apply Theorem 1.10. In this case it doesn't matter: using multiplicities we could arrive at the same result in all characteristics.

As before, we can write

$$\gamma_f = c_0\alpha^r\beta^{s-r} + c_1\alpha^{r-1}\beta^{s-r+1} + c_2\alpha^{r-2}\beta^{s-r+2} + \cdots + c_r\beta^s$$

for some $c_0, \ldots, c_r \in \mathbb{Z}$, and as before we can determine the coefficients c_i in this expression by intersecting both sides with a cycle of complementary dimension:

$$c_i = (\gamma_f \cdot \alpha^i\beta^{r-i}) = \#\big(\Gamma_f \cap (\Lambda \times \Phi)\big)$$

for general linear subspaces $\Lambda \cong \mathbb{P}^{r-i}$ and $\Phi \cong \mathbb{P}^{s-r+i} \subset \mathbb{P}^s$ By Theorem 1.10 the intersection $\Gamma_f \cap (\Lambda \times \Phi)$ is generically transverse.

Finally, $\Gamma_f \cap (\Lambda \times \Phi)$ is the zero locus, in Λ, of $r-i$ general linear combinations of the polynomials $F_0, \ldots, F_r$. By Bertini's Theorem (Hartshorne [1977] Theorem ****) the corresponding hypersurfaces will intersect transversely, and by Bézout's Theorem the intersection will consist of d^{r-i} points. Thus we arrive at the formula:

Proposition 1.29. *If $f : \mathbb{P}^r \to \mathbb{P}^s$ is a regular map given by polynomials of degree d on $\mathbb{P}^r$, the class γ_f of the graph of f is given by*

$$\gamma_f = \sum_{i=0}^{r} d^i\alpha^i\beta^{s-i} \in A^s(\mathbb{P}^r \times \mathbb{P}^s).$$

Using this formula, we can answer a general form of Keynote Question (c). We'll again assume characteristic zero. A sequence $F_0, \ldots, F_r$ of general homogeneous polynomials of degree d in $r+1$ variables defines a map $f : \mathbb{P}^r \to \mathbb{P}^r$, and we can count the fixed points $\{t = (t_0, \ldots, t_r) \in \mathbb{P}^r \mid f(t) = t\}$. Since the F_i are general, we could take them to be general translates under $GL_{r+1} \times GL_{r+1}$ of arbitrary polynomials so the cardinality of this set is the degree of the intersection of the graph γ_f of f with the diagonal $\Delta \subset \mathbb{P}^r \times \mathbb{P}^r$. This is

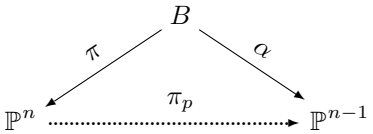
$$\begin{aligned}\delta \cdot \gamma_f &= (\alpha^r + \alpha^{r-1}\beta + \cdots + \beta^r) \cdot (d^r\alpha^r + d^{r-1}\alpha^{r-1}\beta + \cdots + \beta^r) \\ &= d^r + d^{r-1} + \cdots + d + 1,\end{aligned}$$

In particular, if A, B, C are general quadratic forms then there are exactly 7 points $t = (t_0, t_1, t_2) \in \mathbb{P}^2$ such that $(A(t), B(t), C(t)) = (t_0, t_1, t_2)$.

Note that in case $d = 1$ and $s = r$, Proposition 1.29 implies that a general $(r+1) \times (r+1)$ matrix has $r+1$ eigenvalues. It also follows that an arbitrary matrix has at least one eigenvalue.

1.2.4 The blowup of $\mathbb{P}^n$ at a point

The *blowup* of $\mathbb{P}^n$ at a point p is a morphism $\pi : B \to \mathbb{P}^n$ where $B \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ is the closure of the graph of the projection $\pi_p : \mathbb{P}^n \to \mathbb{P}^{n-1}$ from p (a rational map), and π is the projection on the first factor (see for example Section IV.2 of Eisenbud and Harris [2000]).



The *exceptional divisor* $E \subset B$ is defined to be $\pi^{-1}(p)$, the preimage of p in B, which, as a subset of $\mathbb{P}^n \times \mathbb{P}^{n-1}$, is $\{p\} \times \mathbb{P}^{n-1}$. Some other obvious divisors on B are the pullbacks of the hyperplanes of $\mathbb{P}^n$. If the hyperplane $H \subset \mathbb{P}^n$ contains p, then its preimage is the sum of two irreducible divisors, E and H'; the latter is called the *strict transform* of H. More generally, if $Z \subset \mathbb{P}^n$ is any subvariety, we define the strict transform of Z to be the closure in B of the preimage $\pi^{-1}(Z \setminus \{p\})$. See Figure 1.4.

To compute the Chow ring of B, we start from a stratification of B, using the geometry of the projection map $\alpha : B \to \mathbb{P}^{n-1}$ to the second factor. We do this by first choosing a stratification of the target $\mathbb{P}^{n-1}$, and taking the preimages in B of these strata. Then we choose a cross section Λ of α, and add the intersections of these preimages with Λ as strata.

We will choose as our section the preimage $\Lambda = \pi^{-1}(\Lambda')$ of a hyperplane $\Lambda' \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ not containing the point p. The projection $\alpha : B \to \mathbb{P}^{n-1}$ induces an isomorphism from Λ to $\mathbb{P}^{n-1}$; that is, Λ is a cross section of α. (There are other possible choices of a section, such as the exceptional divisor $E \subset B$. See Exercise 1.65).)

To carry this out, let

$$\Gamma'_0 \subset \Gamma'_1 \subset \cdots \subset \Gamma'_{n-2} \subset \Gamma'_{n-1} = \mathbb{P}^{n-1}.$$

be a flag of linear subspaces and, for $k = 1, 2, \ldots, n$ let

$$\Gamma_k = \alpha^{-1}(\Gamma'_{k-1}) \subset B.$$

Since the fibers of $\alpha : B \to \mathbb{P}^{n-1}$ are projective lines, the dimension of Γ_k is k. Next, for $k = 0, 1, \ldots, n-1$ we set

$$\Lambda_k = \Gamma_{k+1} \cap \Lambda,$$

so that Λ_k is the preimage of Γ'_k under the map α.

The subvarieties $\Gamma_1, \ldots, \Gamma_n, \Lambda_0, \ldots, \Lambda_{n-1}$ are the closed strata of a stratification of B, with inclusion relations

$$\begin{array}{ccccccccccccc}
\Lambda_0 & \hookrightarrow & \Lambda_1 & \hookrightarrow & \Lambda_2 & \hookrightarrow & \cdots & \hookrightarrow & \Lambda_{n-2} & \hookrightarrow & \Lambda_{n-1} & & \\
 & \searrow & & \searrow & & & & & & \searrow & & \searrow & \\
 & & \Gamma_1 & \hookrightarrow & \Gamma_2 & \hookrightarrow & \cdots & \hookrightarrow & \Gamma_{n-2} & \hookrightarrow & \Gamma_{n-1} & \hookrightarrow & \Gamma_n
\end{array}$$

To visualize this, take the standard picture of the blowup of $\mathbb{P}^n$ at a point (shown in Fig. 1.4 in case $n = 2$:

If we rotate this picture and "untwist" it, we realize the blow-up as the total space of a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$, via the projection map α, as in Fig. 1.5:

As we'll see shortly, this is an affine stratification, so that the classes of the closed strata generate the Chow group $A(B)$. (In fact, the open strata are isomorphic to affine spaces, so that it follows from **?**] that they generate $A_*(B)$ freely; we will verify this independently when we determine the intersection products.) We describe the Chow ring of B as follows:

Proposition 1.30. *Let B be the blowup of $\mathbb{P}^n$ at a point. With notation as above, the Chow ring $A(B)$ is the free abelian group on the generators $[\Lambda_k] = [\Lambda_{n-1}]^{n-k}$ for $k = 0, \ldots, n-1$ and $[\Gamma_k] = [\Gamma_{n-1}]^{n-k}$ for $k = 1, \ldots, n$. The class of the exceptional divisor E is $[\Lambda_{n-1} - \Gamma_{n-1}]$. If we set $\lambda = [\Lambda_{n-1}]$ and $e = [E]$, then*

$$A(B) \cong \frac{\mathbb{Z}[\lambda, e]}{(\lambda e, \lambda^n + (-1)^n e^n)}$$

as rings.

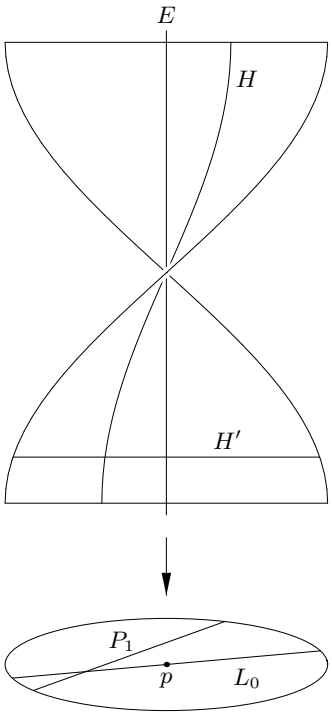
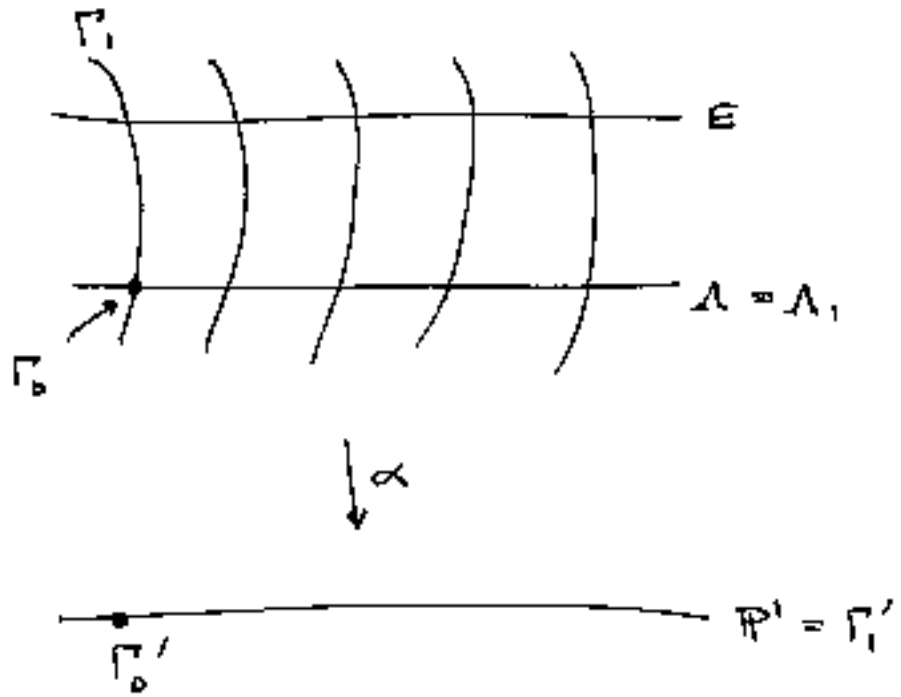


FIGURE 1.4. Blowup of $\mathbb{P}^2$****Relabel: P_1 should be Λ'; H should be Λ; L_0 should be L and H' should be $\tilde{L}$.****


Γ1
E
Λ = Λ1
Γ0
α
P' = Γ1'
Γ0'

Proof. We start by verifying that the open strata $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_n, \tilde{\Lambda}_0, \ldots, \tilde{\Lambda}_{n-1}$ of the stratification of B with closed strata Γ_k, Λ_k are isomorphic to affine spaces. This is immediate for the strata $\tilde{\Lambda}_k$. For the strata $\tilde{\Gamma}_k$, we choose coordinates $(x_0, \ldots, x_n)$ on $\mathbb{P}^n$ so that $p = (1, 0, \ldots, 0)$ and $\Lambda' \subset \mathbb{P}^n$ is the hyperplane $x_0 = 0$. By definition, $B \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ is the locus

$$B = \{((x_0, \ldots, x_n), (y_1, \ldots, y_n)) \mid x_i y_j = x_j y_i \ \forall i, j \geq 1\}.$$

Say the $(k-1)$-plane $\Gamma'_{k-1} \subset \mathbb{P}^{n-1}$ is given by $y_1 = \cdots = y_{n-k} = 0$. We can write the open stratum $\tilde{\Gamma}_k = \alpha^{-1}(\Gamma'_{k-1} \setminus \Gamma'_{k-2}) \cap (B \setminus \Lambda)$ as the locus

$$\tilde{\Gamma}_k = \{((1, 0, \ldots, 0, \lambda, \lambda y_{n-k+2}, \ldots, \lambda y_n), (0, \ldots, 0, 1, y_{n-k+2}, \ldots, y_n))\};$$

the functions $\lambda, y_{n-k+2}, \ldots, y_n$ then give an isomorphism of $\tilde{\Gamma}_k$ with $\mathbb{A}^k$.

It follows that the classes

$$\lambda_k = [\Lambda_k] \quad \text{and} \quad \gamma_k = [\Gamma_k] \in A_k(B)$$

generate the Chow groups of B; we now want to say what their intersection products are.

Since Λ_k is the the preimage of a k-plane in $\mathbb{P}^n$ not containing p, and any two such are linearly equivalent in $\mathbb{P}^n$, the class of the pullback of any k-plane in $\mathbb{P}^n$ not containing p is also equal to λ_k. Similarly, the class of the proper transform of any k-plane in $\mathbb{P}^n$ containing p. is γ_k. Having all these representative cycles for the classes λ_k and γ_k makes it easy to determine their intersection products.

For example, a general k-plane in $\mathbb{P}^n$ intersects a general l-plane transversely in a general $(k + l - n)$-plane; thus

$$\lambda_k \lambda_l = \lambda_{k+l-n} \quad \text{for all } k + l \geq n.$$

Similarly, the intersection of a general k-plane in $\mathbb{P}^n$ containing p with a general l-plane in not containing p is a general $(k + l - n)$-plane not containing p, so that

$$\gamma_k \lambda_l = \lambda_{k+l-n} \quad \text{for all } k + l \geq n,$$

and likewise

$$\gamma_k \gamma_l = \gamma_{k+l-n} \quad \text{for all } k + l \geq n + 1.$$

Note the restriction $k+l \geq n+1$ on the last set of products: in case $k+l = n$, the proper transforms of a general k-plane through p and a general l-plane through p are disjoint.

This determines the Chow ring of B: the pairing $A_k(B) \times A_{n-k}(B) \to A_0(B) \cong \mathbb{Z}$ is given by

$$\lambda_k \lambda_{n-k} = \lambda_k \gamma_{n-k} = \gamma_k \lambda_{n-k} = 1 \quad \text{and} \quad \gamma_k \gamma_{n-k} = 0;$$

we see in particular that it is nondegenerate, so that the classes $\lambda_0, \ldots, \lambda_{n-1}$ and $\gamma_1, \ldots, \gamma_n$ freely generate $A(B)$.

It follows that we can express the class of the exceptional divisor E in terms of the generators Λ_{n-1} and Γ_{n-1} of $A_{n-1}(B)$. The most geometric way to do this is to observe that Λ'_{n-1} is linearly equivalent in $\mathbb{P}^n$ to a hyperplane $\Sigma \subset \mathbb{P}^n$ containing p so the pullback of Σ is linearly equivalent to the union of the exceptional divisor E and a divisor D. Since D projects to a hyperplane of $\mathbb{P}^{n-1}$, it is contained in the preimage Γ of such a hyperplane. Since Γ is a $\mathbb{P}^1$ bundle over its image it is irreducible, and comparing dimensions, we see that $D = \Gamma$. Since any two hyperplanes in $\mathbb{P}^{n-1}$ are rationally equivalent, so are their pullbacks to B; thus $\Lambda_{n-1}\ D+E\ \Gamma_{n-1}+E$, or $[E] = \lambda_{n-1} - \gamma_{n-1}$.

We now turn to the ring structure, and let $\lambda = [\Lambda_{n-1}]$ and $e = [E] = \lambda - \gamma_{n-1}$. Since $\Lambda_{n-1} \cap E = \varnothing$ we have

$$\lambda e = 0.$$

Also,

$$\lambda_k = \lambda^{n-k}, \quad \text{for } k = 0, \ldots, n-1$$

and since $\gamma_{n-1} = \lambda - e$,

$$\gamma_k = \gamma_{n-1}^{n-k} = (\lambda - e)^{n-k} = \lambda^{n-k} + (-1)^{n-k} e^{n-k}, \quad \text{for } k = 1, \ldots, n.$$

It follows that λ and e generate $A(B)$ as a ring. In addition to the relation $\lambda e = 0$, they satisfy the relation

$$0 = \gamma_{n-1}^n = (\lambda - e)^n = \lambda^n + (-1)^n e^n.$$

Thus the Chow ring is a homomorphic image of the ring $A' := \mathbb{Z}[\lambda, e]/(\lambda e, \lambda^{n-1} - (-1)^n e^n)$. It is clear that every homogeneous element of degree m in A' is a $\mathbb{Z}$-linear combination of e^m and λ^m. Since $A^m(B)$ is a free $\mathbb{Z}$-module of rank 2, this implies that the map $A' \twoheadrightarrow A$ is an isomorphism. □

We have computed the intersection products of the Λ_k and Γ_k by taking representatives that meet transversely (and the possibility of doing this motivated our choice of Λ as a cross section of α above.) Since E is the only irreducible variety in the class $[E]$ we cannot give a representative for e^2 quite as easily. But as we have seen, $E\ \langle \Lambda_{n-1} \rangle - \langle \Gamma_{n-1} \rangle$ and both Λ_{n-1} and Γ_{n-1} are transverse to E (this illustrates the conclusion of the moving Lemma!) It follows that

$$e^2 = [E \cap (\Lambda - \Gamma)] = -[E \cap \Gamma_{n-1}].$$

Since E projects isomorphically to $\mathbb{P}^{n-1}$ and Γ projects to a hyperplane in $\mathbb{P}^{n-1}$ we see that $E \cap \Gamma_{n-1}$ is a hyperplane in E; that is, $[E]^2$ is the *negative* of the class of a hyperplane in E.

The Chow ring of the blow-up of $\mathbb{P}^3$ along a line is worked out in Exercises 1.66-1.68. More generally, we will see how to describe the Chow ring of a general projective bundle in Chapter 11, and the Chow ring of a more general blow-up in Chapter 15.

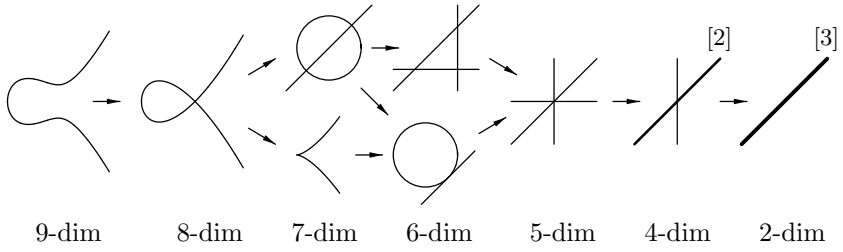


FIGURE 1.6. Hierarchy of singular plane cubic curves.

1.2.5 Loci of singular plane cubics

This section represents an important shift in viewpoint, from studying the Chow rings of common and useful algebraic varieties to studying varieties in families as loci in varieties we understand. It is a hallmark of algebraic geometry that the set of varieties with given numerical invariants—and more generally, schemes, morphisms, bundles and other geometric objects—is often itself an algebraic variety, sometimes called a *parameter space.* Applying intersection theory to the study of such a parameter space, we learn something about the geometry of the objects parametrized, and about geometrically characterized classes of these objects. This was one of the principal motivations for the development of intersection theory in the 19th century.

By way of illustration we will focus on the family of curves of degree 3 in $\mathbb{P}^2$: plane cubics. Plane cubics are parametrized by the set of homogeneous cubic polynomials $F(X, Y, Z)$ in three variables, modulo scalars; that is, by $\mathbb{P}^9$. *For simplicity, we will assume throughout this section that the characteristic of the ground field is zero.*

There are only a finite number of isomorphism classes of singular plane cubics (see Figures 1.6–1.8):

- irreducible plane cubics with a node;
- irreducible plane cubics with a cusp;
- plane cubics consisting of a smooth conic and a line meeting transversely;
- plane cubics consisting of a smooth conic and a line tangent to it;
- plane cubics consisting of three non-concurrent lines ("triangles");
- plane cubics consisting of three concurrent lines ("asterisks");
- cubics consisting of a double line and a line; and finally
- cubics consisting of a triple line.

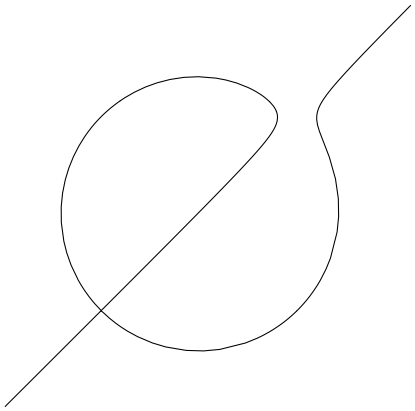


FIGURE 1.7. Nodal cubic about to become the union of a conic and a transverse line: $(y^2-x^2(x+1))+100(x-y)((x-\frac{1}{2})^2+(y-\frac{1}{2})^2-\frac{1}{2})$

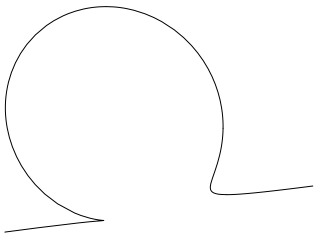


FIGURE 1.8. Cuspidal cubic about to become the union of a conic and a tangent line: $y^2-x^3+7y(x^2+(y-1)^2-1)$

The locus in $\mathbb{P}^9$ of curves of each type is an orbit of PGL_3 and a locally closed subset of $\mathbb{P}^9$; their closures give a stratification of $\mathbb{P}^9$. What are these loci like? What are their dimensions? Which lie in the closures of which others? Where is each one smooth and singular? What are their tangent spaces and tangent cones? What are their degrees as subvarieties of $\mathbb{P}^9$?

Some of these questions are easy to answer. For example, the dimensions are given in Figure **??**, and the reader can verify them as an exercise. The inclusion relationships, also indicated in the chart, are likewise not hard; to establish that one orbit lies in the closure of another, it suffices to exhibit a one-parameter family specializing from one type to the other. The non-inclusion relations can be more subtle—why is a triangle not a specialization of a cuspidal cubic?—but can also be done by focussing on the singularities of the curves. The tangent spaces require more work; we'll give some examples in Exercises 1.71-1.72, in the context of establishing a transversality statement, and we'll see more of these, as well as some tangent cones, in the discussion in Section 9.6.3.

While we can answer almost any question of this sort about plane cubics, the answers to analogous questions for higher degree curves or hypersurfaces of higher dimension, for example about the stratification by singularity type, remain mysterious. Even questions about the dimension and irreducibility of these loci are mostly open; they are a topic of active research.

For example, it is known that the locus of plane curves of degree d having exactly δ nodes is irreducible of codimension δ in the projective space $\mathbb{P}^N$ of all plane curves of degree d (see for example Harris and Morrison [1998]), and its degree has also been determined (Caporaso and Harris [1998]). But we don't know the answers to the analogous questions for plane curves with δ nodes and κ cusps; and when it comes to more complicated singularities even existence questions are open.

****This might merit an "open problems" discussion—in any event, we should be more specific about what's known and what isn't.****

In the rest of this section we will determine the degrees of the loci of reducible cubics, triangles and asterisks. In the exercises we indicate how to compute the degrees of the other loci of plane cubics, except for the loci of irreducible cubics with a node, and of irreducible cubics with a cusp; these will be computed in Section 9.3.2 and Section 13.4 respectively.

The locus of reducible cubics. Let $\Gamma \subset \mathbb{P}^9$ be the locus of reducible cubics and/or non-reduced cubics. We can describe Γ as the image of the map

$$\tau : \mathbb{P}^2 \times \mathbb{P}^5 \to \mathbb{P}^9$$

from the product of the space $\mathbb{P}^2$ of homogeneous linear forms and the space $\mathbb{P}^5$ of homogeneous quadratic polynomials to $\mathbb{P}^9$, given simply by multiplication: $([F],[G]) \mapsto [FG]$. Inasmuch as the coefficients of the product FG are bilinear in the coefficients of F and G, the pullback $\tau^*(\zeta)$ of the hyperplane class $\zeta \in A^1(\mathbb{P}^9)$ is the sum

$$\tau^*(\zeta) = \alpha + \beta$$

where α and β are the pullbacks to $\mathbb{P}^2 \times \mathbb{P}^5$ of the hyperplane classes on $\mathbb{P}^2$ and $\mathbb{P}^5$.

By unique factorization of polynomials, the map τ is birational onto its image; it follows that the degree of Γ is given by

$$\begin{aligned}\deg(\Gamma) &= \big(\tau^*(\zeta)\big)^7 \\ &= (\alpha+\beta)^7 \in A^7(\mathbb{P}^2 \times \mathbb{P}^5) \\ &= 21\end{aligned}$$

and this is the answer to Keynote Question (a).

Another way to calculate the degree of Γ is described in Exercises 1.71–1.73.

Triangles. A similar analysis gives the answer Keynote Question (b)—how many cubics in a three-dimensional linear system factor completely, as a product of three linear forms. Here, the key object is the locus $\Sigma \subset \mathbb{P}^9$ of such totally reducible cubics, which we may call *triangles*; the keynote question asks us for the number of points of intersection of Σ with a general 3-plane, which by Bertini's Theorem is just the degree of Σ.

Since Σ is the image of the map

$$\begin{aligned}\mu : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 &\to \mathbb{P}^9 \\ ([L_1],[L_2],[L_3]) &\mapsto [L_1L_2L_3].\end{aligned}$$

We can proceed as before, with the one difference that the map is now no longer birational, but rather is 6-to-one. Thus if α_1, α_2 and $\alpha_3 \in A^1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$ are the pullbacks of the hyperplane class, so that

$$\mu^*(\zeta) = \alpha_1 + \alpha_2 + \alpha_3,$$

we get

$$\begin{aligned}\deg(\Sigma) &= \frac{1}{6}\,(\alpha_1+\alpha_2+\alpha_3)^6 \\ &= \frac{1}{6}\binom{6}{2,2,2} \\ &= 15.\end{aligned}$$

This is the answer to Keynote Question (b): in a general three-dimensional linear system of cubics, there will be exactly 15 triangles.

Asterisks. By an *asterisk*, we mean a cubic consisting of the sum of three concurrent lines. To see that this locus is indeed a subvariety of $\mathbb{P}^9$ and to calculate its degree, let

$$\mu : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$$

be as above, and consider the subset

$$\Phi = \{(L_1, L_2, L_3) : L_1 \cap L_2 \cap L_3 \neq \varnothing\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2;$$

the locus $A \subset \mathbb{P}^9$ of asterisks is then the image $\mu(\Phi)$ of Φ under the map μ. If we write the line L_i as the zero locus of the linear form

$$a_{i,1}X + a_{i,2}Y + a_{i,3}Z$$

then the condition that $L_1 \cap L_2 \cap L_3 \neq \varnothing$ is equivalent to the vanishing

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = 0.$$

This is a homogeneous trilinear form on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, from which we see that Φ is a closed subset of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, and A is a closed subset of $\mathbb{P}^9$. Moreover, we see that the class of Φ is

$$[\Phi] = \alpha_1 + \alpha_2 + \alpha_3 \in A^1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2),$$

so that the pullback via μ of 5 general hyperplanes in $\mathbb{P}^9$ will intersect Φ in

$$[\Phi]\,(\alpha_1 + \alpha_2 + \alpha_3)^5 = (\alpha_1 + \alpha_2 + \alpha_3)^6 = \binom{6}{2,2,2} = 90$$

points. Since the map $\mu|_\Phi : \Phi \to A$ has degree 6, it follows that the degree of the locus $A \subset \mathbb{P}^9$ of asterisks is 15.

1.3 Chern Classes

******** Unless we can prove Lemma **??** we should change this section to just deal with bundles with "enough" sections in the sense of Ch 7********

In Section 1.1.3 we defined the first Chern class $c_1(L)$ of a line bundle L on a variety X by choosing a rational section σ of L, and setting

$$c_1(L) = [(\sigma)_0 - (\sigma)_\infty] \in A_{\dim X - 1}(X),$$

the divisor of zeros of σ minus the divisor of poles. We will define Chern classes $c_i(E) \in A_{\dim X - i}(X)$ for any vector bundle E and any $i = 0, 1, \ldots$. (The reason that we haven't talked about all the $c_i(L)$ of a line bundle L is that, quite generally, $c_i(E) = 0$ when $i > \operatorname{rank} E$, while $c_0(E) = [X]$ for any E.)

The theory of Chern classes is perhaps the most powerful tool in intersection theory, and will be the main focus of this book starting with Chapter 7. Many of the loci whose classes we will want to compute turn out to come from Chern classes, and the theory can be used to "linearize" problems in a sense exemplified in Section **??**. Fortunately, there are also very strong techniques (the Whitney sum formula, the splitting principle, etc) for computing Chern classes. In this section we will sketch the beginning of the theory. Detailed proofs of the results, and much more, will be found in Chapter 7. In the next section we explain some examples based on the Chern classes of the Tangent bundle of a smooth projective variety X, which are so important in geometry that they are usually just called the *Chern classes of* X.

1.3.1 Chern Classes as Dependency Loci

Although the theory of Chern classes is quite general (and we will give a formula that can serve as a general definition in Chapter 11), we will restrict ourselves here to bundles on smooth projective varieties. The advantage is that we can begin by tensoring E with $\mathcal{O}_X(m)$ for large m, and get a bundle with lots of sections. It turns out that the Chern classes of E can be computed from those of $E(m) = E \otimes \mathcal{O}_X(m)$, so it will suffice to define Chern classes for bundles generated by their global sections. This simplifies our task: for example, in the case of a line bundle L with a global section σ, we have $c_1(L) = [(\sigma)_0]$, since there are no poles.

We begin with two special cases. Let E be a vector bundle of rank r on a variety X of dimension n, and suppose that E is generated by its global sections.

$c_r(E)$: Let σ be any global section of E. Since E is locally trivial, σ can be given locally as a sequence of $r = \operatorname{rank} E$ functions on X, and thus the vanishing locus of σ could have codimension r in X. When this happens—and it will always happen for a general section when E is generated by its global sections—then (as we will show) the class $[(\sigma)_0]$ of the locus where σ vanishes is independent of the choice of σ, and we set

$$c_r(E) = [(\sigma)_0].$$

Note that this is just what we did in the case of a line bundle, where $r = 1$. One way to define the vanishing locus of σ globally is to give its ideal sheaf: this is the image of the dual map $E^* \to \mathcal{O}_X$.

$c_1(E)$: The r-th exterior power $\wedge^r E$ is a line bundle, and we define $c_r(E)$ to be $c_1 \wedge^r E$, the vanishing locus of a global section of $\wedge^r E$.

To understand the generalization, consider again the second case above. If $\sigma_1, \ldots, \sigma_r$ is a collection of $r = \operatorname{rank} E$ sections of E, then $(\sigma_1, \ldots, \sigma_r)$ defines a CO_X-linear transformation $\mathcal{O}_X^r \to E$ sending the i-th basis vector to σ_i. The determinant of this transformation is the induced map

$$\wedge^r \sigma : \mathcal{O}_X = \wedge^r(\mathcal{O}_X^r) \to \wedge^r E.$$

The image under $\wedge^r \sigma$ of the global section 1 of $\mathcal{O}_X$ is a global section of $\wedge^r E$, and if it doesn't vanish identically, the class of its zero locus is, by definition, $c_1(E)$.

Assuming that E is generated by its global sections, we can ensure that $\wedge^r \sigma(1)$ is not identically zero by choosing a point x in X, trivializing E in a neighborhood of this point, and choosing $\sigma_1, \ldots, \sigma_r$ to be sections that restrict to a basis at x; then the determinant will be nonzero at x.

Since the determinant measures linear dependence, the locus where $\sigma(1)$ vanishes can be described as the locus where the r sections σ_i become linearly independent. As before, the the ideal defining the zero locus of σ can be realized as the image of the dual map

$$(\wedge^r \sigma)^* : \wedge^r E^* \to \wedge^r (\mathcal{O}_X^r)^* = \mathcal{O}_X.$$

Here is the general situation:

Proposition-Definition 1.31. Let E be a vector bundle of rank r on a variety X of dimension n. The class of the locus where $r - i + 1$ general global sections of E become dependent is independent of the choice of the sections so long as the codimension of the locus where they are dependent is i (or is empty). If E is generated by its global sections, then this condition is satisfied by any sufficiently general choice of $r - i + 1$ sections, and we define

$$c_i(E) \in A_{n-i}(X)$$

to be this class.

This is proven in Lemma **??**. ****Unfortunately, as of 6/25/11 it's stated but NOT proven.****

As in the special cases treated above, the notion of the *locus where sections* $\sigma_1, \ldots, \sigma_{r-i+1}$ *become dependent* may be clarified as follows: let

$$\sigma : \mathcal{O}_X^{r-i+1} \to E$$

be the map of vector bundles whose i-th component is the map sending $1 \in \mathcal{O}_X$ to σ. This transformation is given locally by an $r \times (r - i + 1)$ matrix of functions, and we define the dependency locus to be the scheme defined by the $(r - i + 1) \times (r - i + 1)$ minors of this matrix. Alternatively

we may define the locus to be the scheme corresponding to the ideal that is the image of the natural map

$$\wedge^{r-i+1}E^* \to \wedge^{r-i+1}(\mathcal{O}_X^{r-i+1}) = \mathcal{O}_X,$$

The locus we have just defined is often referred to as the *degeneracy locus* of the map σ, since it is the locus of points $x \in X$ where the rank of the map $\sigma|_x : K^{r-i+1} \to E|_x \cong K^r$ has less than maximal rank. We shall return to degeneracy loci in Chapter 14.

Adding to our list of special cases, we note that, for any rank r vector bundle E on X, the class c_0E is the class where $r+1$ sections are dependent. But this happens everywhere, so $c_0E = [X]$. When X is smooth and projective the class $[X]$ is the identity element of the Chow ring, so it is usual to write this as $c_0E = 1$.

Also, if $j > r = \operatorname{rank} E$, then $r - j + 1 \leq 0$, so $c_j(E)$ would be the dependency locus of no sections, or of a negative number of sections, whatever that means. We make the convention that $c_jE = 0$ in these cases.

1.3.2 Whitney's Formula and the Splitting Principle

Next we describe the two most useful tools for computing Chern classes. We first define the *total Chern class* of a bundle E to be

$$c(E) := c_0E + c_1E + c_2E \cdots = 1 + c_1E + c_2E \cdots \in A(X),$$

Theorem 1.32 (Whitney's Formula)**.** *If*

$$0 \to E' \to E \to E'' \to 0$$

is a short exact sequence of vector bundles on X then

$$c(E) = c(E')c(E'').$$

We postpone a precise statement (and proof) of the Splitting Principle to Lemma 7.4, but it may be stated informally as:

Informal Splitting Principle. *Any formula identity about the Chern classes of bundles that is true for a bundles that are iterated extensions of line bundles is true in general.*

A third point that is extremely useful in computation is that if $Y \subset X$ is a subvariety and E is a bundle on X, then the pullback of $c(E)$ to Y is equal to $c(E|_Y)$; this follows at once (when E has enough global sections) from the description of the Chern classes as degeneracy loci, and can be proven in general by reducing to this special case.

We will illustrate the use of these methods in the next section.

1.3.3 The Chern Character

The most natural operations on vector bundles are the direct sum and the tensor product while the most natural operations on Chow classes are the sum and the intersection product, so it is natural to ask how they compare. Another natural question is: how much information about a vector bundle is contained in its Chern classes? Both these questions can be answered in terms of a combination of the Chern classes of a vector bundle called the *Chern character*,

$$\mathrm{Ch}(E) = \mathrm{rank}(E) + c_1(E) + \frac{c_1(E)^2 - 2c_2(E)}{2} + \cdots \in \mathbb{Q} \otimes A(X),$$

Whitney's formula shows that the Chern classes are really defined on the classes of bundles in the *Grothendieck group* $K(X)$ of X, which is the abelian group with a generator $[E]$ for each vector bundle E on X and a relation $[E] = [E'] + [E'']$ for each short exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0.$$

The tensor product operation on vector bundles makes the Grothendieck group $K(X)$ of vector bundles on X into a ring, and the Chern character is a ring homomorphism. On a smooth projective variety the Chern Character preserves all the information of the class of a bundle in $K(X)$, and every class in $A(X)$ is the Chern character of some bundle:

Theorem 1.33 (Grothendieck)**.** *If X is a smooth projective variety, then the map*

$$\mathrm{Ch} : \mathbb{Q} \otimes K(X) \to \mathbb{Q} \otimes A(X)$$

is an isomorphism of rings.

1.4 Chern Class Examples

1.4.1 The 27 Lines on a Cubic Surface

To explain how Chern classes can be used to linearize otherwise difficult problems, we explain the Chern class approach to a famous classical result:

Theorem 1.34. *Each smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 distinct lines.*

One way to prove Theorem 1.34 is to show that any smooth cubic surface can be realized as the blowup of $\mathbb{P}^2$ in 6 fairly general points, and to analyze the geometry of such a blowup in detail (see for example Manin [1986]or Reid [1988b]. The Chern class approach that we will now indicate has the

advantage of applying equally to related results where no such analysis is available. For example, the Chern class method will also show that a general quintic threefold in $\mathbb{P}^4$ contains exactly 2875 lines (a computation that played an important role in the discovery of mirror symmetry; see for example Morrison [1993]), or even that a general hypersurface of degree 37 in $\mathbb{P}^{20}$ contains exactly

$$479849240965383456367278060519107076039364076181726998551 5$$

lines.

Sketch of the Chern Class Approach. The first ingredient in the Chern class approach is to make the set of lines in $\mathbb{P}^3$ into an algebraic variety, the Grassmannian $\mathbb{G}(1,3)$, as described in detail in Chapter 2. Given a smooth cubic surface X, one could imagine writing down the system of equations defining the subset of $\mathbb{G}(1,3)$ consisting of lines lying on X in terms of the homogeneous form F defining X, and computing its multiplicity. This system of equations is large, and highly nonlinear. Done directly this is a computation suitable only for a computer, and one that cannot currently be carried out even for the case of a quintic threefold in $\mathbb{P}^4$.

Rather, we will linearize the problem using the observation that, if we give a particular line L in $\mathbb{P}^3$, the condition that L lie on X can be expressed as 4 linear conditions on the coefficients of F: To see this, note that restricting a cubic form on $\mathbb{P}^3$ to a line $L \cong \mathbb{P}^1$ is a linear map from the 20-dimensional space of cubic forms on $\mathbb{P}^3$ onto the 4-dimensional vector space V_L of cubic forms on L, and the condition $L \subset X$ is that F maps to 0 in V_L.

The next step in this approach is to show that, as the line L varies over $\mathbb{G}(1,3)$, the 4-dimensional spaces of cubic forms on the varying L's "fit together" to form a vector bundle $\mathcal{V}$ of rank 4 on $\mathbb{G}(1,3)$. A cubic form F on $\mathbb{P}^3$, through its restriction to each V_L, defines an algebraic global section σ_F of this vector bundle.

Thus the number of lines contained in the cubic surface X is the number of zeros of the section σ_F, and if this number is finite then at least the scheme of zeros will have multiplicity $c_4(\mathcal{V})$. Once we have understood a little bit about vector bundles on Grassmannians it will be quite easy to show, using Whitney's formula and the splitting principle explained in Section 1.3.2, that

$$\deg c_4(\mathcal{V}) = 27.$$

Of course to prove Theorem 1.34 one still has to show that the number of lines on any smooth cubic surface is finite, and that the zeros all occur with multiplicity 1. This whole series of steps will be carried out in detail in Chapter 8. □

1.4.2 First Computations

In the rest of this Chapter we will assume that X is a smooth projective variety and we will compute the total Chern classes of various important bundles. We begin with the simplest illustration of the use of Whitney's formula and the Splitting Principle:

Sums of Line Bundles. When L is a line bundle our definitions give us

$$c(L) = 1 + c_1 L.$$

Using Whitney's formula, we can extend this to any bundle E that is an iterated extension of line bundles L_i (for example, E might be the direct sum $E = \oplus_{i=0}^{r} L_i$):

$$\begin{aligned} c(E) = \prod_{1\le i\le r} c(L_i) &= \prod_{1\le i\le r} (1 + c_1 L_i) \\ &= 1 + \sum_{1\le i\le r} c_1 L_i + \sum_{1\le i\le j\le r} (c_1 L_i)(c_1 L_j) + \cdots . \end{aligned}$$

It follows that $c_i E$ is the i-th elementary symmetric function of the $c_1 L_i$.

For example, since the unit element is a nowhere vanishing global section of the structure sheaf $\mathcal{O}_X$ of X, we have $c(\mathcal{O}_X) = 1 + c_1 \mathcal{O}_X = 1$, and thus $c(\mathcal{O}_X^m) = 1$ (the Chern class of a trivial bundle is trivial!)

Duals. Since $c_1 L^* = -c_1 L$ when L is a line bundle, Whitney's formula shows that $c_i E^* = (-1)^i c_i E$ for a bundle E that is an iterated extension of line bundles. By the Splitting principle, this last formula follows for any bundle:

Proposition 1.35. *If E is any vector bundle on X then $c_i E^* = (-1)^i c_i E$.* □

1.4.3 The Cotangent Bundle

The most important vector bundles on a smooth variety X are its tangent and cotangent bundles T_X and Ω_X. One of the bundles made from these comes up so often it too has its own name: the highest nonvanishing exterior power of Ω_X is called the *canonical bundle* $\omega_X := \wedge^{\dim X} \Omega_X$. The first Chern class $c_1(\omega_X) = c_1 \Omega_X \in A^1(X)$ is called the *canonical divisor* or *canonical class* of X, and is sometimes denoted K_X.

As an illustration of the importance of the canonical class, we recall that if X is a (smooth projective) curve over the complex numbers then the topology of X is determined by the *genus* $g(X)$ of X, which satisfies the formula

$$\deg c_1 \Omega_X = 2g - 2.$$

(see for example **?**] *** and ***, and for the generalization to higher dimensions see Theorem 1.41.)

Example 1.36 ($\mathbb{P}^n$)**.** The *Euler sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0$$

(described in Section 2.1.4) makes it easy to compute the Chern classes of the tangent and cotangent bundles of $\mathbb{P}^n$: applying Whitney's Formula, we see that

$$c(T_{\mathbb{P}^n}) = \frac{c(\mathcal{O}_{\mathbb{P}^n}(1)^{n+1})}{c(\mathcal{O}_{\mathbb{P}^n})} = c(\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}) = (1+\zeta)^{n+1} \in A(\mathbb{P}^n),$$

where $\zeta = c_1\mathcal{O}_{\mathbb{P}^n}(1)$ is the class of a hyperplane. To compute the Chern classes of $\Omega_{\mathbb{P}^n} = T^*_{\mathbb{P}^n}$ we could use the general formula for the Chern classes of a dual, or we could dualize the sequence above; either way, we obtain

$$c(\Omega_{\mathbb{P}^n}) = (1-\zeta)^{n+1}.$$

In particular, the canonical class of $\mathbb{P}^n$ is given by $c_1\Omega_{\mathbb{P}^n} = -(n+1)\zeta$, a formula that will be familiar as the divisor associated to a differential form (see for instance **?**] Example 8.20.1.)

If $Y \subset X$ is a smooth subvariety of a smooth variety X then there is an inclusion of bundles $T_Y \subset T_X|_Y$ and the quotient bundle

$$N_{Y/X} := T_X|_Y/T_Y$$

is called the *normal bundle* of Y in X. Dualizing, we get an exact sequence expressing the cotangent bundle of X as a cokernel,

$$0 \to N^*_{Y/X} \to \Omega_X|_Y \to \Omega_Y \to 0.$$

Thus

$$c(\Omega_Y) = \frac{c(\Omega_X|_Y)}{c(N^*_{Y/X})}.$$

Note that dividing by a the total Chern class of a bundle E makes sense because $c(E) = 1 + c_1E + c_2E + \cdots$, and all the (finitely many) nonzero terms but the first are nilpotent in $A(X)$, so the inverse is given by the formula

$$\frac{1}{c(E)} = 1 - (c_1E + c_2E + \cdots) + (c_1E + c_2E + \cdots)^2 - \cdots,$$

which has only finitely many nonzero terms.

These formulas immediately yield a formula for the canonical class of any smooth subvariety in terms of its normal bundle and the canonical class of the ambient variety:

Proposition 1.37 (Adjunction Formula)**.** *If X is a smooth variety, and $Y \subset X$ is a smooth subvariety of codimension m, then*

$$c_1\Omega_Y = c_1(\Omega_X|_Y) - c_1(N^*_{Y/X}) = c_1(\Omega_X|_Y) + c_1(N_{Y/X}).$$

The sheaf of sections of the cotangent bundle is the sheaf of *Kähler differentials* (Eisenbud [1995] ****). If $Y \subset X$ is a smooth subvariety with ideal sheaf $\mathcal{I}_{Y/X}$ then there is an exact sequence

$$0 \to \frac{\mathcal{I}_{Y/X}}{\mathcal{I}^2_{Y/X}} \to \Omega_X|_Y \to \Omega_Y \to 0,$$

(Eisenbud [1995] Theorem ****) and we see that $N^*_{Y/X} = \mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}$, which can also be written as

$$N_{Y/X} = (\mathcal{I}_{Y/X}|_Y)^*.$$

For example, if Y is a divisor on X then $\mathcal{I}_{Y/X} = \mathcal{O}_X(-Y)$, so we have

$$N_{Y/X} = (\mathcal{O}_X(-Y)|Y)^* = (\mathcal{O}_X(-Y))^*|Y = \mathcal{O}_Y(Y).$$

Example 1.38 (Canonical Bundles of Complete Intersections)**.** If $Y \subset \mathbb{P}^n$ is a hypersurface of degree d, then the ideal sheaf of Y is $\mathcal{O}_{\mathbb{P}^n}(-d)$, so

$$N_{Y/X} = (\mathcal{I}_{Y/X}|_Y)^* = (\mathcal{O}_{\mathbb{P}^n}(-d)|_Y)^* = \mathcal{O}_Y(d).$$

so $c_1(N_{Y/X}) = d\zeta_Y$, where ζ_Y is the class of the hyperplane section of Y. On the other hand, The Chern class of the restriction of $\Omega_{\mathbb{P}^n}$ to Y is the pullback to Y of the Chern class of $\Omega_{\mathbb{P}^n}$, so

$$c_1(\Omega_{\mathbb{P}^n}|Y) = c_1(\Omega_{\mathbb{P}^n})|Y = -(n+1)\zeta|_Y.$$

Applying Proposition 1.37, we get

$$c_1\Omega_Y = (d-n-1)\zeta_Y$$

or, equivalently,

$$\omega_Y \cong \mathcal{O}_Y(d-n-1).$$

More generally, the ideal sheaf of a complete intersection $Y = Z_1 \cap \cdots \cap Z_k \subset \mathbb{P}^n$ of k hypersurfaces of degrees $d_1, \ldots, d_k$ defined by forms F_i of degrees d_i, such that the relations among the F_i are generated by the Koszul relations $F_jF_i - F_iF_j = 0$. This means that if we restrict to Y, where the F_i vanish, we get

$$\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X} = \mathcal{I}_{Y/X}|Y = \oplus_i \mathcal{O}_Y(-d_i),$$

so $c_1(N_{Y/X}) = (d_1 + \cdots d_k)\zeta_Y$. Thus from the Adjunction Formula,

$$c_1\Omega_Y = (d_1 + \cdots + d_k - n - 1)\zeta_Y,$$

or equivalently,

$$\omega_Y \cong \mathcal{O}_Y(d_1 + \cdots + d_k - n - 1).$$

Example 1.39 (Chern Classes of a Product of Varieties)**.** If X and Y are smooth varieties and

$$X \xleftarrow{\pi_x} X \times Y \xrightarrow{\pi_Y}$$

are the projection maps, it is obvious that $T_{X\times Y} = \pi_X^*(T_X) \oplus \pi_Y^*(T_Y)$. Thus Whitney's formula yields

$$c(T_{X\times Y} = (\pi_X^* c(T_X))(\pi_Y^* c(T_Y)).$$

In particular,

$$K_{X\times Y} = c_1(T_{X\times Y} = (\pi_X^* c_1(T_X)) + (\pi_Y^* c_1(T_Y)) = (\pi_X^* K_X) + (\pi_Y^* K_Y).$$

The Genus of a Curve on a Surface. We can apply these ideas to divisors or complete intersections on varieties other than $\mathbb{P}^n$, and the simplest case is that of a smooth curve $C \subset S$ on a smooth surface S. Since the ideal sheaf of C on S may be written $\mathcal{I}_{C/S} = \mathcal{O}_S(-C)$, an argument analogous to the one above gives $N_{C/S} = \mathcal{O}_S(C)|_C = \mathcal{O}_C(C)$. Thus the first Chern class of $N_{C/S}$ may be written as

$$c_1 N_{C/S} = [C]|_C.$$

If we write $K_C = c_1\Omega_C$ and $K_S = c_1\Omega_S$ for the canonical divisor classes, the Adjunction formula becomes

$$K_C = K_S|_C + [C]|_C, so$$

$$\begin{aligned} g &= \frac{1}{2}\deg K_C + 1 \\ &= \frac{\deg(K_S|_C + [C])|_C)}{2} + 1 \\ &= \frac{\deg(K_S\,[C]) + \deg([C]^2))}{2} + 1. \end{aligned}$$

Informally we write

$$g = \frac{K_S \cdot C + C^2}{2} + 1;$$

this is known as the *genus formula*.

By way of examples, consider first a smooth curve $C \subset \mathbb{P}^2$ of degree d. If we let $\zeta \in A^1(\mathbb{P}^2)$ be the class of a line, we have $C \sim d\zeta$ and $K_{\mathbb{P}^2} \sim -3\zeta$, so the genus of C is

$$g = \frac{-3d + d^2}{2} + 1 = \frac{(d-1)(d-2)}{2}.$$

Likewise, suppose that $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth curve of bidegree (a, b)—in other words, if ζ_1 and $\zeta_2 \in A^1(\mathbb{P}^1 \times \mathbb{P}^1)$ are the pullbacks of the point

classes from the two factors, $C \sim a\zeta_1 + b\zeta_2$. From Example 1.39 we get $K_{\mathbb{P}^1\times\mathbb{P}^1} = -2\zeta_1 - 2\zeta_2$. Thus

$$\begin{aligned} g &= \frac{(-2\zeta_1 - 2\zeta_2)(a\zeta_1 + b\zeta_2) + (a\zeta_1 + b\zeta_2)^2}{2} + 1 \\ &= \frac{2ab - 2a - 2b}{2} + 1 \\ &= (a-1)(b-1). \end{aligned}$$

The Self-Intersection of a Curve on a Surface. We can sometimes use the genus formula to determine the self-intersection of a curve on a surface. For example, suppose that $S \subset \mathbb{P}^3$ is a smooth surface of degree d and $L \subset S$ is a line. Letting $\zeta \in A(\mathbb{P}^3)$ denote the plane class, we have $K_S \sim (d-4)\zeta$, so that $\deg(L \cdot K_S) = d - 4$; since the genus of L is zero, the genus formula yields

$$0 = \frac{L^2 + d - 4}{2} + 1$$

or

$$L^2 = 2 - d.$$

The cases $d = 1$ (a line on a plane) and $d = 2$ are probably familiar already; in the latter we are looking at a line on a quadric surface, and such a line moves in a rational family of lines that are disjoint from one another, so the self-intersection of any of them is 0. In case $d \geq 3$ the formula implies the qualitative statement that *a smooth surface $S \subset \mathbb{P}^3$ of degree 3 or more can contain only finitely many lines.* (See Exercise 1.90 below for a sketch of a proof; see also Exercise 1.89 for an alternative derivation of $L^2 = 2 - d$.)

Linked curves in $\mathbb{P}^3$. Another application of the genus formula yields a classical relation between what are called *linked curves* in $\mathbb{P}^3$.

Let S and $T \subset \mathbb{P}^3$ be smooth surfaces of degrees s and t, and suppose that the scheme-theoretic intersection $S \cap T$ consists of the union of two smooth curves C and D. Let the degrees of C and D be c and d, and let their genera be g and h respectively. By Bézout's Theorem, we have

$$c + d = st,$$

so that the degree of C determines the degree of D. What is much less obvious is that the degree and genus of C determine the degree and genus of D. Here is how the formula is derived.

To start, we use the genus formula as above to determine the self-intersection of C on S: since $K_S \sim (s-4)\zeta$, we have

$$g = \frac{C^2 + K_S \cdot C}{2} + 1 = \frac{C^2 + (s-4)c}{2} + 1$$

and hence

$$C^2 = 2g - 2 - (s-4)c$$

(generalizing our formula above for the self-intersection of a line). Next, since $C + D \sim t\zeta \in A^1(S)$, we can write the intersection number of C and D on S as

$$C \cdot D = C(t\zeta - C) = tc - \big(2g - 2 - (s-4)c\big) = (s+t-4)c - (2g-2).$$

This in turn allows us to determine the self-intersection of D on S:

$$D^2 = D(t\zeta - C) = td - \big((s+t-4)c - (2g-2)\big).$$

Applying the genus formula to D we arrive at

$$\begin{aligned} h &= \frac{D^2 + K_S \cdot D}{2} + 1 \\ &= \frac{td - \big((s+t-4)c - (2g-2)\big) + (s-4)d}{2} + 1. \end{aligned}$$

Simplifying, this says

$$h - g = \frac{s+t-4}{2}(d-c); \tag{1.1}$$

in English, *the difference in the genera of C and D is proportional to the difference in their degrees, with ratio $(s+t-4)/2$.*

As is often the case with enumerative formulas, this is just the beginning of a whole theory. The theory of Liaison describes the relationship between the geometry of linked curves such as C and D above. The theory in general is far more broadly applicable (the curves C and D need only be Cohen-Macaulay, and we need no hypotheses at all on S and T) and ultimately provides a complete answer to the question of when two given curves $C, D \subset \mathbb{P}^3$ can be connected by a series of curves $C = C_0, C_1, \ldots, C_{n-1}, C_n = D$ with C_i and C_{i+1} linked as above. We will see a typical application of the notion of linkage in Exercise 1.94 below; references for the general theory are ****References?****

The topological Euler Characteristic. So far we have only seen applications of the first Chern class, c_1. Here is an application involving instead the top Chern class of the cotangent bundle, $c_{\dim X}\Omega_X$, derived from the Poincaré-Hopf Index Theorem. Recall that the *topological Euler characteristic* of a manifold M is by definition $\chi_{\text{top}}(M) := \sum_i (-1)^i \dim_{\mathbb{Q}} H^i(M;\mathbb{Q})$.

Theorem 1.40 (Poincaré-Hopf Theorem)**.** *If M is a smooth compact orientable manifold, and σ is a vector field with isolated zeros, then*

$$\chi_{\text{top}}(M) = \sum_{x|\sigma(x)=0} \text{index}_x(\sigma).$$

Now suppose that M is a smooth complex projective variety. If the tangent bundle T_X is generated by global sections then it will have a section σ that vanishes at only finitely many points, and vanishes simply there. Since this section is represented locally by complex analytic functions, its index at each of its zeros will be 1, and we may replace the sum in the Poincaré-Hopf Theorem by the number of its zeros. This is, by definition, the top Chern class of T_X. An elementary topological argument, given in Chapter 21, shows that this is true more generally:

Theorem 1.41. *If X is a smooth projective variety then*

$$\chi_{\text{top}}(X) = \deg\bigl(c_{\dim X}(T_X)\bigr).$$

Example 1.42 (Euler characteristic of $\mathbb{P}^n$)**.** Since $c(T_{\mathbb{P}^n}) = (1+\zeta)^{n+1}$, where $\zeta = c_1(\mathcal{O}_{\mathbb{P}^n}(1)$ is the class of a hyperplane, we deduce that

$$\chi_{\text{top}}(\mathbb{P}^n) = \deg\bigl(c_n(T_{\mathbb{P}^n})\bigr) = n+1.$$

Of course this is immediate from the fact that $H^{2i}(\mathbb{P}^n, \mathbb{Q}) = \mathbb{Q}$ for $i = 0, \ldots, n$ while $H^{2i+1}(\mathbb{P}^n, \mathbb{Q}) = 0$ for all i.

Example 1.43 (Euler characteristic of a hypersurface)**.** Now let X be a smooth hypersurface of degree d in $\mathbb{P}^n$. From the exact sequence

$$0 \to T_X \to T_{\mathbb{P}^n}|_X \to N_{X/\mathbb{P}^n} \to 0.$$

The normal bundle of X is $N_{X/\mathbb{P}^n} = \mathcal{O}_X(d)$ so $c(N_{X/\mathbb{P}^n} = 1 + d\zeta_X$, where ζ_X is the hyperplane class on X. By Whitney's formula

$$c(T_X) = \frac{(1+\zeta_X)^{n+1}}{(1+d\zeta_X)} = ((1+\zeta_X)^{n+1})(1 - d\zeta_X + d^2(\zeta_X)^2 + \cdots).$$

Taking the component of degree $\dim X = n-1$, we get

$$c_{n-1}(T_X) = \sum_{i=0}^{n-1} (-1)^i \binom{n+1}{n-1-i} d^i \zeta_X^{n-1}.$$

Since the degree of ζ_X^{n-1} is the number of points of intersection of $n-1$ general hyperplanes on the $(n-1)$ dimensional variety X, we have $\zeta_X^{n-1} = d$. Thus, finally,

$$\chi_{\text{top}}(X) = \deg(c_{n-1}T_X) = \sum_{i=0}^{n-1} (-1)^i \binom{n+1}{n-1-i} d^{i+1}.$$

For example, we know that the Euler characteristic of a Riemann surface of genus g is $2-2g$. If X is a smooth plane curve of degree d then, plugging $n = 2$ into the formula above, we see that

$$\chi_{\text{top}}(X) = 2 - 2g = 3d - d^2,$$

whence the familiar formula

$$g = \frac{d^2 - 3d + 2}{2} = \binom{d-1}{2}.$$

1.5 Exercises

Exercise 1.44. Let $\nu = \nu_{2,2} : \mathbb{P}^2 \to \mathbb{P}^5$ be the quadratic Veronese map. If $C \subset \mathbb{P}^2$ is a plane curve of degree d, show that the image $\nu(C)$ has degree $2d$. (In particular, this means that the Veronese surface $S \subset \mathbb{P}^5$ contains only curves of even degree!)

Exercise 1.45. More generally, let $\nu = \nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$ be the degree d Veronese map. If $X \subset \mathbb{P}^n$ is a variety of dimension k and degree e, show that the image $\nu(X)$ has degree $d^k e$.

Exercise 1.46. Let $\sigma = \sigma_{r,s} : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{(r+1)(s+1)-1}$ be the Segre map, and let $X \subset \mathbb{P}^r \times \mathbb{P}^s$ be a subvariety of codimension k. If the class $[X] \in A^k(\mathbb{P}^r \times \mathbb{P}^s)$ is given by

$$[X] = c_0\alpha^k + c_1\alpha^{k-1}\beta + \cdots + c_k\beta^k$$

(where α and $\beta \in A^1(\mathbb{P}^r \times \mathbb{P}^s)$ are the pullbacks of the hyperplane classes, and we take $c_i = 0$ if $i > s$ or $k - i > r$),

(a) Show that all $c_i \geq 0$.

(b) Calculate the degree of the image $\sigma(X) \subset \mathbb{P}^{(r+1)(s+1)-1}$; and, using this and the first part,

(c) Show that any linear space $\Lambda \subset \Sigma_{r,s} \subset \mathbb{P}^{(r+1)(s+1)-1}$ contained in the Segre variety lies in a fiber of a projection map.

Exercise 1.47. Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ be the rational map given by

$$\varphi : (x_0, x_1, x_2) \mapsto \left(\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}\right),$$

or, equivalently,

$$\varphi : (x_0, x_1, x_2) \mapsto (x_1x_2, x_0x_2, x_0x_1)$$

and let $\Gamma_\varphi \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the graph of φ. Find the class

$$[\Gamma_\varphi] \in A^2(\mathbb{P}^2 \times \mathbb{P}^2).$$

Exercise 1.48. Let $X_1, \ldots, X_n \subset \mathbb{P}^n$ be hypersurfaces of degrees $d_1, \ldots, d_n$. Let $p \in \mathbb{P}^n$ be a point, and suppose that the hypersurface X_i has multiplicity m_i at p; suppose moreover that the intersection of the projective tangent cones $\mathbb{P}TC_pX_i$ to X_i at p is empty. Use the description of the Chow ring of the blow-up of $\mathbb{P}^n$ at p to show that the number of points of intersection of the X_i away from p is

$$\#\left(\bigcap(X_i \setminus \{p\})\right) = \prod d_i - \prod m_i.$$

Exercise 1.49. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. Suppose that X has an ordinary double point (that is, a point $p \in X$ such that the projective tangent cone $\mathbb{P}TC_pX$ is a smooth quadric) and is otherwise smooth. What is the degree of the dual hypersurface $X^* \subset \mathbb{P}^{n*}$?

Exercise 1.50. Let $X \subset \mathbb{P}^n$ be a variety of degree d and dimension k; suppose that $p \in X$ is a point of multiplicity m. Let $B = Bl_p(\mathbb{P}^n)$ be the blow-up of $\mathbb{P}^n$ at the point p, and $\tilde{X} \subset B$ the proper transform of X in B. Find the class $[\tilde{X}] \in A^{n-k}(B)$.

Exercise 1.51. Let $p \in X \subset \mathbb{P}^n$ be as in the preceding Exercise, and suppose that the projection map $\pi_p : X \to \mathbb{P}^{n-1}$ is birational onto its image. What is the degree of $\pi_p(X)$?

Exercise 1.52. Show that if X is an irreducible plane cubic with a node, then $c_1 : Pic(X) \to A_1(X)$ is not a monomorphism as follows. Show that there is no birational map from X to $\mathbb{P}^1$. Use this to show that if $p \neq q \in X$ are smooth points then the line bundles $\mathcal{O}_X(p)$ and $\mathcal{O}_X(q)$ are non-isomorphic. Show, however, that the zero loci of their unique sections, the points p and q, are rationally equivalent.

Exercise 1.53. Let X be a quadric cone in $\mathbb{P}^3$ and let B be the blowup of X at the origin. Show that B is a smooth surface. Show that if $X \subset \mathbb{P}^3$ a quadric cone with vertex p then $A_1(X) = \mathbb{Z}$, generated by the class of a line, and the image of $c_1 : Pic(X) \to A_1(X)$ is $2\mathbb{Z}$ by showing that the image consists of the subgroup of classes of curves lying on X that have even degree as curves in $\mathbb{P}^3$. In particular, the class of a line on X is not in the image. Do this by showing that any curve C passing through the vertex of X no curve $C \subset X$ of odd degree can be a Cartier divisor on X: if such a curve meets the general line of the ruling of X at δ points away from p and has multiplicity m at p, then intersecting C with a general plane through p we see that $\deg(C) = 2\delta + m$; it follows that m is odd, and hence that C cannot be Cartier at p. Thus, the class $[M]$ of a line of the ruling cannot be $c_1(L)$ for any line bundle L.

Exercise 1.54. Show that the Chow ring of a product of projective spaces $\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k}$ is

$$\begin{aligned} A(\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k}) &= \bigotimes A(\mathbb{P}^{r_i}) \\ &= \mathbb{Z}[\alpha_1, \ldots, \alpha_k]/(\alpha_1^{r_1+1}, \ldots, \alpha_k^{r_k+1}), \end{aligned}$$

where $\alpha_1, \ldots, \alpha_k$ are the pullbacks of the hyperplane classes from the factors. Use this to calculate the degree of the image of the Segre embedding

$$\sigma : \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k} \hookrightarrow \mathbb{P}^{(r_1+1)\cdots(r_k+1)-1}$$

corresponding to the multilinear map $V_1 \times \cdots \times V_k \to V_1 \otimes \cdots \otimes V_k$.

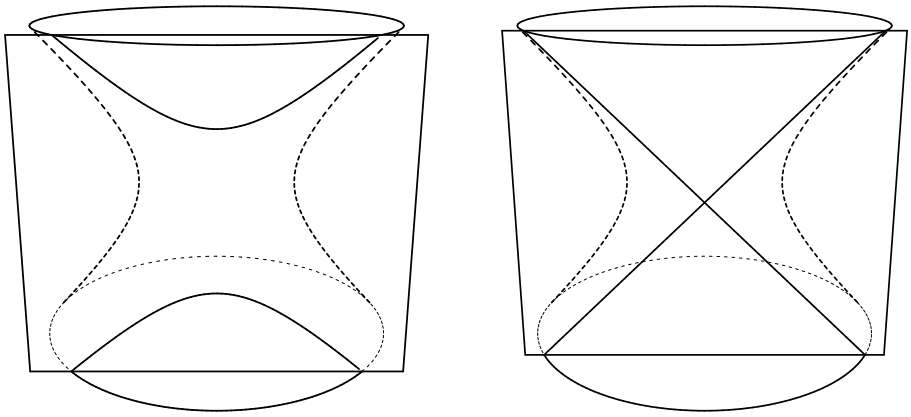


FIGURE 1.9. The diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ is equivalent to a sum of fibers.

Exercise 1.55. For $t \neq 0$, let $A_t : \mathbb{P}^r \to \mathbb{P}^r$ be the automorphism

$$[X_0, X_1, X_2, \ldots, X_r] \mapsto [X_0, tX_1, t^2X_2, \ldots, t^rX_r].$$

Let $\Phi \subset \mathbb{A}^1 \times \mathbb{P}^r \times \mathbb{P}^r$ be the closure of the locus

$$\tilde{\Phi} = \{(t, p, q) : t \neq 0 \text{ and } q = A_t(p)\}.$$

Describe the fiber of Φ over the point $t = 0$, and deduce once again the formula of Section 1.2.3 for the class of the diagonal in $\mathbb{P}^r \times \mathbb{P}^r$.

In the simplest case, this construction is a rational equivalence between a smooth plane section of a quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ (the diagonal, in terms of suitable identifications of the factors with $\mathbb{P}^1$), and a singular one (the sum of a line from each ruling), as in Figure 1.9.

Exercise 1.56. Let

$$\Psi = \{(p, q, r) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n \mid p, q \text{ and } r \text{ are collinear in } \mathbb{P}^n\}.$$

(Note that this includes all diagonals.) Show that this is a closed subvariety of codimension $n - 1$ in $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$.

Exercise 1.57. With $\Psi \subset \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ as in the preceding exercise, use the method of undetermined coefficients to find the class

$$\psi = [\Psi] \in A^{n-1}\left(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n\right).$$

Assume characteristic 0, and use Kleiman's theorem as necessary.

Exercise 1.58. To extend the result of the preceding exercise to characteristic p, show in arbitrary characteristic that for a general product of subspaces $\mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^k \subset \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, the intersection $\Psi \cap (\mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^k)$ is transverse.

(We will see a way to calculate the class ψ using Porteous' formula in Exercise 14.6)

Exercise 1.59. Suppose that $(F_0, \ldots, F_r)$ and $(G_0, \ldots, G_r)$ are general $(r+1)$-tuples of homogeneous polynomials in $r+1$ variables, of degrees d and e respectively, so that in particular the maps $f : \mathbb{P}^r \to \mathbb{P}^r$ and $g : \mathbb{P}^r \to \mathbb{P}^r$ sending x to $(F_0(x), \ldots, F_r(x))$ and $(G_0(x), \ldots, G_r(x))$ are regular. For how many points $x = (x_0, \ldots, x_r) \in \mathbb{P}^r$ do we have $f(x) = g(x)$?

Exercise 1.60. For $i = 0, \ldots, r+s+1$, let $F_i([X_0, \ldots, X_r], [Y_0, \ldots, Y_s])$ be a general bihomogeneous polynomial of bidegree (d, e). For how many $X = [X_0, \ldots, X_r]$ and $Y = [Y_0, \ldots, Y_s]$ do we have

$$[F_0(X, Y), \ldots, F_r(X, Y)] \sim [X_0, \ldots, X_r]$$

and

$$[F_{r+1}(X, Y), \ldots, F_{r+s+1}(X, Y)] \sim [Y_0, \ldots, Y_s]?$$

The next two exercises set up the following one, which considers when a point $p \in \mathbb{P}^2$ is collinear with its images under several maps:

Exercise 1.61. Consider the locus $\Phi \subset (\mathbb{P}^2)^4$ of fourtuples of collinear points. Find the class $\varphi = [\Phi] \in A^2((\mathbb{P}^2)^4)$ of Φ by the method of undetermined coefficients; that is, by intersecting with cycles of complementary dimension.

Exercise 1.62. With $\Phi \subset (\mathbb{P}^2)^4$ as in the preceding problem, calculate the class $\varphi = [\Phi]$ by using the result of Exercise 1.57 on the locus $\Psi \subset (\mathbb{P}^2)^3$ of triples of collinear points, and considering the intersection of the loci $\Psi_{1,2,3}$ and $\Psi_{1,2,4}$ of fourtuples (p_1, p_2, p_3, p_4) with p_1, p_2, p_3 colinear and with p_1, p_2, p_4 colinear.

Exercise 1.63. Let A, B and $C : \mathbb{P}^2 \to \mathbb{P}^2$ be three general automorphisms. For how many points $p \in \mathbb{P}^2$ are the points $p, A(p), B(p)$ and $C(p)$ collinear? Assume characteristic 0, and use Kleiman's theorem as necessary.

Exercise 1.64. Consider the map $f : \mathbb{P}^1 \to \mathbb{P}^1$ given by

$$f : [X_0, X_1] \mapsto \left[X_0^p, X_1^p\right].$$

Assuming that the ground field has characteristic p, show that the intersection of Γ_f with $\mathbb{P}^1 \times \{x\}$ is non-reduced for every x, but that the class $\gamma_f = [\Gamma_f]$ is $p\alpha + \beta$, just as it would be in characteristic zero.

Exercise 1.65. Let B be the blowup of $\mathbb{P}^n$ at a point p, with exceptional divisor E as in Section 1.2.4. With notation as in that section, show that there is an affine stratification with closed strata Γ_k for $k = 1, \ldots n$ and, $E_k := \Gamma_k \cap E$ for $k = 0, \ldots, n-1$. Let e_k be the class of E_k. Show that $e_{n-1} = \lambda_{n-1} - \gamma_{n-1}$ to describe the classes γ_k in terms of λ_k and e_k and vice versa. Conclude that the classes $\gamma_k = [\Gamma_k]$ and e_k form a basis for the Chow group $A(B)$.

The next few exercises deal with the blow-up of $\mathbb{P}^3$ along a line. To fix notation, let $\pi : X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along a line $L \subset \mathbb{P}^3$; that is, the graph $X \subset \mathbb{P}^3 \times \mathbb{P}^1$ of the rational map $\pi_L : \mathbb{P}^3 \to \mathbb{P}^1$ given by projection from L; let $\alpha : X \to \mathbb{P}^1$ be projection on the second factor.

Exercise 1.66. Let $H \subset \mathbb{P}^3$ be a plane containing L, and $\tilde{H} \subset X$ its proper transform. Let $J \subset \mathbb{P}^3$ be a plane transverse to L, $\tilde{J} \subset X$ its proper transform (which equals its preimage in X), and let $M \subset J$ be a line not meeting L. Show that the subvarieties

$$X, \quad \tilde{H}, \quad \tilde{J}, \quad \tilde{J} \cap \tilde{H}, \quad M, \quad M \cap \tilde{H}$$

are the closed strata of an affine stratification of X, with open strata isomorphic to affine spaces. In particular, since only one $(M \cap \tilde{H})$ is a point, deduce that $A^3(X) \cong \mathbb{Z}$.

Exercise 1.67. Let $h = [\tilde{H}]$, $j = [\tilde{J}] \in A^1(X)$ and $m = [M] \in A^2(X)$ be the classes of the corresponding strata. Show that

$$h^2 = 0, \quad j^2 = m, \quad \text{and} \quad \deg(jm) = \deg(hm) = 1.$$

Conclude that

$$A(X) = \mathbb{Z}[h, j]/(h^2, j^3 - hj^2).$$

Exercise 1.68. Now let $E \subset X$ be the exceptional divisor of the blow-up, and $e = [E] \in A^1(X)$ its class. What is the class e^2?

Exercise 1.69. Let $\mathbb{P}^5$ be the space of conic curves in $\mathbb{P}^2$.

(a) Find the dimension and degree of the locus of double lines (in characteristic $\neq 2$).

(b) Find the dimension and degree of the locus $\Delta \subset \mathbb{P}^5$ of singular conics (that is, line pairs and double lines).

Exercise 1.70. Let $\mathbb{P}^5$ be the space of conic curves in $\mathbb{P}^2$.

(a) Find the dimension and degree of the locus of double lines (in characteristic $\neq 2$).

(b) Find the dimension and degree of the locus $\Delta \subset \mathbb{P}^5$ of singular conics (that is, line pairs and double lines).

The following exercises deal with some of the loci in the space $\mathbb{P}^9$ of plane cubics described in Section 1.2.5.

Exercise 1.71. Let $\mathbb{P}^9$ be the space of plane cubics, and $\Gamma \subset \mathbb{P}^9$ the locus of reducible cubics. Let L and $C \subset \mathbb{P}^2$ be a line and a smooth conic intersecting transversely at two points $p, q \in \mathbb{P}^2$; let $L + C$ be the corresponding point of Γ. Show that Γ is smooth at $L + C$, with tangent space

$$\mathbb{T}_{L+C}\Gamma = \mathbb{P}\{\text{homogeneous cubic polynomials } F : F(p) = F(q) = 0\}.$$

Exercise 1.72. Using the preceding exercise, show that if $p_1, \ldots, p_7 \in \mathbb{P}^2$ are general points, and $H_i \subset \mathbb{P}^9$ is the hyperplane of cubics containing p_i, then the hyperplanes $H_1, \ldots, H_7$ intersect Γ transversely—that is, the degree of Γ is the number of reducible cubics through $p_1, \ldots, p_7$.

Exercise 1.73. Calculate the number of reducible plane cubics passing through 7 general points $p_1, \ldots, p_7 \in \mathbb{P}^2$, and hence, by the preceding exercise, the degree of Γ.

Exercise 1.74. We can also calculate the degree of the locus $\Sigma \subset \mathbb{P}^9$ of triangles (that is, totally reducible cubics) directly, as in the series of exercises starting with (1.71). To start, show that if $C = L_1 + L_2 + L_3$ is a triangle with the three vertices—that is, points $p_{i,j} = L_i \cap L_j$ of pairwise intersection—distinct, then Σ is smooth at C with tangent space

$$\mathbb{T}_{L+C}\Sigma = \mathbb{P}\{\text{homogeneous cubic polynomials } F : F(p_{i,j}) = 0 \ \forall\, i, j\}.$$

Exercise 1.75. Using the preceding exercise,

(a) Show that if $p_1, \ldots, p_6 \in \mathbb{P}^2$ are general points, then the degree of Σ is the number of triangles containing $p_1, \ldots, p_6$; and

(b) Calculate this number directly.

Exercise 1.76. Consider a general asterisk—that is, the sum $C = L_1 + L_2 + L_3$ of three distinct lines all passing through a point p. Show that the variety $\Sigma \subset \mathbb{P}^9$ of triangles is smooth at C, with tangent space the space of cubics double at p. Deduce that the space $A \subset \mathbb{P}^9$ of asterisks is also smooth at C.

Exercise 1.77. Let $p_1, \ldots, p_5 \in \mathbb{P}^2$ be general points. Show that any asterisk containing $\{p_1, \ldots, p_5\}$ consists, after possibly relabelling the points, of the sum of the line $L_1 = \overline{p_1, p_2}$, the line $L_2 = \overline{p_3, p_4}$ and the line $L_3 = \overline{p_5, (L_1 \cap L_2)}$.

Exercise 1.78. Using the preceding two exercises, show that for $p_1, \ldots, p_5 \in \mathbb{P}^2$ general points, the hyperplanes H_{p_i} intersect the locus $A \subset \mathbb{P}^9$ of asterisks transversely, and calculate the degree of A accordingly.

Exercise 1.79. Show that (in characteristic $\neq 3$) the locus $Z \subset \mathbb{P}^9$ of triple lines is a cubic Veronese surface, and deduce that its degree is 9.

Exercise 1.80. Let $X \subset \mathbb{P}^9$ be the locus of cubics of the form $2L + M$ for L and M lines in $\mathbb{P}^2$.

(a) Show that X is the image of $\mathbb{P}^2 \times \mathbb{P}^2$ under a regular map such that the pullback of a general hyperplane in $\mathbb{P}^9$ is a hypersurface of bidegree $(2, 1)$.

(b) Use this to find the degree of X.

Exercise 1.81. If you try to find the degree of the locus X of the preceding problem by intersecting X with hyperplanes $H_{p_1}, \ldots, H_{p_4}$, where

$$H_p = \{C \in \mathbb{P}^9 : p \in C\},$$

you get the wrong answer (according to the preceding problem). Why? Can you account for the discrepancy?

Exercise 1.82. Let $\mathbb{P}^2$ denote the space of lines in the plane, and $\mathbb{P}^5$ the space of plane conics. Let $\Phi \subset \mathbb{P}^2 \times \mathbb{P}^5$ be the closure of the locus of pairs

$$\{(L, C) : \ C \text{ is smooth, and } L \text{ is tangent to } C\}.$$

Show that Φ is a hypersurface; and, assuming characteristic 0, find its class $[\Phi] \in A^1(\mathbb{P}^2 \times \mathbb{P}^5)$.

Exercise 1.83. Now let $\mathbb{P}^9$ be the space of plane cubic curves as before, and let $Y \subset \mathbb{P}^9$ be the closure of the locus of reducible cubics consisting of a smooth conic and a tangent line. Use the result of the first part to determine the degree of B.

Exercise 1.84. Let $\mathbb{P}^{14}$ be the space of quartic curves in $\mathbb{P}^2$, and let $\Sigma \subset \mathbb{P}^{14}$ be the closure of the space of reducible quartics. What are the irreducible components of Σ, and what are their dimensions and degrees?

Exercise 1.85. Find the dimension and degree of the locus $\Omega \subset \mathbb{P}^{14}$ of totally reducible quartics (that is, quartic polynomials that factor as a product of four linear forms).

Exercise 1.86. Again let $\mathbb{P}^{14}$ be the space of plane quartic curves, and let $\Theta \subset \mathbb{P}^{14}$ be the locus of sums of four concurrent lines. Using the result of Exercise 1.61, find the degree of Θ.

Exercise 1.87. Find the degree of the locus $A \subset \mathbb{P}^{14}$ of the preceding problem, this time by calculating the number of sums of four concurrent lines containing six general points $p_1, \ldots, p_6 \in \mathbb{P}^2$, assuming transversality.

A natural generalization of the locus of asterisks, or of sums of four concurrent lines, would be the locus, in the space $\mathbb{P}^N$ of hypersurfaces of degree d in $\mathbb{P}^n$, of *cones*. We will indeed be able to calculate the degree of this locus in general, but it will require more advanced techniques than we have at our disposal here. ******** Joe will write this into Section 12.3********.

Exercise 1.88. More generally, let $\mathbb{P}^N$ be the space of hypersurfaces of degree d in $\mathbb{P}^n$.

(a) Find the dimension and degree of the locus $\Gamma \subset \mathbb{P}^N$ of reducible and/or nonreduced hypersurfaces. (Note that this locus is in general reducible; by its degree we mean the degree of the union of its components of maximal dimension.)

(b) Find the dimension and degree of the locus $\Sigma \subset \mathbb{P}^N$ of totally reducible hypersurfaces.

Exercise 1.89. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d and $L \subset S$ a line. Calculate the degree of the self-intersection of the class $\lambda = [L] \in A^1(S)$ by considering the intersection of S with a general plane $H \subset \mathbb{P}^3$ containing L.

Exercise 1.90. Let S be a smooth surface. Show that if $C \subset S$ is any curve, and the corresponding point in the Hilbert scheme $\mathcal{H}$ of curves on S lies on a positive-dimension irreducible component of $\mathcal{H}$, then the degree $\deg(\gamma^2)$ of the self-intersection of the class $\gamma = [C] \in A^1(S)$ is nonnegative. Using this and the preceding Exercise, prove the statement made in Section 1.4.3 that *a smooth surface $S \subset \mathbb{P}^3$ of degree 3 or more can contain only finitely many lines.*

Exercise 1.91. Let $C \subset \mathbb{P}^3$ be a smooth quintic curve. Show that

(a) if C has genus 2, it must lie on a quadric surface;

(b) if C has genus 1, it cannot lie on a quadric surface; and

(c) if C has genus 0, it may or may not lie on a quadric surface (that is, some rational quintic curves do lie on quadrics and some don't).

Exercise 1.92. Let $C \subset \mathbb{P}^3$ be a smooth quintic curve. Show that C lies on a quadric surface Q and a cubic surface S with intersection $Q \cap S$ consisting of the union of C and a line.

Exercise 1.93. Use the result of Exercise 1.94—showing that a smooth quintic curve of genus 2 is linked to a line in the complete intersection of a quadric and a cubic—to find the dimension of the subset of the Hilbert scheme corresponding to smooth curves of degree 5 and genus 2.

Exercise 1.94. Let $C \subset \mathbb{P}^3$ be a smooth quintic curve. Show that C lies on a quadric surface Q and a cubic surface S with intersection $Q \cap S$ consisting of the union of C and a line; and use this to calculate the dimension of the open subset of the Hilbert scheme parametrizing such curves.

Exercise 1.95. Use the Poincaré-Hopf Theorem to compute the topological Euler characteristic of a smooth variety $Y = X_1 \cap X_2 \subset \mathbb{P}^n$ where X_i is a hypersurface of degree d_i.

2

Introduction to Grassmannians and Lines in $\mathbb{P}^3$

Keynote Questions

(a) Given four general lines $L_1, \ldots, L_4 \subset \mathbb{P}^3$, how many lines will meet all four?

(b) If C and $C' \subset \mathbb{P}^3$ are two general twisted cubic curves, how many lines will meet each twice?

(c) If $Q_1, \ldots, Q_4 \subset \mathbb{P}^3$ are four general quadric surfaces, how many lines are tangent to all four?

2.1 Introduction to Grassmann Varieties

A *Grassmann variety*, or *Grassmannian*, is a projective variety whose closed points correspond to the vector subspaces of a certain dimension in a given vector space. This chapter is devoted to describing some features of the geometry on Grassmannian varieties, including their Chow rings. Along with projective spaces, of which they are a generalization, Grassmannians are the most fundamental objects in algebraic geometry and, as parameter spaces, they are of special interest in enumerative geometry. We will write $G(k, V)$ for the Grassmann variety of k-dimensional subspaces of V. The Grassmannians we treat in this Chapter are the first and simplest cases of the homogeneous spaces defined from algebraic groups. Such spaces play a

key role in all of representation theory. A good reference for this point of view is ****. What we do here is just the beginning.

A k-dimensional sub vector space of an n-dimensional vector space V is the same as a $(k-1)$-dimensional linear subspace of $\mathbb{P}(V) \cong \mathbb{P}^{n-1}$, so the Grassmannian $G(k, V)$ could also be thought of as parametrizing $(k-1)$-dimensional subspaces of $\mathbb{P}V$. We will write the Grassmannian $G(k, V)$ as $\mathbb{G}(k-1, \mathbb{P}V)$ when we wish to think of it this way. When there's no need to specify the vector space V but only its dimension, say n, we'll write simply $G(k, n)$ or $\mathbb{G}(k-1, n-1)$. Note also that there is a natural identification

$$G(k, V) = G(n-k, V^*)$$

sending a k-dimensional subspace $\Lambda \subset V$ to its annihilator $\Lambda^\perp \subset V^*$.

After establishing some basic ideas, we will study the geometry of the Grassmannian $\mathbb{G}(1, 3)$ of lines in $\mathbb{P}^3$, applying it to questions such as the three Key Questions above; in the following chapter we'll turn to the Grassmannians $\mathbb{G}(k, n)$ in general. Our treatment will be rather brief. For more results and more details, the reader may consult the books of Harris [1992] (for basic geometry of the Grassmannian); **?**] for the basics of the Schubert calculus and Fulton [1997] for combinatorial formulas, as well as the classic treatment in the second volume of Hodge and Pedoe [1952]. Further references will be given in the text.

There are two points of potential confusion in the notation. First, if $\Lambda \subset V$ is a k-dimensional vector subspace of an n-dimensional vector space V, we will often use the same symbol Λ to denote the corresponding point in $G = G(k, V)$. When we need to make the distinction explicit, we'll write $[\Lambda] \in G$ for the point corresponding to the plane $\Lambda \subset V$. Second, when we consider the Grassmannian $G = \mathbb{G}(k, \mathbb{P}V)$ we will sometimes need to work with the corresponding vector subspaces of V. In these circumstances, if $\Lambda \subset \mathbb{P}V$ is a k-plane, we'll write $\tilde{\Lambda}$ for the corresponding $(k+1)$-dimensional vector subspace of V.

2.1.1 The Plücker embedding

As a set, we take the Grassmann variety $G = G(k, V)$ to be the set of k-dimensional vector subspaces of the vector space V. We give this set the structure of a projective variety by giving an inclusion in a projective space, called the *Plücker embedding*, and showing that the image is the zero locus of a certain collection of homogeneous polynomials. We shall soon see that this embedding is fundamental: it is given by the complete linear series of sections of the unique ample generator of the Picard group of G; in more elementary terms, every map of a Grassmannian to projective space is the Plücker embedding followed by some Veronese re-embedding followed by a projection.

To do this, we associate to a k-dimensional subspace $\Lambda \subset V$ the one-dimensional subspace

$$\wedge^k \Lambda \subset \wedge^k V;$$

that is, if Λ has basis $v_1, \ldots, v_k$, we associate to it the point $[v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\wedge^k V)$. Concretely, if we take $V = \mathbb{C}^n$, we may represent Λ as the row space of a $k \times n$ matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & \cdots & a_{k,n} \end{pmatrix}.$$

If we choose a basis $\{e_1, \ldots, e_n\}$ for V, then a basis for $\wedge^k V$ is given by the set of products

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \le i_1 < \cdots < i_k \le n},$$

and if $v_1, \ldots, v_k$ is a basis for Λ then we may write a nonzero element of $\wedge^k \Lambda$ in the form

$$v_1 \wedge \cdots \wedge v_k = \sum_{1 \le i_1 < \cdots < i_k \le n} p_{i_1,\ldots,i_k} e_{i_1} \wedge \cdots \wedge e_{i_k};$$

(This is unique up to scalars.) Here the scalar $p_{i_1,\ldots,i_k}$ is the determinant of the submatrix (that is, *minor*) of A made from columns $i_1, \ldots, i_k$. These $p_{i_1,\ldots,i_k}$ are called the *Plücker coordinates* of Λ.

While the matrix A is hardly unique—we can multiply on the left by any invertible $k \times k$ matrix Ω without changing the row space—the collection of $k \times k$ minors of A, viewed as a vector in $\mathbb{C}^{\binom{n}{k}}$, is well-defined up to scalars: multiplying by Ω simply multiplies each such minor by $\det(\Omega)$. Conversely, if another matrix A' has the same row space, then we can write $A' = \Omega A$ for some invertible $k \times k$ matrix Ω. This shows that the "Plücker embedding" is a well-defined map of sets from $G(k, n)$ to $\mathbb{P}^{\binom{n}{k}-1}$. On the other hand, it is not hard to show that if $v_1, \ldots, v_k$ are independent vectors in V, then a vector v annihilates $v_1 \wedge \cdots \wedge v_k$ in the exterior algebra if and only if v is in the span of $v_1, \ldots, v_k$. This proves that the "Plücker embedding" is one-to-one as a map of sets from $G(k, n)$ to $\mathbb{P}^{\binom{n}{k}-1}$.

A quick-and-dirty way to see that the image $G \hookrightarrow \mathbb{P}(\wedge^k V)$ of the Plücker embedding—the locus of vectors $\eta \in \wedge^k V$ that are expressible as a wedge product $v_1 \wedge \cdots \wedge v_k$ of k vectors $v_i \in V$—is an algebraic set is to use the ring structure of the exterior algebra $\wedge V$. Writing out an element $\eta \in \wedge^k V$ in coordinates as above, we see that $e_i \wedge \eta = 0$ if and only if η can be written as $e_i \wedge \eta'$ for some $\eta' \in \wedge^{k-1} V$. Since there is nothing special about the vector $e_i \in V$ we could replace it by any nonzero element $v \in V$. Repeating this idea, we see that an element $0 \neq \eta \in \wedge^k V$ can be written in the form

$v_1 \wedge \cdots \wedge v_k$ for some (necessarily independent) $v_1, \ldots, v_k \in V$ if and only if the kernel of the multiplication map

$$V \xrightarrow{-\wedge\eta} \wedge^{k+1} V$$

has dimension at least k. That is, the image of the Plücker embedding is

$$G = \{[\eta] \mid \eta \in \wedge^k V \text{ and } \operatorname{rank}(V \xrightarrow{-\wedge\eta} \wedge^{k+1} V) \leq n-k\}.$$

and this is the zero locus of the homogeneous polynomials of degree $n-k+1$ on $\wedge^k V$ that are the $n-k+1$-order minors of the map $V \xrightarrow{-\wedge\eta} \wedge^{k+1} V$, written out as a matrix.

Once we know that G is an algebraic set, it follows that G is a variety: Its ideal is the kernel of the map of polynomial rings

$$K[p_{i_1,\ldots,i_k}]_{1\leq i_1<\cdots<i_k\leq n} \to K[x_{i,j}]_{\substack{1\leq i\leq k\\ 1\leq j\leq n}}$$

sending $p_{i_1,\ldots,i_k}$ to the corresponding Plücker coordinate of the generic matrix $(x_{i,j})$, and is thus prime.

In fact, the generators of the ideal of homogeneous forms vanishing on $G \subset \mathbb{P}(\wedge^k V)$ are quadratic polynomials, known as the *Plücker relations*. We will be able to describe these quadratic polynomials explicitly following Proposition 2.2; for fuller accounts (including a proof that they do indeed generate the homogeneous ideal of $G \subset \mathbb{P}(\wedge^k V)$) and some of their beautiful combinatorial structure, we refer the reader to de Concini et al. [1980] Section 2 or Fulton [1997] Section 9.1.

From here on, we will view $G(k, V)$ as endowed with the structure of a projective variety via the Plücker embedding. As will follow from the description of its covering by affine spaces in the following subsection, it's a smooth variety of dimension $k(n-k)$. The first of these statements follows in any case from the fact that $GL(V)$ acts transitively on it by linear transformations of the projective space $\mathbb{P}(\wedge^k V)$.

Example 2.1. The first example of a Grassmannian other than projective space is the Grassmannian $G(2,4)$. Let V be a 4-dimensional vector space, and consider the Plücker embedding of $G(2,V) = G(2,4) = \mathbb{G}(1,3)$ in $\mathbb{P} \wedge^2 V \cong \mathbb{P}^5$. Since $\dim G(2,4) = 4$, this will be a hypersurface. From the discussion above we know that the equation of $G(2,4)$ in this embedding is a polynomial relation among the minors $p_{i,j}$ of a generic 2×4 matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{pmatrix}.$$

One way to obtain this relation is to note that the determinant of the 4×4 matrix with repeated rows

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{pmatrix}$$

must be zero. Expanding this determinant as a sum of products of minors of the first two rows and of the last two rows, all of which are Plücker coordinates, we obtain

$$p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{1,4}p_{2,3} = 0. \tag{2.1}$$

As this is an irreducible polynomial, it generates the homogeneous ideal of $G(2,4) \subset \mathbb{P}^5$, which is thus a smooth quadric.

In fact, for any n, the ideal of the Grassmannian of lines $\mathbb{G}(1,n)$ is cut out by quadratic polynomials in the Plücker coordinates similar to the polynomial (2.1) above. More precisely, if $e_1, \ldots, e_n$ is a basis of V and $\eta = \sum_{1 \le a < b \le n+1} p_{a,b} e_a \wedge e_b \in \wedge^2 V$, then the polynomials

$$\{g_{a,b,c,d} := p_{a,b}p_{c,d} - p_{a,c}p_{b,d} + p_{a,d}p_{b,c} = 0 \mid 1 \le a < b < c < d \le n+1\}$$

minimally generate the ideal of the Grassmannian. These are the *Plücker relations* in the special case of the Grassmannian $\mathbb{G}(1,n)$ of lines in $\mathbb{P}^n$. We'll describe the Plücker relations in general following Proposition 2.2.

Another way to characterize the collection of polynomials $\{g_{a,b,c,d}\}$, in characteristic not equal to 2, is that they are tantamount to the equation $\eta^2 = 0 \in \wedge^4 V$. The relation between these equations in characteristic 2 is described in Exercise 2.10.

Exercises 2.11-2.19 describe a number of aspects of the projective geometry of the Grassmannian in the Plücker embedding.

2.1.2 Covering by affine spaces; local coordinates

Like a projective space, a Grassmann variety $G = G(k,V)$ can be covered by Zariski open subsets isomorphic to affine space. To see this, fix an $(n-k)$-dimensional subspace $\Gamma \subset V$, and let U_Γ be the subset of k-planes complementary to Γ

$$U_\Gamma = \{\Lambda \subset V : \Lambda \cap \Gamma = \{0\}\} \subset G.$$

This is a Zariski open subset of G: in fact, if we take $\eta \in \wedge^{n-k}\Gamma$ to be any generator—that is, take $w_1, \ldots, w_{n-k}$ any basis for Γ and set $\eta = w_1 \wedge \cdots \wedge w_{n-k}$—then we have

$$U_\Gamma = \{[\omega] \in G \subset \mathbb{P} \wedge^k V : \omega \wedge \eta \neq 0\}$$

from which we see that U_Γ *is the complement of a hyperplane section of* G *under the Plücker embedding.* Note: not all hyperplane sections of G are the same; see Exercise 2.20 for more on this.

We claim now that the open set $U_\Gamma \subset G(k,n)$ is isomorphic to affine space $\mathbb{A}^{k(n-k)}$. To see this, we first choose an arbitrary point $[\Omega] \in U_\Gamma$ that will play the role of the origin; that is, fix a k-plane $\Omega \subset V$ complementary to Γ, so that we have a direct-sum decomposition $V = \Omega \oplus \Gamma$. Any k-dimensional subspace $\Lambda \subset V$ complementary to Γ projects to Γ modulo Ω—call this map π_Γ—and projects isomorphically to Ω mod Γ—call this map π_Ω. Thus Λ is the graph of the linear map

$$\varphi : \Omega \xrightarrow{\pi_\Omega^{-1}} \Lambda \subset V = \Omega \oplus \Gamma \xrightarrow{\pi_\Gamma} \Gamma$$

Conversely, the graph of any map $\varphi : \Omega \to \Gamma$ is a subspace $\Lambda \subset \Omega \oplus \Gamma = V$ complementary to Γ. These two correspondences establish a bijection

$$U_\Gamma \cong \mathrm{Hom}(\Omega, \Gamma) \cong \mathbb{A}^{k(n-k)}.$$

To make this explicit, suppose we choose a basis for V consisting of a basis $e_1, \ldots, e_k$ for Ω followed by a basis $e_{k+1}, \ldots, e_n$ for Γ. If $\Lambda \in U_\Gamma$ is a k-plane then the projections to Λ of the basis vectors $e_1, \ldots, e_k$, form a basis for Λ. Thus Λ has a matrix representative whose first $k \times k$ submatrix is the identity: that is,

$$B = \begin{pmatrix} 1 & 0 & \ldots & 0 & a_{1,1} & a_{1,2} & \ldots & a_{1,n-k} \\ 0 & 1 & \ldots & 0 & a_{2,1} & a_{2,2} & \ldots & a_{2,n-k} \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & a_{k,1} & a_{k,2} & \ldots & a_{k,n-k} \end{pmatrix}.$$

Here B is the matrix $C_{\Omega,\Gamma}^{-1} A$, where $C_{\Omega,\Gamma}$ is the projection of Λ to Ω with kernel Γ, so (given the basis for V) this matrix representation is unique. The bijection defined above sends $\Lambda \in U_\Gamma$ to the linear transformation $\Omega \to \Gamma$ given by the transpose of the matrix $A = (a_{i,j})$.

If we start with any representation of Λ as the span of the rows of a $k \times (n-k)$ matrix B' with respect to the given basis of V, then the Plücker coordinate $p_{1,2,\ldots,k}$, which is the determinant of the submatrix consisting of the first k columns of B' will be nonzero. Multiplying B on the left by the inverse of this submatrix gives us back the matrix B. In particular, the entries $a_{i,j}$ of B are ratios of Plücker coordinates of Λ: specifically, $a_{i,j}$ is the $(1, \ldots, i-1, \hat{i}, i+1, \ldots, k, k+j)$-th minor divided by the $(1, 2, \ldots, k)$-th minor—so each $a_{i,j}$ is a regular function on U_Γ, and the bijection $U_\Gamma \cong \mathbb{A}^{k(n-k)}$ is a biregular isomorphism.

It is useful to note that, more generally, the Plücker coordinates of Λ (divided by $p_{1,2,\ldots,k}(\Lambda)$) are precisely the determinants of submatrices (of all sizes) of A. To express the result, suppose that I and J are sets of indices.

Write B_I^J for the minor of the matrix B involving rows with indices in I and columns with indices in J. Write I' for the complement (in the set of row indices) of I. With this notation, each $t \times t$ minor of A can be expressed as $B_{I'}^J$ for some set of k column indices J all of which are $> k$, and some set of $k - t$ row indices I.

Proposition 2.2. *With notation as above, suppose that $I = \{i_1, \ldots, i_{k-t}\}$ are row indices and $J = \{j_1, \ldots, j_t\}$ are column indices with each $j_i > k$. We have*

$$\frac{p_{I \cup J}(\Lambda)}{p_{1,\ldots,k}(\Lambda)} = \det B_{I'}^J.$$

Proof. The ratio on the left is the (signed) determinant of B involving the columns with indices in I and J. Expanding this minor in terms of the $s \times s$ minors in columns I of the identity matrix that is the left-most part of B and the $t \times t$ minors of B in columns J, we find that all but the given term are zero. □

Once we have established Proposition 2.2, we can describe readily the *Plücker relations*, the quadratic polynomials in the Plücker coordinates that cut out $G(k, V) \subset \mathbb{P}(\wedge^k V)$. We have seen that the determinant of any square submatrix of A is a ratio of two Plücker coordinates with denominator $p_{1,\ldots,k}(\Lambda)$; we can take any such submatrix of A and expand its determinant by cofactors along any row or column to arrive at a homogeneous quadratic polynomial in the p_I satisfied identically in U_Γ and hence in all of $G(k, V)$. Again, for a fuller account of these relations we refer the reader to de Concini et al. [1980] Section 2 or Fulton [1997] Section 9.1.

2.1.3 Universal sub and quotient bundles

Let V be an n-dimensional vector space, $G = G(k, V)$ the Grassmannian of k-planes in V, and $\overline{V} = V \otimes \mathcal{O}_G$ the trivial vector bundle of rank n on G whose fiber at every point is just the vector space V. We will define a sub-vector bundle $S \subset \overline{V}$ of rank k whose fiber at a point $[\Lambda] \in G$, the fiber of S is the subspace Λ itself; that is,

$$S_{[\Lambda]} = \Lambda \subset V.$$

S is called the *universal sub-bundle* on G; the quotient $Q = \overline{V}/S$ is called the *universal quotient bundle*. In case $k = 1$—that is, $G = \mathbb{P}V \cong \mathbb{P}^{n-1}$—the universal sub-bundle S is the line bundle $\mathcal{O}_{\mathbb{P}V}(-1)$; similarly, in case $k = n - 1$, so that $G = \mathbb{P}V^*$, the universal quotient bundle Q is the line bundle $\mathcal{O}_{\mathbb{P}V^*}(1)$.

These constructions are a good example of the "naturality principle"; who could doubt that they describe vector bundles? Nevertheless, it is

important to see what would go into a formal demonstration. It is enough to do this just for the universal sub-bundle S, since the universal quotient bundle can be described in terms of it and the trivial bundle, and we will do this in the proof below. The second part of the result is the reason that we refer to S as the "universal" sub-bundle.

Theorem 2.3. *There is a sub-bundle $S \subset \overline{V}$ on $G(k, V)$ whose fiber over a point $[\Lambda] \in G(k, V)$ is the subspace $\Lambda \subset V$. If X is any scheme then the morphisms from X to $G(k, V)$ are in one-to-one correspondence with rank k sub-bundles $\mathcal{F} \subset V \times X$.*

Proof. Let $S \subset V \times G(k, V)$ be the subset consisting of points $(v, [\Lambda])$ such that $v \in \Lambda$. (This subset E is sometimes called the *incidence correspondence* or the *universal subspace*. But the reader should beware: the term "incidence correspondence" is used in algebraic geometry—and in this book—for many similar but different constructions.)

It is easy to see that E is an algebraic subset (actually a smooth subvariety, as will follow from what is proven below): if Λ is the row space of the matrix A as in Section 2.1.2 then the condition $v \in \Lambda$ is equivalent to the vanishing of the $(k+1)^{\text{st}}$-order minors of the matrix obtained from A' by adjoining v as the $(k+1)^{\text{st}}$ row. These minors can be expressed (by expanding along the new row of A' as bilinear functions in the coordinates of v and the Plücker coordinates, proving that S is an algebraic subset.

Now pick a subspace $\Gamma \subset V$ of dimension $n-k$ and consider the preimage of $U_\Gamma \subset G(k, V)$. Choosing a complement Ω to Γ as before, we can identify U_Γ with $\text{Hom}(\Omega, \Gamma)$. Moreover if $\Lambda \in U_\Gamma$ then the projection $\beta_{\Omega,\Gamma} : V \to \Omega$ with kernel Γ takes $S_{[\Lambda]} = \Lambda \subset V$ isomorphically to Ω. In other words, this projection gives an isomorphism S_{U_Γ} to the trivial bundle $\Omega \times U_\Gamma$. This proves that S is actually a vector bundle, as claimed.

****add proof of the universal property: assume the version for projective space. Make the map to the ambient projective space via the Plücker coordinates of $\mathcal{F} \subset V \times X$ and note that the image is inside G, since we know the equations as the relations satisfied by indeterminate Plücker coords.****

□

The universal bundles on the Grassmannian G are of fundamental importance in studying the geometry of the Grassmannian, and indeed in the study of vector bundles in general: in the topological or $\mathcal{C}^\infty$ categories it's the case that *any vector bundle E on a space X is the pullback of a universal bundle via some map $\varphi : X \to G$ to a Grassmannian*; in the algebraic geometry setting, this is true for all bundles generated by global sections.

2.1.4 The Tangent Bundle of the Grassmann Variety

As with any manifold, knowledge of the tangent bundle of the Grassmannian is the key to its geometry. It turns out that this can be expressed in terms of the tautological bundles S and Q:

Theorem 2.4. *The tangent bundle T_G to the Grassmannian $G = G(k, V)$ is isomorphic to $\mathrm{Hom}_G(S, Q)$, where S and $Q = (V \times G)/S$ are the universal sub and quotient bundles.*

Proof. Consider the open affine set

$$U_\Gamma = \{[\Lambda] \mid \Lambda \cap \Gamma = \{0\}\} \subset G,$$

described in Section 2.1.2, where Γ is a subspace of V of dimension $n - k$. Fixing a point $[\Omega] \in U_\Gamma$ and decomposing $V = \Omega \oplus \Gamma$ we get an identification of U_Γ with the vector space $\mathrm{Hom}(\Omega, \Gamma)$ under which the point $[\Omega]$ goes to the linear transformation 0. In particular, the tangent bundle T_G restricted to U_Γ is the trivial bundle, and the fiber over $[\Omega]$ is $\mathrm{Hom}(\Omega, \Gamma)$.

The bundle $S|_{U_\Gamma}$ is isomorphic to the trivial bundle $\Omega \times U_\Gamma$ by the composite map

$$S|_{U_\Gamma} \to V \times U_\Gamma \to V/\Gamma \times U_\Gamma = \Omega \times U_\Gamma,$$

and the bundle $Q|_{U_\Gamma}$ is isomorphic to the trivial bundle $\Gamma \times U_\Gamma$ via the tautological projection $V \times G \to Q$. This gives an identification of fibers, depending on Γ:

$$(T_G)_{[U]} = \mathrm{Hom}(\Omega, \Gamma) = \mathrm{Hom}(S_{[U]}, Q_{[U]}).$$

To prove that these identifications extend to an isomorpshism $T_G \cong \mathrm{Hom}_G(S, Q)$ we must check that the gluing map for T_G and that for $\mathrm{Hom}_G(S, Q)$ on an intersection $U = U_\Gamma \cap U_{\Gamma'}$ containing the point $[\Omega]$ agree on the fiber over $[\Omega]$ (and thus agree as maps of bundles). We may regard $U \subset U_\Gamma = \mathrm{Hom}(\Omega, \Gamma)$ as the set of linear transformations whose graphs do not meet Γ', and this representation is related to the representation of $U \subset U_{\Gamma'}$ by the isomorphisms $\Gamma \to V/\Omega \leftarrow \Gamma'$. The gluing of $T_G|_{U_\Gamma}$ to $T_G|_{U_{\Gamma'}}$ along this set is by the differential of the linear transformation

$$\mathrm{Hom}(\Omega, \Gamma) \to \mathrm{Hom}(\Omega, V/\Omega) \leftarrow \mathrm{Hom}(\Omega, \Gamma')$$

induced by these isomorphisms. Of course the differential of a linear transformation is the same linear transformation. The same isomorphisms give the gluing of the bundle $\mathrm{Hom}_G(S, Q)$ at the point $[\Omega]$, so we are done. □

One special case of the isomorphism of Theorem 2.4 that is of particular interest—both because it can be readily visualized, and because it's invoked so frequently—is the case of projective space $\mathbb{P}^n = \mathbb{G}(0, n)$ itself. Here

Theorem 2.4 gives rise to the *Euler sequence*; we'll say a few words about that here.

To start with, let V be an $(n+1)$-dimensional vector space, and $\mathbb{P}^n = \mathbb{P}V$ its projectivization; we consider the quotient map

$$q : U = V \setminus \{0\} \to \mathbb{P}^n$$

sending a nonzero vector $v \in V$ to the corresponding point $p = [v] \in \mathbb{P}^n$. The tangent space to U at v is the same as the tangent space to V at v, which is to say the vector space V itself, and in these terms the kernel of the differential

$$dq_v : T_v U \to T_p \mathbb{P}^n$$

is just the one-dimensional subspace $\tilde{p} = \langle v \rangle \subset V$ spanned by v. Thus dq_v induces an isomorphism

$$V/\tilde{p} \xrightarrow{\cong} T_p \mathbb{P}^n.$$

****draw and insert picture****

This does not, however, give a natural identification of the vector spaces $V/\tilde{p}$ and $T_p\mathbb{P}^n$! Even though these vector spaces depend only on the point $p \in \mathbb{P}^n$, the isomorphism dq_v between them does depend on the choice of the vector v. Indeed, if λ is any nonzero scalar, the differential $dq_{\lambda v}$ is equal to dq_v divided by λ. In other words, if $l : \langle v \rangle \to K$ is any linear functional, the map $l(v) \cdot dq_v$ actually *is* independent of the choice of v, and so we have a natural identification

$$\langle v \rangle^* \otimes V/\langle v \rangle \xrightarrow{\cong} T_{[v]} \mathbb{P}^n,$$

and this is the identification asserted in Theorem 2.4, in the case of projective space.

To put it another way, in terms of coordinates $x_0, \ldots, x_n$ on V, a constant vector field $\partial/\partial x_i$ on V does not give rise to a vector field on $\mathbb{P}V$, but the vector field

$$w(x) = x_j \frac{\partial}{\partial x_i}$$

does. This gives us a map

$$\mathcal{O}_{\mathbb{P}^n}(1) \otimes V \to T_{\mathbb{P}V},$$

whose kernel is the Euler vector field

$$e(x) = \sum x_i \frac{\partial}{\partial x_i},$$

or in other words, an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}V} \to \mathcal{O}_{\mathbb{P}^n}(1) \otimes V \to T_{\mathbb{P}V} \to 0,$$

called the *Euler seuence*. To relate this to the isomorphism given in Theorem 2.4, start with the universal sequence on $\mathbb{P}V$:

$$0 \to S \to \mathcal{O}_{\mathbb{P}V} \otimes V \to Q \to 0.$$

Now tensor with the line bundle $S^* = \mathcal{O}_{\mathbb{P}V}(1)$; since $S \otimes S^* \cong \mathcal{O}_{\mathbb{P}V}$, we arrive at the sequence

$$0 \to \mathcal{O}_{\mathbb{P}V} \to \mathcal{O}_{\mathbb{P}^n}(1) \otimes V \to S^* \otimes Q \to 0.$$

Applying the isomorphism of Theorem 2.4 to the last term, this is the Euler sequence.

We can use the universal property of the Grassmannian to make the correspondence between $\mathrm{Hom}(S, Q)$ and the tangent spaces of the Grassmannian asserted in Theorem 2.4 clearer; this will be useful when we come to computing intersections, for which we will need to prove the transversality of certain subvarieties.

If $\mathcal{O}_{X,x}$ is the local ring of a smooth variety X at a point x, with corresponding maximal ideal $\mathfrak{m}_{X,x}$ and residue field $F = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$, then the tangent space $T_{X,x}$ to X at x can be identified with the vector space of linear transformations $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 \to F$.

An immediate computation shows that this is again the same as the set of F-algebra homomorphisms $\mathcal{O}_{X,x} \to T := F[\epsilon]/(\epsilon^2)$. (The same thing works for arbitrary schemes and the Zariski tangent space.) This identifies the tangent space to X at x with the set of morphisms

$$\mathcal{T} := \mathrm{Spec}(F[\epsilon]/(\epsilon^2) \to X$$

sending the closed point to x. When $X = G(k, V)$, and $x = [\Lambda]$, Theorem 2.3 shows that this is the same as the set of vector bundles of rank k over $\mathcal{T}$ with constant rank k maps to the trivial bundle V that specialize to the inclusion $\Lambda \subset V$ modulo ϵ. Since $\mathcal{T}$ is affine, a vector bundle over $\mathcal{T}$ is the same as a locally free module over $F[\epsilon]/(\epsilon^2)$. Since this ring is local, Nakayama's Lemma shows that such a module is free (see for example Eisenbud [1995] Exercise 4.11). Thus only the inclusion $\Lambda \times \mathcal{T} \to V \times \mathcal{T}$ varies.

Putting this together, we get a new way to look at the identification of the tangent spaces to the Grassmannian.

Proposition 2.5. *Let $\Lambda \subset V$ be a k-dimensional subspace, and let $\varphi \in Hom(\Lambda, V/\Lambda)$ be a homomorphism. As an element of the tangent space to the Grassmannian $G(k, V)$ at the point $[\Lambda]$, φ corresponds to the sub free module*

$$\Lambda \otimes F[\epsilon]/(\epsilon^2) \to V \otimes F[\epsilon]/(\epsilon^2) \quad v \otimes 1 \mapsto v \otimes 1 + \varphi'(v) \otimes \epsilon.$$

where $\varphi' : \Lambda \to V$ is any lifting of φ.

Proof. Any map $\Lambda \times \mathcal{T} \to V \times \mathcal{T}$ that reduces to the inclusion modulo ϵ has the form $v \otimes 1 \mapsto v \otimes 1 + \varphi'(v) \otimes \epsilon$ for some φ'. If we work in the affine coordinates corresponding to a subspace Γ complementary to Λ and use the splitting $V = \Lambda \oplus \Omega$, then the point $\Lambda \subset V$ corresponds to the matrix

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}.$$

In this matrix representation, φ is represented by the last $n-k$ columns of φ', and taking a different lifting of φ corresponds to making a different choice of the first $k \times k$ block of φ'.

We can do row operations to clear all the ϵ terms from the first $k \times k$ block, adding a multiple of ϵ times certain rows to other rows. This corresponds to composing with to an automorphism of $\Lambda \times \mathcal{T}$, and thus does not change the image of $\Lambda \times \mathcal{T} \to V \times \mathcal{T}$. Since we add after multiplying by ϵ, this does not change the block representing φ. Thus we may assume that the first $k \times k$ block of φ' is zero; equivalently, the first $k \times k$ block corresponding to the map $\Lambda \times \mathcal{T} \to V \times \mathcal{T}$ is the identity. □

See Eisenbud and Harris [2000]Chapter *** for another account of this phenomenon.

Kleiman's Theorem does not include the case of the Grassmannian in characteristic > 0; and in fact it fails there in certain geometrically interesting cases. **** put in Kleiman's example and Joel Roberts' example, referenced in his review of Kleiman's paper****. Nevertheless, the conclusions it gives for intersections of Schubert cycles are correct in all characteristics (and will be proven by a different method.)

2.2 The Chow Ring of $\mathbb{G}(1,3)$

We start our discussion of intersection theory on Grassmannians with the simplest case beyond projective space, the variety $G = \mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$. This will be a template for the analysis of Grassmannians in general, with the additional feature that we can at least delude ourselves into thinking we can visualize what's going on. Accordingly, we'll devote the remainder of this chapter to the special case of $G = \mathbb{G}(1,3)$ (and will in particular write simply G for $\mathbb{G}(1,3)$ henceforth); the case of general Grassmannians will be the subject of the next chapter.

Note that, as we observed above, G is realized via the Plücker embedding as the nonsingular quadric in $\mathbb{P}(\wedge^2 V) \cong \mathbb{P}^5$ given by the equation (2.1).

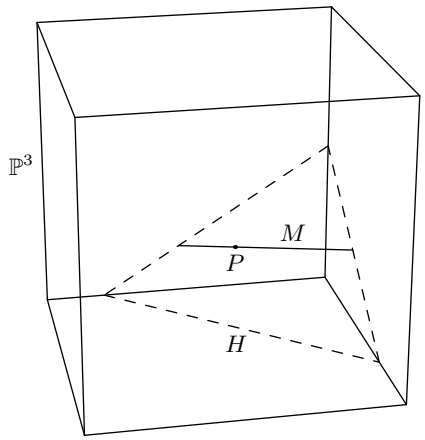


FIGURE 2.1. A complete flag $p \subset M \subset H$ in $\mathbb{P}^3$

2.2.1 Schubert Cycles in $G = \mathbb{G}(1,3)$

****Notation: we keep Λ for the variable line in $\mathbb{P}^3$; use $p \in L \subset H \subset \mathbb{P}^3$ for the flag. We will have to ask Silvio to change the figures accordingly.****

To start, we fix a *flag* $\mathcal{V}$ on $\mathbb{P}^3$; that is, a choice of a point $p \in \mathbb{P}^3$, a line $L \subset \mathbb{P}^3$ containing p, and a plane $H \subset \mathbb{P}^3$ containing L (Figure 2.1).

We can give a decomposition of G as a disjoint union of locally closed subsets by considering the loci of lines $\Lambda \in G$ having specified dimension of intersection with each of the subspaces p, L and H. These are called *Schubert cells*, and their closures, or (often) the classes of their closures in the Chow groups, are called *Schubert cycles*. As we shall see, the Schubert cells form an affine stratification of G, and it follows from Theorem **??** that the Schubert cycles generate the Chow group $A_*(G)$. Using intersection theory we will be able to show that in fact $A_*(G)$ is a free $\mathbb{Z}$-module having the Schubert cycles as free generators. In the following chapter we will see that the same situation is repeated for all the Grassmann varieties.

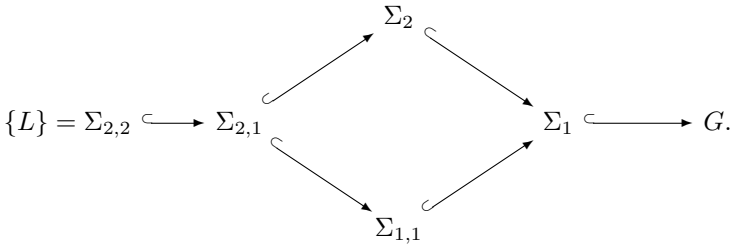
We begin by naming and describing the Schubert cycles:

$$\begin{aligned}
\Sigma_{0,0} &= G \\
\Sigma_{1,0} &= \{\Lambda : \Lambda \cap L \neq \varnothing\}; \\
\Sigma_{2,0} &= \{\Lambda : p \in \Lambda\}; \\
\Sigma_{1,1} &= \{\Lambda : \Lambda \subset H\}; \\
\Sigma_{2,1} &= \{\Lambda : p \in \Lambda \subset H\}; \text{ and} \\
\Sigma_{2,2} &= \{\Lambda : \Lambda = L\}.
\end{aligned}$$

Thus $\Sigma_{a,b}$ denotes the set of lines meeting the $(2-a)$-dimensional plane of $\mathcal{V}$ in a point, and the $(3-b)$-dimensional plane of $\mathcal{V}$ in a line. The indexing may seem peculiar at first, but there are good reasons for it; for example, as we'll see in the next Chapter when we introduce Schubert cycles and classes on general Grassmannians, it behaves well with respect to the pullback maps associated to the inclusions of sub-Grassmannians $G(k,n) \hookrightarrow G(k,n+1)$ and $G(k,n) \hookrightarrow G(k+1,n+1)$. One convenient aspect of the indexes is that the codimension of $\Sigma_{a,b}$ is $a+b$, as we will soon see.

We often drop the second index when it is zero, writing for example Σ_1 instead of $\Sigma_{1,0}$. When the choice of flag is relevant, we'll sometimes indicate the dependence by writing $\Sigma(\mathcal{V})$, or simply note the dependence on the relevant flag elements by writing, for example, $\Sigma_1(L)$ for the cycle of lines Λ meeting L.

It is easy to see that there are inclusions:



For each index (a,b) we define the Schubert cell $\tilde{\Sigma}_{a,b}$ to be the complement in $\Sigma_{a,b}$ of the union of all the other Schubert cycles properly contained in $\Sigma_{a,b}$. To show that the $\Sigma_{a,b}$ form an affine stratification, it suffices to show that each $\tilde{\Sigma}_{a,b}$ is isomorphic to an affine space. We will do the most complicated case, leaving the others for the reader (the general case of a Schubert cycle in $\mathbb{G}(k,n)$ is done in Proposition 3.1).

Example 2.6. We will show that the set

$$\begin{aligned}\tilde{\Sigma}_1 &= \Sigma_1 \setminus (\Sigma_2 \cup \Sigma_{1,1}) \\ &= \{\Lambda : \Lambda \cap L \neq \varnothing; \text{ but } p \notin \Lambda \text{ and } \Lambda \not\subset H\}\end{aligned}$$

is isomorphic to $\mathbb{A}^3$. Let H' be a general plane containing the point p but not containing the line L. Any line meeting L but not passing through p, and not contained in H, meets H' in a unique point contained in $H' \setminus (H' \cap H)$ (Figure 2.2). Thus we have maps

$$\tilde{\Sigma}_1 \to (L \setminus \{p\}) \cong \mathbb{A}^1 \quad \text{and} \quad \tilde{\Sigma}_1 \to (H' \setminus (H \cap H')) \cong \mathbb{A}^2$$

sending Λ to $\Lambda \cap L$ and $\Lambda \cap H'$ respectively. The product of these maps gives us an isomorphism

$$\tilde{\Sigma}_1 \cong \mathbb{A}^1 \times \mathbb{A}^2 = \mathbb{A}^3.$$

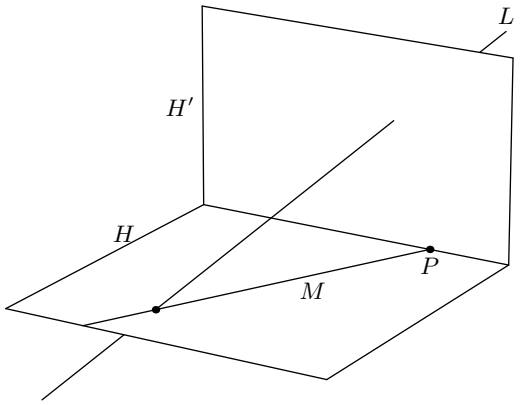


FIGURE 2.2. $L \mapsto (L \cap M, L \cap H')$ defines an isomorphism $\tilde{\Sigma}_1 \to \mathbb{A}^1 \times \mathbb{A}^2 = \mathbb{A}^3$.

Note that by Proposition 5.21, the class $[\Sigma_{a,b}] \in A^{a+b}(G)$ doesn't depend on the choice of flag, since any two flags differ by a transformation in GL_4; we'll denote this class by

$$\sigma_{a,b} = [\Sigma_{a,b}] \in A^{a+b}(G).$$

By ****, the groups $A^0(G)$ and $A^4(G)$ are both isomorphic to $\mathbb{Z}$, generated by the class $\sigma_{0,0}[G]$ and the class $\sigma_{2,2}$ of a point, respectively. In particular any two points in G are linearly equivalent.

2.2.2 Equations of the Schubert Cycles

Let V be a 4-dimensional vector space and $G = G(1, \mathbb{P}V)$. If we choose a basis $e_1, \ldots, e_4$ of V and a flag of subspaces

$$V_1 = \langle e_1 \rangle \subset V_2 = \langle e_1, e_2 \rangle \subset V_3 = \langle e_1, e_2, e_3 \rangle \subset V_4 = \langle e_1, e_2, e_3, e_4 \rangle = V$$

then the Schubert cycle $\Sigma_{a,b}$ consists of those 2-dimensional subspaces that are contained in V_{4-b} and meet V_{3-a} nontrivially. The ideal of this subvariety in the Plücker embedding is generated by a subset of the Plücker coordinates:

Proposition 2.7. *With coordinates as above, the ideal of*

$$\Sigma_{a,b} \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$$

is generated by the Plücker relation $g := p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{14}p_{2,3}$ *together with the coordinates* $p_{i,j}$ *such that* $i > 3 - a$ *or* $j > 4 - b$. *Specifically:*

- Σ_1 *is the hyperplane section* $p_{3,4} = 0 \subset G$; *that is, it's the cone over the nonsingular quadric* $\overline{g} = -p_{1,3}p_{2,4} + p_{14}p_{2,3}$ *in* $\mathbb{P}^3$;

- Σ_2 *is is the plane* $p_{2,3} = p_{2,4} = p_{3,4} = 0$.
- $\Sigma_{1,1}$ *is the plane* $p_{1,4} = p_{2,3} = p_{2,4} = 0$.
- $\Sigma_{2,1}$ *is the line* $p_{1,4} = p_{2,3} = p_{2,4} = p_{3,4} = 0$.
- $\Sigma_{2,2}$ *is the point* $p_{1,3} = p_{1,4} = p_{2,3} = p_{2,4} = p_{3,4} = 0$.

Proof. A subspace $L \in \Sigma_{a,b}$ has a basis whose first vector is in V_{3-a}, and therefore has its last $a+1$ coordinates zero; and whose second vector is in V_{4-b} and thus has its last b coordinates equal to zero. If B is the matrix whose rows are the coordinates of these two vectors, then $p_{i,j}$ is (up to sign) the determinant of the submatrix of B involving columns i, j. It follows that if $i > 3-a$ or $j > 4-b$ then $p_{i,j}(L) = 0$, so the given subsets of Plücker coordinates do vanish on the Schubert cycles as claimed.

To show that the ideals of the Schubert cycles are generated by the relation g and the given subsets, observe that each of the subsets is the ideal of the irreducible subvariety described in the Proposition, and these have the same dimensions as the Schubert cycles. For example, we know that $\dim \Sigma_{1,1} = \dim G(2,4) - (1+1) = 2$; and the ideal

$$(g, p_{1,4}, p_{2,3}, p_{2,4}) = (p_{1,4}, p_{2,3}, p_{2,4}) \subset K[p_{1,2}, \ldots, p_{3,4}]$$

is the entire ideal of a plane. □

2.2.3 Ring Structure

We can now determine the structure of the Chow ring of $G = \mathbb{G}(1,3)$ completely.

Theorem 2.8. *The six Schubert cycles* $\sigma_{a,b} \in A^{a+b}(G)$ *with* $0 \leq b \leq a \leq 2$ *freely generate* $A^*(G)$ *as a graded abelian group, and satisfy the multiplicative relations:*

$$\begin{aligned} \sigma_1^2 &= \sigma_{1,1} + \sigma_2; \\ \sigma_1\sigma_{1,1} &= \sigma_1\sigma_2 = \sigma_{2,1}; \\ \sigma_1\sigma_{2,1} &= \sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2}; \\ \sigma_{1,1}\sigma_2 &= 0. \end{aligned}$$

Since $A^m(G) = 0$ for $m > 4$, any product that would have degree > 4, such as $\sigma_2\sigma_{2,1}$, is actually zero.

Proof of Theorem 2.8. Except in one case, we will prove the formulas by considering the intersections of pairs of cycles, taking these with respect to generically situated flags $\mathcal{V}, \mathcal{V}'$. To simplify the notation we will henceforth write $\Sigma_{a,b}$ and $\Sigma'_{a,b}$ for $\Sigma_{a,b}(\mathcal{V})$ and $\Sigma'_{a,b}(\mathcal{V}')$, respectively.

We will use the fact that the Schubert cycles $\Sigma_{a,b}$ and $\Sigma'_{a',b'}$ are generically transverse. This follows from Kleiman's Theorem 5.20 in characteristic zero; to prove generic transversality in all characteristics, we need to know the tangent spaces to the Schubert cycles. We first give the proof in characteristic 0 using Kleiman's Theorem, and then show how to describe the tangent spaces to Schubert cycles and deduce the result in all characteristics. The determination of tangent spaces to Schubert cycles is useful and interesting in its own right.

We begin with the case of cycles of complementary dimension, starting with the intersection number of σ_2 with itself. First, by generic transversality we have

$$\sigma_2^2 = \#\big(\Sigma_2 \cap \Sigma'_2\big) \cdot \sigma_{2,2};$$

and since the intersection

$$\Sigma_2 \cap \Sigma'_2 = \{\Lambda : p \in \Lambda \text{ and } p' \in \Lambda\};$$

consists of one point (corresponding to the unique line $\Lambda = \overline{p,p'}$ through p and p'), we conclude that

$$\sigma_2^2 = \sigma_{2,2}.$$

Similarly, we have by transversality

$$\sigma_{1,1}\sigma_{1,1} = \#\big(\Sigma_{1,1} \cap \Sigma'_{1,1}\big) \cdot \sigma_{2,2};$$

since

$$\Sigma_{1,1} \cap \Sigma'_{1,1} = \{\Lambda : \Lambda \subset H \text{ and } \Lambda \subset H'\}$$

consists of the unique line $\Lambda = H \cap H'$, we conclude that

$$\sigma_{1,1}^2 = \sigma_{2,2}$$

as well. On the other hand, $\Sigma_2 = \{\Lambda : p \in \Lambda\}$ and $\Sigma'_{1,1} = \{\Lambda : \Lambda \subset H'\}$ will be disjoint, since $p \notin H'$, so that

$$\sigma_2\sigma_{1,1} = 0.$$

These three computations together show that $A^2(G)$, which we knew to be generated by σ_2 and $\sigma_{1,1}$, is in fact freely generated by them, since their intersection matrix is nonsingular.

Finally in complementary dimension, we have

$$\Sigma_1 \cap \Sigma'_{2,1} = \{\Lambda : \Lambda \cap L \neq \varnothing \text{ and } p' \in \Lambda \subset H'\}.$$

Since L will intersect H' in one point q, and any line Λ satisfying all the above conditions can only be the line $\overline{p',q}$ (Figure 2.3), this intersection is again a single point. Thus

$$\sigma_1\sigma_{2,1} = \sigma_{2,2}$$

as well.

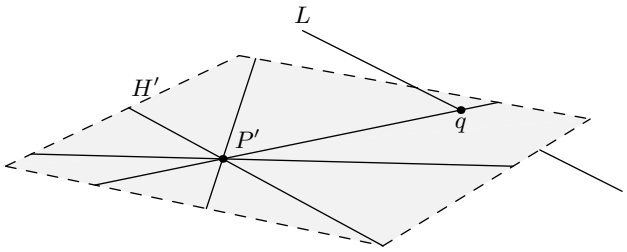


FIGURE 2.3. $\Sigma_1(L) \cap \Sigma'_{2,1}(p', H') = \{\overline{p', q}\}$

In particular we see that each of σ_1 and $\sigma_{2,1}$ generate a group isomorphic to $\mathbb{Z}$, and this completes the proof that the Schubert classes form a free basis of $A^*(G)$ as a group.

We now turn to the intersections of cycles whose codimensions sum to a number < 4. The intersection $\Sigma_1 \cap \Sigma'_2$ is the locus of lines Λ meeting L and containing the point p', which is to say the Schubert cycle $\Sigma_{2,1}$ with respect to a flag containing the point p' and the plane $\overline{p', L}$, so we have $\sigma_1\sigma_2 = \sigma_{2,1}$. In similar fashion, the intersection of Σ_1 and $\Sigma'_{1,1}$ is a cycle of the form $\Sigma_{2,1}$, with respect to a certain flag; specifically, it is the locus of lines containing the point $L \cap H'$ and lying in H', so that $\sigma_1\sigma_{1,1} = \sigma_{2,1}$.

The last and most interesting computation to be made is the product σ_1^2: the locus $\Sigma_1 \cap \Sigma'_1$ of lines meeting each of the two general lines L and L' is not a Schubert cycle, so something more is needed to evaluate the class. We will introduce here a new technique, called the method of *undetermined coefficients*; it's tremendously useful and will be used frequently throughout the rest of this book. Briefly: we know to begin with that the product $\sigma_1^2 \in A^2(G)$ is a linear combination

$$\sigma_1^2 = \alpha\sigma_2 + \beta\sigma_{1,1}, \tag{2.2}$$

and we simply need to determine the coefficients α and β. To do this, we can take the product of both sides in (2.2) with a class of complementary dimension 2: for example, if we take the product with σ_2 we have

$$\sigma_1^2 \cdot \sigma_2 = (\alpha\sigma_2 + \beta\sigma_{1,1}) \cdot \sigma_2 = \alpha\sigma_{2,2}$$

and so we have

$$\alpha = \#\left\{\Lambda : \begin{array}{c} \Lambda \cap L \neq \varnothing; \\ \Lambda \cap L' \neq \varnothing; \text{ and} \\ p'' \in \Lambda \end{array}\right\}$$

for L and L' general lines and p'' a general point in $\mathbb{P}^3$. This last cardinality is easy to evaluate: any line Λ satisfying the three conditions must lie in each of the planes $\overline{p'', L}$ and $\overline{p'', L'}$, and so must be their intersection; thus

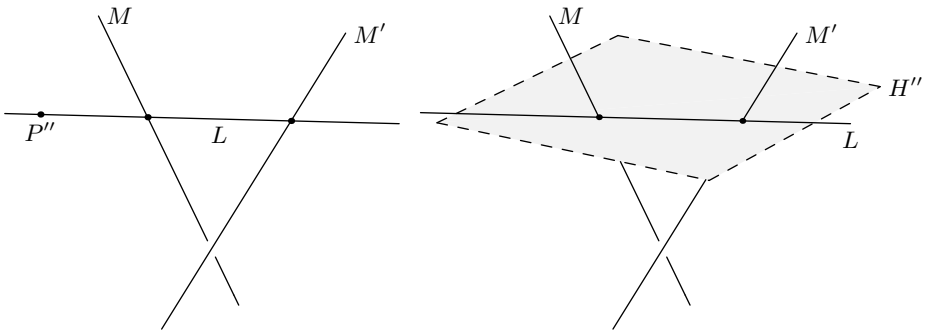


FIGURE 2.4. $\Sigma_2(p'')\cap\Sigma_1(M)\cap\Sigma_1(M') = \{L\}$; $\Sigma_{1,1}(H'')\cap\Sigma_1(M)\cap\Sigma_1(M') = \{L\}$

$\alpha = 1$. Similarly, to determine β we intersect both sides of the equation (2.2) with $\sigma_{1,1}$; we have

$$\sigma_1^2 \cdot \sigma_{1,1} = (\alpha\sigma_2 + \beta\sigma_{1,1}) \cdot \sigma_{1,1} = \beta\sigma_{2,2}$$

so

$$\beta = \#\left\{\begin{array}{c} \\ \Lambda : \Lambda \cap L \neq \varnothing; \\ \Lambda \cap L' \neq \varnothing; \text{ and} \\ \Lambda \subset H'' \\ \\ \end{array}\right\}$$

for L and L' general lines and H'' a general plane in $\mathbb{P}^3$. Again, this is elementary: any line Λ satisfying these conditions can only be the line joining the points $L \cap H''$ and $L' \cap H''$, so again $\beta = 1$ (Figure 2.4).

In sum, we have

$$\sigma_1^2 = \sigma_2 + \sigma_{1,1},$$

and this completes our description of the Chow ring $A^*(G)$, at least in characteristic 0.

Note that we could also have evaluated the triple products in the last two calculations simply by invoking the associativity of $A(G)$ and the previous calculations: for example, we have

$$\sigma_1^2 \cdot \sigma_2 = \sigma_1(\sigma_1\sigma_2) = \sigma_1\sigma_{2,1} = \sigma_{22}.$$

Tangent spaces to Schubert cycles. To extend the calculation above to characteristic p, where we can no longer use Kleiman to deduce transversality for pairs of Schubert cycles defined relative to general flags, we have to identify the tangent spaces to the Schubert cycles in question at their points of intersection, and show directly that they're transverse. We'll carry this out in full just in one case, that of the intersection $\Sigma_2 \cap \Sigma_2'$. Tangent spaces to other Schubert cycles in $\mathbb{G}(1,3)$ are described in Exercise **??**; they will be treated in general in Theorem 3.1.

Identifying the tangent space to G at a point $[\Lambda]$ as $\mathrm{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda})$ (where the $\tilde{}$ denotes the corresponding vector subspace), we see that a tangent vector $\varphi : \tilde{\Lambda} \to V/\tilde{\Lambda}$ represented by a lifting $\varphi' : \tilde{\Lambda} \to V$ lies in the tangent space to Σ_2 at Λ if the deformed two-dimensional free summand

$$\tilde{\Lambda}_\epsilon := \{v + \epsilon\varphi'(v) \mid v \in \tilde{\Lambda}\} \subset V + \epsilon V = V \otimes_F F[\epsilon]/(\epsilon^2)$$

contains the (un-deformed) summand $\tilde{p} + \epsilon\tilde{p} \subset V + \epsilon V$.

********What's missing here is a description of the functor of the Schubert cycle. An alternative would be to look at the Plücker equations cutting out the Schubert cycle and show they don't vanish on the tangent vector unless the given condition is satisfied.********

Since $\tilde{p} \subset \tilde{\Lambda}$, we certainly have $\epsilon\tilde{p} \subset \epsilon\tilde{\Lambda} = \epsilon\tilde{\Lambda}_\epsilon$. The condition for this to happen is simply that $\varphi'(\tilde{p}) \subset \tilde{\Lambda}$. But $\tilde{p} \subset \tilde{\Lambda}_\epsilon$ if and only if $\varphi'(\tilde{p}) \subset \tilde{\Lambda}$, that is, $\varphi(\tilde{p}) = 0$. Since $p \neq p'$, and these two points span Λ, the subspaces $\tilde{p}$ and $\tilde{p}'$ span $\tilde{\Lambda}$, so the spaces of maps $\{\varphi \mid \varphi(\tilde{p}) = 0\}$ and $\{\varphi \mid \varphi(\tilde{p}') = 0\}$ are transverse (they span all of $\mathrm{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda})$), proving that $\sigma_2^2 = \sigma_{2,2}$ □

As a consequence of Theorem 2.8, we have the following description of the Chow ring of $\mathbb{G}(1,3)$:

Corollary 2.9. *The Chow ring of the Grassmannian $G = \mathbb{G}(1,3)$ may be presented as*

$$A^*(G) = \frac{\mathbb{Z}[\sigma_1, \sigma_2]}{(\sigma_1^3 - 2\sigma_1\sigma_2,\ \sigma_1^2\sigma_2 - \sigma_2^2)}.$$

There is an analogous representation of the Chow ring of any Grassmannian, which we'll prove as an easy application of the theory of Chern classes in Chapter 7. A point to note is that the given presentation of the Chow ring has the same number of generators as relations—that is, given that the Chow ring $A^*(\mathbb{G}(1,3))$ has dimension 0, it is a complete intersection. The analogous statement is true for all of the Grassmann varieties.

2.2.4 A Specialization Argument

In calculating five of the six intersection products $\sigma_{a,b}\sigma_{a',b'}$ in the proof of Theorem 2.8, we were able to take general translates $\Sigma_{a,b}$ and $\Sigma'_{a',b'}$ of Schubert cycles representing these classes, and recognize the intersection $\Sigma_{a,b} \cap \Sigma'_{a',b'}$—necessarily generically transverse, by Kleiman's Theorem—as another Schubert cycle. There was one case, however, where this didn't work: the intersection $\Sigma_1 \cap \Sigma'_1$ of general translates of the Schubert cycle Σ_1 is not itself a Schubert cycle. We were forced, accordingly, to use the indirect method of undetermined coefficients to evaluate the product $\sigma_1^2 \in A(G)$.

As you may expect, this is by far the more common situation: even when we are fortunate enough to be able to find explicit cycles A and B representing two given classes $\alpha, \beta \in A(X)$ and intersecting generically transversely, it may not be obvious what is the class of the intersection $A \cap B$. This is one of the reasons the method of undetermined coefficients is so ubiquitous in the subject.

There is, however, another approach to evaluating products, which is also very frequently used: the method of *specialization*. This is also one of the fundamental tools of intersection theory—in the context of Chow rings of Grassmannians, for example, it's the starting point for the general algorithms of Coskun and Vakil—and we'll illustrate here a simple case of this method by revisiting the calculation of $\sigma_1^2 \in A(\mathbb{G}(1,3))$. We'll see other examples of this technique in the following chapter when we consider intersections in the Chow rings of more general Grassmannians.

In the present context, the idea of specialization is extremely simple. If we want to evaluate the product σ_1^2 of the Schubert class σ_1 with itself we could take two general Schubert cycles representing this class—that is, the cycles

$$\Sigma_1(L) = \{\Lambda : \Lambda \cap L \neq \varnothing\} \quad \text{and} \quad \Sigma_1(L') = \{\Lambda : \Lambda \cap L' \neq \varnothing\}$$

for a general pair of lines $L, L' \subset \mathbb{P}^3$. The intersection $\Sigma_1(L) \cap \Sigma_1(L')$ would then be generically transverse, and the problem is to evaluate its class; this is what we did above. Now, instead of this, we propose to choose a *special* pair of lines L, L': specifically, we want to choose L and L' special enough that the class of the intersection $\Sigma_1(L) \cap \Sigma_1(L')$ is readily identifiable, but at the same time not so special that the intersection fails to be generically transverse.

The solution is to choose L and L' to be distinct, but incident. (We really don't have a lot of options in this instance: if L and L' are disjoint, they are effectively a general pair of lines; if they're equal, of course, the cycles $\Sigma_1(L)$ and $\Sigma_1(L')$ won't intersect properly.) Now the intersection $\Sigma_1(L) \cap \Sigma_1(L')$ is easy to describe: if we let $p = L \cap L'$ be the point of intersection of the lines and $H = \overline{L, L'}$ the plane they span, then any line Λ meeting L and L' but not passing through p must meet L and L' in distinct points, and so must lie in H. Thus, as sets, we have

$$\begin{aligned}\Sigma_1(L) \cap \Sigma_1(L') &= \{\Lambda : \Lambda \cap L \neq \varnothing \text{ and } \Lambda \cap L' \neq \varnothing\} \\ &= \{\Lambda : p \in \Lambda \text{ or } \Lambda \subset H\} \\ &= \Sigma_2(p) \cup \Sigma_{1,1}(H).\end{aligned}$$

Now we just need to show that the intersection is generically transverse, and we'll be able to deduce the formula $\sigma_1^2 = \sigma_2 + \sigma_{1,1} \in A(\mathbb{G}(1,3))$. To check this, we'll use the description of the tangent spaces to $\Sigma_1(L)$ and $\Sigma_1(L')$ given in Exercise 2.24. First, suppose Λ is a general point of the

component $\Sigma_2(p)$ of $\Sigma_1(L) \cap \Sigma_1(L')$, that is, a general line through p; we'll let $K = \overline{\Lambda, L}$ and $K' = \overline{\Lambda, L'}$ be the planes spanned by Λ together with L and L'. Viewing the tangent space $T_\Lambda(G)$ as the vector space of linear maps $\varphi : \tilde{\Lambda} \to V/\tilde{\Lambda}$, we have

$$T_\Lambda(\Sigma_1(L)) = \{\varphi : \varphi(\tilde{p}) \subset \tilde{K}/\tilde{\Lambda}\}$$

and

$$T_\Lambda(\Sigma_1(L')) = \{\varphi : \varphi(\tilde{p}) \subset \tilde{K'}/\tilde{\Lambda}\}.$$

Since K and K' are distinct, they intersect in Λ, so that the intersection is simply

$$T_\Lambda(\Sigma_1(L)) \cap T_\Lambda(\Sigma_1(L')) = \{\varphi : \varphi(\tilde{p}) = 0\};$$

this being 2-dimensional, we see that the intersection $\Sigma_1(L) \cap \Sigma_1(L')$ is transverse at $[\Lambda]$.

Similarly, if Λ is a general point of the component $\Sigma_{1,1}(H)$ of $\Sigma_1(L) \cap \Sigma_1(L')$, so that Λ meets L and L' in distinct points q and q', we have

$$T_\Lambda(\Sigma_1(L)) = \{\varphi : \varphi(\tilde{q}) \subset \tilde{H}/\tilde{\Lambda}\}$$

and

$$T_\Lambda(\Sigma_1(L')) = \{\varphi : \varphi(\tilde{q'}) \subset \tilde{H}/\tilde{\Lambda}\},$$

so

$$T_\Lambda(\Sigma_1(L)) \cap T_\Lambda(\Sigma_1(L')) = \{\varphi : \varphi(\tilde{\Lambda}) \subset \tilde{H}\};$$

again this is 2-dimensional and we conclude that $\Sigma_1(L) \cap \Sigma_1(L')$ is transverse at $[\Lambda]$.

We should say that this is an example of the simplest kind of specialization argument, what we may call *static specialization*: we are able to find cycles representing the two given classes that are special enough that the class of the intersection is readily identifiable, but general enough that they still intersect properly. In general, we may not be able to find such cycles. Such situations call for a more powerful and broadly applicable technique, called *dynamic specialization*, where we consider a one-parameter family of pairs of cycles specializing from a "general" pair to a special one, and ask not for the intersection of the limiting cycles but for the limit of their intersections—a key difference. We'll see an example of this in Section 3.3.1 of the following chapter, where we consider, in the Grassmannian $\mathbb{G}(1,4)$ of lines in $\mathbb{P}^4$, the self-intersection of the cycle of lines meeting a given line in $\mathbb{P}^4$.

2.3 Applications

In this section we present some simple computations made possible by Theorem 2.8. To assign the full geometric meaning to our assertions, some transversality conditions are necessary. In characteristic zero these are all covered by Kleiman's Transversality Theorem 5.20. In fact, they hold in all characteristics. We supply proofs in a few cases, and others can be done in a similar way; we leave them to the interested reader.

In this section we will generally be interested in the numbers of intersections, in the style of classical enumerative geometry, so we will generally replace a zero-dimensional class $\alpha\sigma_{2,2}$ with its degree α to simplify the notation. As we've observed, $A^4(G) = \mathbb{Z} \cdot \sigma_{2,2}$, so this does not destroy any information.

2.3.1 How Many Lines Meet Four General Lines?

We can use the calculation of the Chow ring $A^*(G)$ to answer the first of the keynote questions of this Chapter: given four general lines $L_1, \ldots, L_4 \subset \mathbb{P}^3$, how many lines will meet all four? By Kleiman's Theorem in characteristic zero, or an auxiliary argument sketched below in general, we know that the intersection $\cap\Sigma_1(L_i)$ will be transverse, so that its cardinality—the answer to the question—is equal to the product $\sigma_1^4 \in A^4(G) \cong \mathbb{Z}$. We can calculate this readily enough: we have

$$\begin{aligned}\sigma_1^4 &= \sigma_1^2\,(\sigma_2 + \sigma_{1,1}) \\ &= \sigma_1\,(\sigma_{2,1} + \sigma_{2,1}) \\ &= 2.\end{aligned}$$

So: *there are exactly two lines meeting each of four general lines in space.*

In fact, we can see this directly, without recourse to intersection theory; we'll outline one way of doing this in Exercise 2.27.

Another question we can ask in connection with the keynote question is what "general" means in this context: more precisely, if $L_1, \ldots, L_4 \subset \mathbb{P}^3$ are lines, and $L \subset \mathbb{P}^3$ a line meeting all four, when is $[L] \in G$ a transverse point of intersection of the four Schubert cycles $\Sigma_1(L_i)$ (in the sense that the intersection scheme is a reduced point), and when is it not? We'll answer this in Exercise 2.28.

As we have seen, the question of how many lines meet each of four general lines is not hard to answer without intersection theory. But there are many other applications that are not so obvious, including the answer to the second and third keynote questions of this Chapter. Here is a sampling.

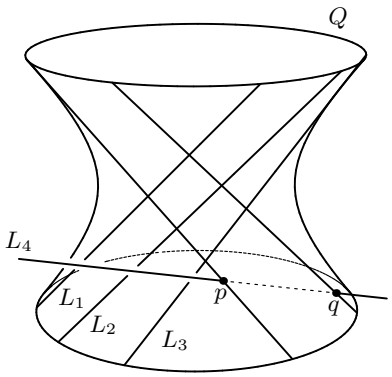


FIGURE 2.5. Two lines that meet each of $L_1, \ldots, L_4$

2.3.2 Lines meeting a curve

Let $C \subset \mathbb{P}^3$ be a curve of degree d. As in the case of C a line, the locus $\Gamma_C \subset G$ of lines meeting C is a divisor in G. (To prove this, consider the incidence correspondence whose points are $\{(p, L) \mid p \in C, L \text{ a line containing } p\}$ and its projections to C and to Γ_C, as in Exercise 2.17.) We now ask for the class $\gamma_C = [\Gamma_C] \in A(G)$.

Since Γ_C has codimension 1, we must have

$$\gamma_C = \alpha \cdot \sigma_1$$

for some $\alpha \in \mathbb{Z}$. To determine α, we intersect both sides with the class $\sigma_{2,1}$ and get

$$\gamma_C \cdot \sigma_{2,1} = \alpha(\sigma_1 \cdot \sigma_{2,1}) = \alpha.$$

Now, if (p, H) is a general pair consisting of a point $p \in \mathbb{P}^3$ and a plane $H \subset \mathbb{P}^3$ containing it, the Schubert cycle

$$\Sigma_{2,1}(p, H) = \{L : p \in L \subset H\}$$

will intersect the cycle Γ_C transversely; this follows from Kleiman's theorem in characteristic zero, and can be proven in general by using the description of the tangent spaces to $\Sigma_1(p, H)$ in Exercise 2.25 and of the tangent spaces to Γ_C in Exercise 2.29. We have therefore

$$\begin{aligned} \alpha &= \#\big(\Gamma_C \cap \Sigma_{2,1}(p, H)\big) \\ &= \#\{L : p \in L \subset H \quad \text{and} \quad L \cap C \neq \varnothing\}. \end{aligned}$$

To evaluate this last number, note that H (being general) will intersect C transversely in d points $\{q_1, \ldots, q_d\}$; and, $p \in H$ being general, no two of them will be collinear with p. Thus the intersection $\Gamma_C \cap \Sigma_{2,1}(p, H)$ will

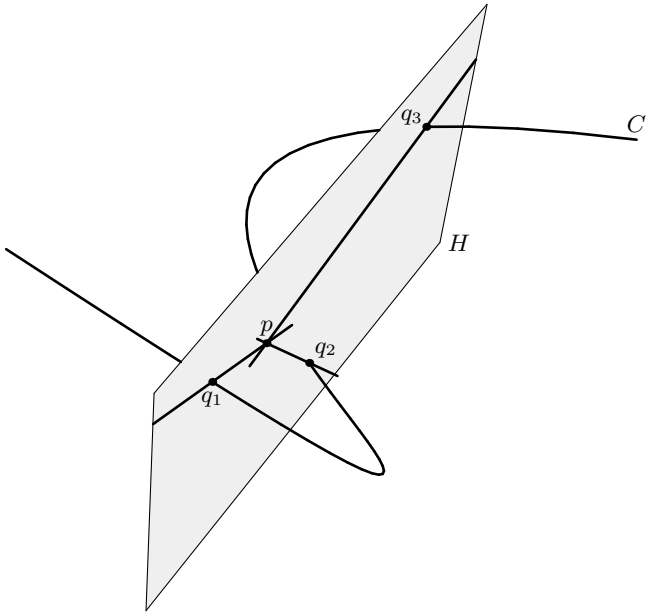


FIGURE 4.7. The intersection of Γ_C with $\Sigma_{2,1}(p, H)$

consist simply of the d lines $\overline{p, q_i}$, as in Figure 4.7; we see accordingly that $\alpha = d$ and so

$$\gamma_C = d \cdot \sigma_1.$$

By way of an example, suppose $C_1, \ldots, C_4 \subset \mathbb{P}^3$ are four general conics (see Chapter 10 for a discussion of a parameter space $\mathcal{H}$ for conics in $\mathbb{P}^3$; for now, we can take a general conic to be the translate of a given conic under a general element $A \in PGL_4$). We ask: how many lines will meet all four?

We answer this exactly as in the case of lines: the class of the cycle Γ_C of lines meeting a conic is $2\sigma_1$; since the cycles Γ_{C_i} are general translates of Γ_C, we can hope that they intersect transversely, and so we'll have

$$\#\left(\bigcap_{i=1}^{4} \Gamma_{C_i}\right) = (2\sigma_1)^4 = 32.$$

Thus, assuming the transversality, there are 32 lines meeting each of four general conics; similarly, there'll be $2\prod d_i$ lines meeting general translates of any four curves C_i of degrees d_i.

What about the transversality? Kleiman's Theorem 5.20 tells us that at least the intersection of the cycles is zero-dimensional, and that in char-

acteristic zero it will be transverse. For the general case, we can verify transversality directly; as a bonus, we can see exactly when transversality fails. This is the content of the series of exercises 2.29-2.32.

2.3.3 Lines meeting a curve via specialization

Before moving on, we want to mention another approach to calculating the class γ_C of the locus $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting a curve $C \subset \mathbb{P}^3$. We determined the class above by the method of undetermined coefficients, but just as in the calculation of the product σ_1^2 we can also do it by specialization.

To carry this out, suppose $C \subset \mathbb{P}^3$ is any curve of degree d; choose a plane $H \subset \mathbb{P}^3$ intersecting C transversely in points $p_1, \ldots, p_d$ and $q \in \mathbb{P}^3$ any point not lying on H. Consider the one-parameter group $\{A_t\} \subset PGL_4$ with repellor plane H and attractor p; that is, choose coordinates $[Z_0, \ldots, Z_3]$ on $\mathbb{P}^3$ such that $q = [1,0,0,0]$ and H is given by $Z_0 = 0$, and consider for $t \neq 0$ the automorphisms of $\mathbb{P}^3$ given by

$$A_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{pmatrix}.$$

Let $C_t = A_t(C)$, and let $\Phi \subset \mathbb{A}^1 \times \mathbb{P}^3$ be the closure of the locus

$$\tilde{\Phi} = \{(t,p) : t \neq 0 \text{ and } p \in C_t\}$$

As we saw in Section 4.1.1 (specifically, Proposition 4.5), the limit $\lim_{t\to 0} C_t$—that is, the fiber of Φ over $t = 0$—is supported on the union of the d lines $\overline{p_i, q}$, and has multiplicity 1 at a general point of each.

Now, just as we used this construction in Section 4.1.1 to give a rational equivalence between the curve C and the union $\bigcup \overline{p_i, q}$, we can use it to give a rational equivalence between the cycle Γ_C and the sum of the Schubert cycles $\Sigma_1(\overline{p_i, q})$ in $\mathbb{G}(1,3)$. Explicitly, take $\Psi \subset \mathbb{A}^1 \times \mathbb{G}(1,3)$ to be the closure of the locus

$$\tilde{\Psi} = \{(t, \Lambda) \mid t \neq 0 \text{ and } \Lambda \cap C_t \neq \varnothing\}.$$

As we'll verify in Exercises 2.33 and 2.34, the fiber Ψ_0 of Ψ over $t = 0$ is supported on the union of the Schubert cycles $\Sigma_1(\overline{p_i, q})$ and has multiplicity one along each, establishing the rational equivalence $\gamma_C = d \cdot \sigma_1$.

****Fig. 4.7.5****

Note that, we observed in Section 4.1.1 it's *not* the case that the fiber of Φ over $t = 0$—that is, the flat limit $\lim_{t\to 0} C_t$ of the curves C_t—is equal to the union of the d lines $\overline{p_i q}$; in general, it may have an embedded point

at the point q. Similarly, the flat limit Ψ_0 of the family of subvarieties $\Gamma_{C_t} \subset \mathbb{G}(1,3)$ will not be the union of the subvarieties $\Sigma_1(\overline{p_i q}) \subset \mathbb{G}(1,3)$; it may likewise have an embedded component supported on the locus of lines passing through q. But—just as in our initial application of dynamic projection in Section 4.1.1—this occurs in strictly smaller dimension, and so doesn't affect the rational equivalence $\Gamma_C \sim \sum \Sigma_1(\overline{p_i q})$. If you want to see this yourself, try Exercise 2.35.

We'll use this specialization again after the following discussion, in Section 2.3.5 (where, interestingly, it won't work so well!).

2.3.4 Chords to a space curve

Consider now a smooth, nondegenerate space curve $C \subset \mathbb{P}^r$, of degree d and genus g. We can define the locus $S_1(C) \subset \mathbb{G}(1,r)$ of *chords*, or *secant lines* to C in either of two ways. We can consider the rational map $\varphi : C^{(2)} \to \mathbb{G}(1,r)$ from the symmetric square $C^{(2)}$ of C to the Grassmannian $\mathbb{G}(1,r)$ sending a pair of distinct points p, q to the line $\overline{pq}$, and take the variety $S_1(C) \subset \mathbb{G}(1,r)$ to be the (closed) image. Alternatively, we can define $S_1(C)$ to be the locus of lines $L \subset \mathbb{P}^r$ such that the scheme-theoretic intersection $L \cap C$ has degree at least 2. As we'll see in Exercise 2.37, these definitions (or their analogs) differ when we consider singular curves, or (as we'll see in Exercise 2.38) higher-dimensional secant planes to curves; but for smooth curves in $\mathbb{P}^r$ we'll show in Exercise 2.36 they agree, and we can adopt either one.

Let's now restrict ourselves to the case $r = 3$ of smooth, nondegenerate space curves $C \subset \mathbb{P}^3$, and ask: what's the class, in $A^2(\mathbb{G})$, of the locus $S_1(C)$ of secant lines to C? We can answer this question by intersecting with Schubert cycles of complementary codimension (in this case, codimension 2). We know that

$$[S_1(C)] = \alpha\sigma_2 + \beta\sigma_{1,1}$$

for some integers α and β. To find the coefficient β we take a general plane $H \subset \mathbb{P}^3$, and consider the Schubert cycle

$$\Sigma_{1,1}(H) = \{L : L \subset H\} \subset \mathbb{G}.$$

Assuming transversality (or, in characteristic zero using Kleiman's theorem),

$$\begin{aligned} \#\left(\Sigma_{1,1}(H) \cap S_1(C)\right) &= \sigma_{1,1} \cdot [S_1(C)] \\ &= \sigma_{1,1} \cdot (\alpha\sigma_2 + \beta\sigma_{1,1}) \\ &= \beta, \end{aligned}$$

The cardinality of this intersection is easy to determine: the plane H will intersect C in d points $p_1, \ldots, p_d$, no three of which will be collinear (**ref

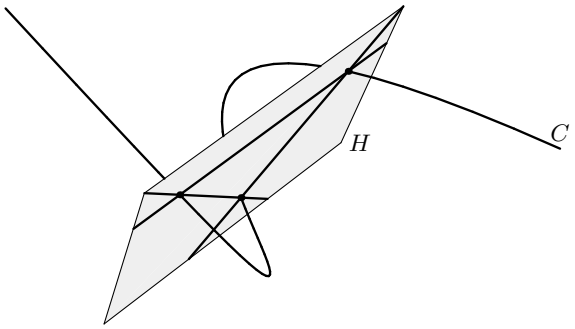


FIGURE 2.7. $\Sigma_{1,1}(H) \cap S_1(C)$ consists of $\binom{\deg C}{2}$ lines.

ACGH**), so that there will be exactly $\binom{d}{2}$ lines $\overline{p_i p_j}$ joining these points pairwise; thus

$$\beta = \binom{d}{2}$$

(Figure 2.7).

Similarly, to find α we let $p \in \mathbb{P}^3$ be a general point and

$$\Sigma_2(p) = \{L : p \in L\} \subset \mathbb{G};$$

we have as before

$$\begin{aligned} \#\left(\Sigma_2(p) \cap S_1(C)\right) &= \sigma_2 \cdot [S_1(C)] \\ &= \sigma_2 \cdot (\alpha\sigma_2 + \beta\sigma_{1,1}) \\ &= \alpha. \end{aligned}$$

To count this intersection—that is, the number of chords to C through the point p—consider the projection $\pi_p : C \to \mathbb{P}^2$. This map is birational onto its image $\overline{C} \subset \mathbb{P}^2$, which will be a curve having only nodes as singularities ****give reference****, and those nodes correspond exactly to the chords to C through p. (These chords were classically called the *apparent nodes* of C (Figure 4.9): if you were looking at C with your eye at the point p, and had no depth perception, they're the nodes you would see.) By the genus formula, this number is the difference between the arithmetic genera of C and $\overline{C}$, that is,

$$\alpha = \binom{d-1}{2} - g$$

In sum,

$$[S_1(C)] = \left(\binom{d-1}{2} - g\right)\sigma_2 + \binom{d}{2}\sigma_{1,1}.$$

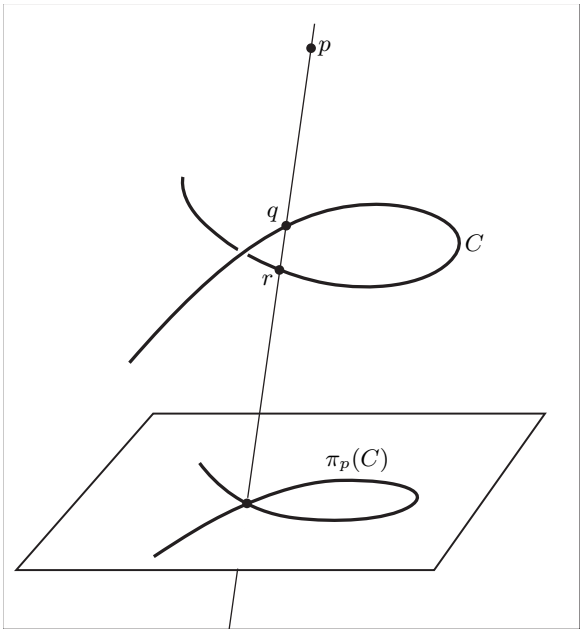


FIGURE 4.9. An "apparent node".

We can use this to answer the second of the keynote questions of this Chapter: if C and C' are general twisted cubic curves, by Kleiman's theorem the cycles $S = S_1(C)$ and $S' = S_1(C')$ will intersect transversely; since by our general formula the class of each is $\sigma_2 + 3\sigma_{1,1}$, we have

$$\begin{aligned} \#(S \cap S') &= (\sigma_2 + 3\sigma_{1,1})^2 \\ &= 10; \end{aligned}$$

in other words, C and C' will have exactly 10 common chords.

As usual, if we want to extend this result to characteristic p we can't use Kleiman, and so have to identify (at least an open subset of) the locus of smooth points of $S_1(C)$, and the tangent spaces to $S_1(C)$ at these points; this is the content of Exercises 2.39 and 2.40.

2.3.5 Chords via specialization

Finally, it's instructive to see what happens if we try to determine the class of $S_1(C)$ by specialization methods, as we did in Section 2.3.3 to find the class of the locus Γ_C of incident lines to C. There, we took a one-parameter subgroup $\{A_t\} \subset PGL_4$ such that the limit of the translates $C_t = A_t(C)$ consisted on the union of d lines $\overline{p_i, q}$, and showed that the corresponding

family of cycles Γ_{C_t} gave a rational equivalence between Γ_C and the sum of the Schubert cycles $\Sigma_1(\overline{p_i, q}$; we concluded that the class $\gamma_C = d\sigma_1$.

What happens if we do the same thing here—that is, consider the limit as $t \to 0$ of the cycles $S_1(C_t)$? We set it up as before: we let $\Pi \subset \mathbb{A}^1 \times G$ be the closure of the locus

$$\tilde{\Pi} = \{(t, \Lambda) : t \neq 0 \text{ and } \Lambda \in S_1(C_t)\},$$

and ask for the fiber Π_0 of this family.

What we find is that, first of all, the support of Π_0 is readily identified. It's certainly contained in the locus of lines having intersection of degree at least 2 with the flat limit $C_0 = \lim_{t\to 0} C_t$, which is to say the union of the Schubert cycles $\Sigma_{1,1}(\overline{p_i, p_j, q})$ of lines lying in a plane spanned by a pair of the lines $\overline{p_i, q}$, and the Schubert cycle $\Sigma_2(q)$ of lines containing the point q. Moreover, we can see that the Schubert cycles $\Sigma_{1,1}(\overline{p_i, p_j, q})$ all appear with multiplicity 1 in the limiting cycle Π_0, from which we can deduce that the coefficient of $\sigma_{1,1}$ in the class of $S_1(C)$ is $\binom{d}{2}$.

The hard part is determining the multiplicity with which the cycle $\Sigma_2(q)$ appears in Π_0: this, it turns out, will depend in part on the multiplicity of the embedded point of C_0 at q, which (as we saw in Exercise **??** **ref to exercise in Ch. 2**) will depend on the genus g of C. (Note that the embedded component of the limit scheme $\lim_{t\to 0} \Gamma_{C_t}$ also depends on the genus of C, but since it occurs in lower dimension won't affect the limiting cycle.)

An alternative approach to this problem would be to use a different specialization to capture the coefficient of σ_2: specifically, we can take the one-parameter subgroup with repellor a general point q and attractor a general plane $H \subset \mathbb{P}^3$. In this setting, the limiting scheme $C_0 = \lim_{t\to 0} C_t$ will be a plane curve of degree d with $\delta = \binom{d-1}{2} - g$ nodes $r_1, \ldots, r_\delta$, with a spatial embedded point of multiplicity 1 at each node. The limit of the corresponding cycles $S_1(C_t) \subset \mathbb{G}(1, 3)$ will correspondingly be supported on the union of the Schubert cycle $\Sigma_{1,1}(H)$ and the δ Schubert cycles $\Sigma_2(r_i)$. In this case the coefficient of the Schubert cycle $\Sigma_{1,1}(H)$ is the mysterious one (though calculable: given that a general line $\Lambda \subset H$ meets C_0 in d points, we can show that it's the limit of $\binom{d}{2}$ chords to C_t as $t \to 0$). On the other hand, we can see that the Schubert cycles $\Sigma_2(r_i)$ all appear with multiplicity 1 in the limit of the cycles $S_1(C_t)$, from which we can read off the coefficient δ of σ_2 in the class of $S_1(C)$.

We will fill in some of the details in this calculation in Exercises **??**-**??**; we describe this approach here to give the reader a further idea of the art of specialization methods.

********need to rewrite relevant exercises********

2.3.6 Lines on a quadric

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. The family $F_1(Q) \subset \mathbb{G}(1,3)$ of lines contained in Q is one-dimensional, with two lines passing through every point $p \in Q$; and we can use this to determine the class of the locus $F_1(Q)$. To begin with, we know

$$[F_1(Q)] = \alpha \cdot \sigma_{2,1}$$

for some integer α. Next, if $L \subset \mathbb{P}^3$ is a general line, and $\Sigma_1(L) \subset \mathbb{G}(1,3)$ the Schubert cycle of lines meeting L, we have

$$\begin{aligned}\alpha &= [F_1(Q)] \cdot \sigma_1 \\ &= \#\left(\Sigma_1(L) \cap F_1(Q)\right) \\ &= \#\{M \in \mathbb{G}(1,3) : M \subset Q \text{ and } M \cap L \neq \varnothing\}.\end{aligned}$$

Now L, being general, will intersect Q in two points, and through each of these points there will be two lines contained in Q; thus we have $\alpha = 4$ and

$$[F_1(Q)] = 4\sigma_{2,1}.$$

The variety $F_1(Q)$ is actually the union of two disjoint curves, corresponding to the two rulings of Q; each of these curves is actually a conic curve on $\mathbb{G}(1,3) \subset \mathbb{P}^5$. For a fuller description of the subschemes $F_1(Q) \subset \mathbb{G}(1,3)$, see Eisenbud and Harris [2000].

2.3.7 Tangent Lines to a Surface

Next, let $S \subset \mathbb{P}^3$ be any smooth surface of degree d, and consider the locus $T_1(S) \subset \mathbb{G}(1,3)$ of tangent lines to S. Let Φ be the incidence correspondence

$$\Phi = \{(p, L) \mid p \in L \subset \mathbb{T}_q S\} \subset S \times \mathbb{G}(1,3),$$

where $\mathbb{T}_q S$ denotes the projective plane tangent to S at q. The projection $\Phi \to S$ on the first factor expresses Φ as a $\mathbb{P}^1$-bundle over S, from which we deduce that Φ, and hence its image $T_1(S)$ in $\mathbb{G}(1,3)$, is irreducible of dimension 3.

To find the class of $T_1(S)$, we write

$$[T_1(S)] = \alpha \cdot \sigma_1,$$

and choose a general plane $H \subset \mathbb{P}^3$ and a general point $p \in H$. Using Kleiman's Theorem 5.20 we may write

$$\begin{aligned}\alpha &= [T_1(S)] \cdot \sigma_{2,1} \\ &= \deg\left(\Sigma_{2,1}(p, H) \cap T_1(S)\right) \\ &= \deg\left\{M \in \mathbb{G}(1,3) : M \subset \mathbb{T}_p S \text{ for some } q \in S \text{ and } p \in M \subset H\right\}.\end{aligned}$$

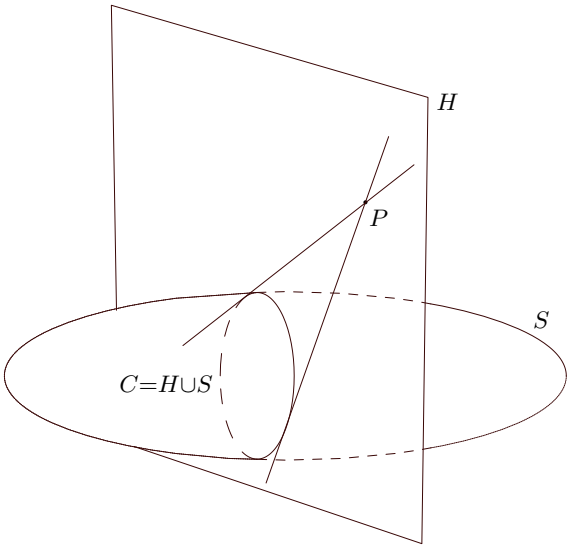


FIGURE 2.9. $\deg\big(\Sigma_{2,1}(p,H)\cap T_1(S)\big)=2$.

Now, H, being general, will intersect S in a smooth plane curve $C\subset H\cong\mathbb{P}^2$ of degree d; and p being general, the line $p^*\subset\mathbb{P}^{2^*}$ dual to p will intersect the dual curve $C^*\subset\mathbb{P}^{2^*}$ tranversely in $\deg(C^*)$ points. By the count of Chapter **, **ref** we have

$$\deg(C^*)=d(d-1)$$

and hence

$$[T_1(S)]=d(d-1)\sigma_1$$

(Figure 2.9).

As an immediate consequence, we have the answer to the last Keynote Question of this chapter: how many lines are tangent to each of four general quadric surfaces Q_i? At least in characteristic 0, where we know the cycles $T_1(Q_i)$ intersect transversely by Kleiman, the answer is just the intersection product

$$\prod[T_1(Q_i)]=(2\sigma_1)^4=32.$$

2.4 Exercises

Exercise 2.10. Let $e_1, \ldots, e_4$ be a basis for a vector space V, and let

$$\eta = \sum_{1 \le a < b \le 4} p_{a,b} e_a \wedge e_b \in \wedge^2 V.$$

Show that the polynomial in the $p_{a,b}$ corresponding to the equation

$$\eta \wedge \eta = 0 \in \wedge^4 V$$

is $2g$, where g is as in (2.1). Thus, in characteristic not 2, the equation $\eta^2 = 0$ characterizes decomposable vectors in the exterior square $\wedge^2 V$, and more generally defines the Grassmannian $\mathbb{G}(1, \mathbb{P}V) \subset \mathbb{P}(\wedge^2 V)$ for a vector space V of any dimension.

In fact, even over the integers, there is an element $\eta^{(2)} \in \wedge^4 V$ called the *divided square* of η, such that $\eta \wedge \eta = 2\eta^{(2)}$, and in every characteristic η is decomposable if and only if $\eta^{(2)} = 0$. The coefficients of $\eta^{(2)}$, when written out in terms of the basis of $\wedge^4 V$, are called the 4×4 Pfaffians of η; they are the polynomials $g_{a,b,c,d}$ of Exercise **??**. The square of the ideal of 4×4 Pfaffians of η is the ideal of 4×4 minors of the generic alternating matrix $(p_{i,j})$. See Eisenbud [1995] Appendix A2.4.

Exercise 2.11. Let Λ and $\Gamma \in G$ be two points in the Grassmannian $G = G(k, V)$. Show that the line $\overline{\Lambda, \Gamma} \subset \mathbb{P}(\wedge^k V)$ is contained in G if and only if the intersection $\Lambda \cap \Gamma \subset V$ of the corresponding subspaces of V has dimension $k - 1$.

Exercise 2.12. Suppose that $L_1, \ldots, L_m \subset V$ are k-planes contained in a vector space, and that for every $i \neq j$ we have $\dim L_i \cap L_j = k - 1$ (equivalently, $\dim L_i + L_j = k + 1$.) Show that either all the L_i contain a subspace M of dimension $k - 1$ or all L_i are contained in a subspace N of dimension $k + 1$.

Exercise 2.13. Using Exercises 2.11 and 2.12, show that the linear subspaces of $\mathbb{P}(\wedge^k V)$ that are maximal among linear spaces contained in $G(k, V)$ are precisely those defined in one of two ways:

(a) $\Gamma_M = \{L \mid M \subset L\}$ for some subspace $M \subset V$ of dimension $k - 1$, so that $\Gamma_M \cong \mathbb{P}(V/M)$; or

(b) $\Sigma_N = \{L \mid L \subset N\}$ for some subspace $N \subset V$ of dimension $k + 1$, so that $\Sigma_N \cong \mathbb{P}(N)$.

Notice that unless $\dim V = 2k$, these two kinds of maximal linear subspaces have different dimensions. Indeed, Exercise (2.13) is key in establishing a basic fact: that for $\dim V \neq 2k$, the automorphism group of the Grassmannian $G = G(k, V)$ is the same as the automorphism group $PGL(V)$

of $\mathbb{P}V$: assuming $2k < \dim V$, we argue that an automorphism of $G(k,V)$ must be projective, that is, induced by an automorphism of the vector space $\wedge^k V$; hence must preserve linear spaces of $\mathbb{P}(\wedge^k V)$ contained in $G(k,V)$; and so induces an automorphism of the Grassmannian $G(k-1,V)$. Continuing in this way, we arrive at an automorphism of $\mathbb{P}V$ that induces the original automorphism of $G(k,V)$. The situation when $\dim V = 2k$ is similar, except that there do exist automorphisms of $G(k,V)$ exchanging the two families of maximal linear subspaces (for example, we can choose a nondegenerate bilinear form on V and send each k-plane Λ to $\Lambda^\perp$); we arrive at an exact sequence

$$1 \to PGL(V) \to Aut(G) \to \mathbb{Z}/2 \to 1.$$

Exercise 2.14. Let $\mathbb{G}(1,3) \subset \mathbb{P}^5$ be the Grassmannian of lines in $\mathbb{P}^3$, embedded in $\mathbb{P}^5$ by the Plücker embedding.

(a) Write down the equations in Plücker coordinates of the maximal linear spaces on $\mathbb{G}(1,3)$, that is, the locus of lines containing a point and the locus lines contained in a 2-plane.

(b) Write down the equations in Plücker coordinates of the subvariety of $\mathbb{G}(1,3)$ consisting of lines meeting a given line.

These spaces are special cases of what we will call *Schubert cycles*.

Exercise 2.15. Using the fact that the Grassmannian

$$G = G(k,V) \subset \mathbb{P}(\wedge^k V)$$

is cut out by quadratic equations, show that if $[\Lambda] \in G$ is the point corresponding to a k-plane Λ then the tangent plane $\mathbb{T}_{[\Lambda]}G \subset \mathbb{P}(\wedge^k V)$ intersects G in the locus

$$G \cap \mathbb{T}_{[\Lambda]}G = \{[\Gamma] : \dim(\Gamma \cap \Lambda) \geq k-1\};$$

that is, the locus of k-planes meeting Λ in codimension 1.

Exercise 2.16. Consider the universal k-plane over $G = \mathbb{G}(k,\mathbb{P}V)$:

$$\Phi = \{(\Lambda, p) : p \in \Lambda\} \subset G \times \mathbb{P}V.$$

whose fiber over a point $[\Lambda] \in G$ is the k-plane $\Lambda \subset \mathbb{P}V$. Show that this is a closed subvariety of $G \times \mathbb{P}V$ of dimension $k + (k+1)(n-k)$, and that it's cut out by bilinear forms on $\mathbb{P}(\wedge^k V) \times \mathbb{P}V$.

Exercise 2.17. Use the preceding exercise to show that if $X \subset \mathbb{P}^n$ is any subvariety of dimension $l < n-k$, then the locus

$$\Gamma_X = \{\Lambda : X \cap \Lambda \neq \varnothing\} \subset \mathbb{G}(k,n)$$

of k-planes meeting X is a closed subvariety of $\mathbb{G}(k,n)$ of codimension $n-k-l$.

Exercise 2.18. Let $l < k < n$, and consider the locus of nested pairs of linear subspaces of $\mathbb{P}^n$ of dimensions l and k:

$$\mathbb{F}(l,k;n) = \{(\Gamma,\Lambda) : \Gamma \subset \Lambda\} \subset \mathbb{G}(l,n) \times \mathbb{G}(k,n).$$

Show that this is a closed subvariety of $\mathbb{G}(l,n) \times \mathbb{G}(k,n)$, and calculate its dimension. (These are examples of a further generalization of Grassmannians called *flag manifolds*.)

Exercise 2.19. Again let $l < k < n$, and for any $m \leq l$ consider the locus of pairs of linear subspaces of $\mathbb{P}^n$ of dimensions l and k intersecting in dimension at least m:

$$\mathbb{F}(l,k;n) = \{(\Gamma,\Lambda) : \dim(\Gamma \cap \Lambda) \geq m\} \subset \mathbb{G}(l,n) \times \mathbb{G}(k,n).$$

Show that this is a closed subvariety of $\mathbb{G}(l,n) \times \mathbb{G}(k,n)$, and calculate its dimension.

Exercise 2.20. Different hyperplane sections of $G(k,n)$ can have different properties; not all are given by the vanishing of a Plücker coordinate.

(a) Use (for example) Exercise 2.15 to show that a hyperplane section of $G(2,4) = \mathbb{G}(1,3)$ defined by a Plücker coordinate is the cone over a nonsingular quadric in $\mathbb{P}^3$. What other varieties appear as a hyperplane section of $\mathbb{G}(1,3)$?

(b) In general, how many isomorphism classes of hyperplane sections of $\mathbb{G}(k,n)$ are there?

Exercise 2.21. We gave a description of the Plücker relations in the special case $\mathbb{G}(1,n)$ following Example 2.1, and in Exercise 2.10 above. Show that the polynomials above, obtained by expanding the determinants of submatrices of A, are the same as the polynomials described there.

Exercise 2.22. By Theorem 2.4, a tangent vector to the Grassmannian $G(k,V)$ at a point $[\Lambda]$ corresponds naturally to a homomorphism $\varphi : \Lambda \to V/\Lambda$, and so we can associate to it its rank. Show that an automorphism of $G(k,n)$ carries tangent vectors to tangent vectors of the same rank, and hence that in case $1 < k < n$ the group of automorphisms of $G(k,n)$ cannot act transitively on nonzero tangent vectors (so that in particular Kleiman's Theorem does not apply in characteristic p). Show, on the other hand, that the group of automorphisms of $G(k,n)$ *does* act transitively on tangent vectors of a given rank.

Exercise 2.23. In Example 2.6, we showed that the open Schubert cell $\tilde{\Sigma}_1 = \Sigma_1 \setminus (\Sigma_2 \cap \Sigma_{1,1})$ is isomorphic to affine space $\mathbb{A}^3$. For each of the remaining Schubert indices a, b, show that the Schubert cell $\tilde{\Sigma}_{a,b} \subset \mathbb{G}(1,3)$ is isomorphic to affine space of dimension $4 - a - b$.

Exercise 2.24. Consider the Schubert cycle

$$\Sigma_1 = \{\Lambda : \Lambda \cap L \neq \varnothing\} \subset G.$$

Suppose $\Lambda \in \Sigma_1$ and that $\Lambda \neq L$, so that $\Lambda \cap L$ is a point q and the span $\overline{\Lambda, L}$ a plane K. Show that Λ is a smooth point of Σ_1, and that its tangent space is

$$T_\Lambda(\Sigma_1) = \{\varphi \in \mathrm{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda}) : \varphi(\tilde{q}) \subset \tilde{K}/\tilde{\Lambda}\}.$$

Exercise 2.25. Consider the Schubert cycle

$$\Sigma_{2,1} = \Sigma_{2,1}(p, H) = \{\Lambda : p \in \Lambda \subset H\} \subset G.$$

Show that $\Sigma_{2,1}$ is smooth, and that its tangent space at a point Λ is

$$T_\Lambda(\Sigma_{2.1}) = \{\varphi \in \mathrm{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda}) : \varphi(\tilde{p}) = 0 \text{ and } \mathrm{Im}(\varphi) \subset \tilde{H}/\tilde{\Lambda}\}.$$

Exercise 2.26. Use the preceding two exercises to show in arbitrary characteristic that general Schubert cycles Σ_1 and $\Sigma_{2,1} \subset \mathbb{G}(1,3)$ intersect transversely, and deduce the equality $\deg(\sigma_1 \cdot \sigma_{2,1}) = 1$.

Exercise 2.27. Let L_1, L_2 and $L_3 \subset \mathbb{P}^3$ be pairwise disjoint lines (Figure 2.5).

(a) Show that the union $L_1 \cup L_2 \cup L_3$ lies on a quadric surface Q;

(b) Observe that any line in $\mathbb{P}^3$ meeting all three lines L_1, L_2 and L_3 must lie in Q; conclude that Q is unique, and consists of the union of the lines meeting L_1, L_2 and L_3;

(c) Show that through any point of Q there is a unique line meeting L_1, L_2 and L_3; and finally

(d) If $L_4 \subset \mathbb{P}^3$ is a general line, deduce that there are exactly two lines meeting L_1, L_2, L_3 and L_4.

Exercise 2.28. Let $L_1, \ldots, L_4 \subset \mathbb{P}^3$ be four pairwise skew lines, and $\Lambda \subset \mathbb{P}^3$ a line meeting all four; set

$$p_i = \Lambda \cap L_i \quad \text{and} \quad H_i = \overline{\Lambda, L_i}.$$

Use the result of Exercise 2.24 to show that $[\Lambda] \in G$ fails to be a transverse point of intersection of the Schubert cycles $\Sigma_1(L_i)$ exactly when the cross-ratio of the four points $p_1, \ldots, p_4 \in \Lambda$ equals the cross-ratio of the four planes $H_1, \ldots, H_4$ in the pencil of planes containing Λ.

The following series of Exercises deals with a question raised in Section 2.3.2: if $C_1, \ldots, C_4 \subset \mathbb{P}^3$ are general translates of four curves in $\mathbb{P}^3$, do the corresponding cycles $\Gamma_{C_i} \subset \mathbb{G}(1,3)$ of lines meeting the C_i intersect transversely?

To start with, we have to identify the smooth locus of the cycle $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting a given curve C, and its tangent spaces at these

points; this is the content of the first exercise, which is a direct generalization of Exercise 2.24 above.

Exercise 2.29. Let $C \subset \mathbb{P}^3$ be any curve, and $L \subset \mathbb{P}^3$ a line meeting C at one smooth point p of C and not tangent to C. Show that the cycle $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting C is smooth at the point $[L]$, and that its tangent space at $[L]$ is the space of linear maps $\tilde{L} \to \mathbb{C}^4/\tilde{L}$ carrying the one-dimensional subspace $\tilde{p} \subset \tilde{L}$ to the one-dimensional subspace $(\tilde{\mathbb{T}}_p C + \tilde{L})/\tilde{L}$ of $\mathbb{C}^4/\tilde{L}$, where we have written $\mathbb{T}_p C$ for the projective line in $\mathbb{P}^3$ that is tangent to C at p.

Next, we have to verify that, for general translates C_i of any four curves, the corresponding cycles Γ_{C_i} are smooth at each of the points of their intersection. A key fact will be the irreducibility of the relevant incidence correspondence:

Exercise 2.30. Let $B_1, \ldots, B_4 \subset \mathbb{P}^3$ be four irreducible curves, and let $\varphi_1, \ldots, \varphi_4 \in PGL_4$ be four general automorphisms of $\mathbb{P}^3$; let $C_i = \varphi_i(B_i)$. Show that the incidence correspondence

$$\Phi = \left\{(\varphi_1, \ldots, \varphi_4, L) \ : \ L \cap \varphi_i(B_i) \neq \varnothing \ \forall i\right\} \subset (PGL_4)^4 \times \mathbb{G}(1,3)$$

is irreducible.

Using this, we can prove the following Exercise—asserting that for general translates C_i of four given curves and any line L meeting all four, the cycles Γ_{C_i} are smooth at $[L]$—simply by exhibiting a single collection $(\varphi_1, \ldots, \varphi_4, L)$ satisfying the conditions in question:

Exercise 2.31. Let $B_1, \ldots, B_4 \subset \mathbb{P}^3$ be four curves, and $\varphi_1, \ldots, \varphi_4 \subset PGL_4$ four general automorphisms of $\mathbb{P}^3$; let $C_i = \varphi_i(B_i)$. Show that the set of lines $L \subset \mathbb{P}^3$ meeting C_1, C_2, C_3 and C_4 is finite; and that for any such L

(a) L meets each C_i at only one point p_i;

(b) p_i is a smooth point of C_i; and

(c) L is not tangent to C_i for any i

Exercise 2.32. Let $C_1, \ldots, C_4 \subset \mathbb{P}^3$ be any four curves, and $L \subset \mathbb{P}^3$ a line meeting all four and satisfying the conclusions of Exercise **??**. Use the result of Exercise **??** to give a necessary and sufficient condition that the four cycles $\Gamma_{C_i} \subset \mathbb{G}(1,3)$ intersect transversely at $[L]$, and show directly that this condition is satisfied when the C_i are general translates of given curves.

The following three exercises deal with the approach, described in Section 2.3.3, to calculating the class of the variety $\Sigma_C \subset \mathbb{G}(1,3)$ of lines

incident to a space curve $C \subset \mathbb{P}^3$ by specialization. Recall from that section that we choose a general plane $H \subset \mathbb{P}^3$ meeting C at d points p_i and a general point $q \in \mathbb{P}^3$, and let $\{A_t\}$ be the one-parameter subgroup of PGL_4 with attractor q and repellor H; we let $C_t = A_t(C)$ and take $\Psi \subset \mathbb{A}^1 \times \mathbb{G}(1,3)$ to be the closure of the locus

$$\tilde{\Psi} = \{(t, \Lambda) \mid t \neq 0 \text{ and } \Lambda \cap C_t \neq \varnothing\}.$$

Exercise 2.33. Show that the support of the fiber Ψ_0 is exactly the union of the Schubert cycles $\Sigma_1(\overline{p_i, q})$

Exercise 2.34. Show that Ψ_0 has multiplicity 1 at a general point of each Schubert cycle $\Sigma_1(\overline{p_i, q})$

Exercise 2.35. Suppose now that $C \subset \mathbb{P}^3$ is a general rational quartic curve. Describe the flat limit of the family of cycles $\Gamma_{C_t} \subset \mathbb{G}(1,3)$, and in particular its embedded components.

Exercise 2.36. Let $C \subset \mathbb{P}^r$ be a smooth curve. Show that the rational map $\varphi : C^{(2)} \to \mathbb{G}(1,r)$ sending a pair of distinct points $p, q \in C$ to the line $\overline{pq}$ actually extends to a regular map on all of $C^{(2)}$ by sending the pair $2p$ to the projective tangent line $\mathbb{T}_p C$. Use this to show that the image of φ coincides with the locus of lines $L \subset \mathbb{P}^r$ such that the scheme-theoretic intersection $L \cap C$ has degree at least 2.

Exercise 2.37. Show by example that the conclusion of the preceding exercise is false in general if we do not assume $C \subset \mathbb{P}^r$ smooth. Is it still true if we allow C to have mild singularities, such as nodes?

Exercise 2.38. Similarly, show by example that the conclusion of Exercise 2.36 is false if we consider higher-dimensional secant planes: for example, the image of the map

$$\varphi : C^{(3)} \to \mathbb{G}(2,r)$$
$$p + q + r \mapsto \overline{p, q, r}$$

need not coincide with the locus of 2-planes $\Lambda \subset \mathbb{P}^r$ whose scheme-theoretic intersection with C has degree at least 3.

Exercise 2.39. Show that the smooth locus of $S = S_1(C)$ contains the locus of lines $L \subset \mathbb{P}^3$ such that the scheme-theoretic intersection $L \cap C$ consists of two reduced points, and for such a line L identify the tangent plane $T_L S$ as a subspace of $T_L \mathbb{G}$. (When is a tangent line to C a smooth point of $S_1(C)$?)

Exercise 2.40. Use the result of the preceding Exercise to show that if C and $C' \subset \mathbb{P}^3$ are two general twisted cubic curves, then the varieties $S_1(C)$ and $S_1(C') \subset \mathbb{G}(1,3)$ of chords to C and C' intersect transversely.

Exercise 2.41. Let $C \subset \mathbb{P}^3$ be a smooth, nondegenerate curve, and let L and $M \subset \mathbb{P}^3$ be general lines.

(a) Find the number of chords to C meeting both L and M by applying the result above; and

(b) Verify this count by considering the product morphism

$$\pi_L \times \pi_M : C \to \mathbb{P}^1 \times \mathbb{P}^1$$

(where $\pi_L, \pi_M : C \to \mathbb{P}^1$ are the projections from L and M) and comparing the arithmetic and geometric genera of the image curve.

Exercise 2.42. Let $C \subset \mathbb{P}^3$ be a twisted cubic curve. Show that the locus $S_1(C) \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ of chords to C is the Veronese surface.

Exercise 2.43. Show that the twisted cubic and the elliptic quartic curves are the only smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^3$ such that $S_1(C)$ is smooth. (Hint: if C has no trisecant lines, the projection $\pi_p : C \to \mathbb{P}^2$ of C to a plane from a point $p \in C$ gives an isomorphism of C with a smooth plane curve of degree $d-1$; now use adjunction in the plane to argue that if this is the case for all $p \in C$ then either C is rational or $d = 4$.)

********need to rewrite the following to reflect the discussion in Section 2.3.5********

Exercise 2.44. Let $C \subset \mathbb{P}^3$ be a smooth, irreducible nondegenerate curve of degree d. In Exercise **??** above, we constructed a family of curves $C_t \subset \mathbb{P}^3$ with $C_1 = C$ and C_0 the union of d lines through a point. If we use the corresponding family of cycles $S_1(C_t) \subset \mathbb{G}(1,3)$ to describe a rational equivalence between $S_1(C)$ and a union of Schubert cycles, one of the two coefficients in the formula above for the class of $S_1(C)$ emerges, but the other doesn't. What's going on?

Exercise 2.45. Let $C \subset \mathbb{P}^3$ be a smooth nondegenerate curve of degree d and genus g, and let $T(C) \subset \mathbb{G}(1,3)$ be the locus of its tangent lines. Find the class $[T(C)] \in A^3(\mathbb{G}(1,3))$ of $T(C)$ in the Grassmannian $G(1,3)$.

Exercise 2.46. Let $C \subset \mathbb{P}^3$ be a smooth nondegenerate curve of degree d and genus g, and let $S \subset \mathbb{P}^3$ be a general surface of degree e. How many tangent lines to C are tangent to S?

3

Grassmannians in General

Keynote Questions

(a) A generalization of the problem of lines meeting four lines in $\mathbb{P}^3$: if $V_1, \ldots, V_4 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ are four general n-planes, how many lines $L \subset \mathbb{P}^{2n+1}$ meet all four?

(b) Let $C \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ be a twisted cubic curve in the Grassmannian of lines in $\mathbb{P}^3$, and let

$$S = \bigcup_{[\Lambda] \in C} \Lambda \subset \mathbb{P}^3$$

be the surface swept out by the lines corresponding to points of C. What is the degree of S? How can we describe the geometry of S?

(c) If Q_1 and $Q_2 \subset \mathbb{P}^4$ are general quadric hypersurfaces, and $S = Q_1 \cap Q_2$ their surface of intersection, how many lines does S contain? More generally, if Q_1 and Q_2 are general quadric hypersurfaces in $\mathbb{P}^{2n}$ and $X = Q_1 \cap Q_2$, how many $(n-1)$-planes does X contain?

(d) If $G \subset \mathbb{P}^N$ is the Grassmannian $G(1,n)$ of lines in $\mathbb{P}^n$, embedded in projective space via the Plücker embedding, what is the degree of G?

We will now extend the ideas developed for lines in $\mathbb{P}^3$ to all dimensions. For simplicity of notation, we work with the formulation for vector subspaces in a vector space, rather than for linear subspaces of a projective space; the easy translation is left to the reader.

3.1 Schubert Cells and Schubert Cycles

Let $G = G(k, V)$ be the Grassmannian of k-dimensional subspaces of an n-dimensional vector space V. As suggested by the example of $\mathbb{G}(1, 3)$, we introduce a *flag* $\mathcal{V}$ in V, that is, a nested sequence of subspaces

$$0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

with $\dim V_i = i$. Now, for any sequence $a = (a_1, \ldots, a_k)$ of integers with

$$n - k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$$

we define the *Schubert cycle* $\Sigma_a(\mathcal{V}) \subset G$ to be the closed subset

$$\Sigma_a(\mathcal{V}) = \{\Lambda : \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i \ \forall i\} \subset G.$$

Exactly as before, Proposition 5.21 shows that the class $[\Sigma_a(\mathcal{V})] \in A(G)$ doesn't depend on the choice of flag, since any two flags differ by an element of GL_n. In general, when dealing with a property independent of the choice of $\mathcal{V}$, we'll shorten the name to Σ_a and we define

$$\sigma_a := [\Sigma_a] \in A(G).$$

To shorten the notation, we generally suppress trailing zeros in the indices, and write $\Sigma_{a_1,\ldots,a_s}$ in place of $\Sigma_{a_1,\ldots,a_s,0,\ldots,0}$. Also, we use the shorthand Σ_{p^r} to denote $\Sigma_{p,\ldots,p}$ with r indices equal to p. We shall see (Theorem **??**) that the classes σ_a form a free basis of $A^*(G(k, n))$.

To elucidate this definition, suppose for the moment that $\Lambda \subset V$ is any k-plane, and consider the sequence of subspaces of Λ:

$$0 \subset (V_1 \cap \Lambda) \subset (V_2 \cap \Lambda) \subset \cdots \subset (V_{n-1} \cap \Lambda) \subset (V_n \cap \Lambda) = \Lambda \tag{3.1}$$

Note that each subspace is either equal to the one before it, or of dimension one greater, the latter occurring exactly k times. For a general k-plane $\Lambda \subset V$, the subspaces $V_i \cap \Lambda$ will all be zero for $i \leq n-k$, with $\dim(V_{n-k+i} \cap \Lambda) = i$ for $i = 1, \ldots, k$; and we can interpret the definition of the Schubert cycle $\Sigma_a(\mathcal{V})$ as saying *it's the locus of planes* Λ *for which the* i^{th} *jump in the sequence (3.1) above occurs at least* a_i *steps early.*

Special cases include:

- The cycle of k-subspaces Λ meeting a given space of dimension l nontrivially is the Schubert cycle

  $$\Sigma_{n-k+1-l}(\mathcal{V}) = \{\Lambda : \Lambda \cap V_l \neq 0\}.$$

  In particular, the Schubert cycle of k-dimensional subspaces meeting a given $(n-k)$-dimensional subspace nontrivially is

  $$\Sigma_1(\mathcal{V}) = \{\Lambda : \Lambda \cap V_{n-k} \neq 0\}.$$

  This is a hyperplane section of G in the Plücker embedding (but not every hyperplane section of G is of this form).

- The sub-Grassmannian of k-subspaces contained in a given l-subspace is the Schubert cycle

$$\Sigma_{(n-l)^k}(\mathcal{V}) = \{\Lambda : \Lambda \subset V_l\}.$$

  Similarly, the sub-Grassmannian of planes containing a given plane is the Schubert cycle

$$\Sigma_{(n-k)^r}(\mathcal{V}) = \{\Lambda : V_r \subset \Lambda\}.$$

The cycles $\Sigma_{i,0,\dots 0}$, defined for $0 \le i \le n-k$ and the cycles Σ_{1^i}, defined for $0 \le i \le k$, are called *special* Schubert cycles. As we shall see in Section 7.5, these classes are intimately connected with the tautological sub and quotient bundles on $G(k,n)$, and each of these sequences of cycles forms a minimal generating set for the algebra $A^*(G(k,n))$.

Our indexing of the Schubert cycles is by no means the only one in use, but it has several good properties:

- It reflects the partial order of the Schubert cycles by inclusion: if we order the indices termwise, that is, $(a_1,\dots,a_k) \le (a'_1,\dots,a'_k)$ if and only if $a_i \le a'_i$ for $1 \le i \le k$, then

$$\Sigma_a \subset \Sigma_b \iff a \ge b.$$

  This follows immediately from the definition.

- It makes the codimension of a Schubert cycle easy to compute: By Theorem 3.1 below

$$\operatorname{codim}(\Sigma_a \subset G) = \sum a_i,$$

  so that $|a| := \sum a_i$ is the degree of $\sigma_a := [\Sigma_a]$ in $A^*(G(k,n))$.

- It is preserved under pullback via the obvious inclusions $i : G(k,n) \hookrightarrow G(k,n+1)$ and $j : G(k,n) \hookrightarrow G(k,n+1)$, that is,

$$i^*(\sigma_a) = \sigma_a \text{ and } j^*(\sigma_a) = \sigma_a.$$

  Here we adopt the convention that when $a_1 > n-k$, or when $a_{k+1} > 0$, we take $\sigma_a = 0$ as a class in $A(G(k,n))$. (This is consistent; for example, $\Sigma_{n-k+1} \subset G(k,n+1)$ is the subset of the k-planes containing a fixed general point, and thus the intersection of Σ_{n-k+1} with the $G(k,n)$ of subspaces in a fixed general n-plane is empty, so that $j^*\sigma_{n-k+1} = 0 \in A^*(G(k,n))$.) It follows that if we establish a formula

$$\sigma_a\sigma_b = \sum \gamma_{a,b;c}\,\sigma_c$$

  in the Chow ring of $G(k,n)$, the same formula holds true in all $G(k',n')$ with $k' \le k$ and $n' \le n$. Whenever it happens that i^* or j^* is an isomorphism on $A^{|a|+|b|}$, the formula will also hold in $A^*(G(k,n+1))$ or $A^*(G(k+1,n+1))$, respectively. Conditions for this are given in Exercise 3.27.

There is a natural isomorphism $G(k,V) \cong G(n-k,V^*)$ obtained by associating to a k-dimensional subspace $\Lambda \subset V$ the $(n-k)$-dimensional subset $\Lambda^\perp \subset V^*$ consisting of all those linear functionals on V that annihilate Λ. To see that this is an algebraic isomorphism most directly, use the result from linear algebra given in Exercise 3.11. (One can also see that these two spaces represent the same functor.) This duality carries each Schubert cycle to another Schubert cycle. For example, one checks immediately that $\Sigma_i(W)$, which is the set of k-planes Λ meeting a fixed $(n-k+1-i)$-plane W nontrivially, is carried into the Schubert cycle Σ_{1^i} of $(n-k)$-planes Λ' such that $\dim(\Lambda' \cap W^\perp) \geq i$; that is, such that $\Lambda' + W^\perp \subsetneq V$.

As in the case of $\mathbb{G}(1,3) = G(2,4)$, the Grassmannian $G(k,n)$ has an affine stratification. To see this, set

$$\tilde{\Sigma}_a = \Sigma_a \setminus \Big(\bigcup_{\substack{b \geq a \\ b \neq a}} \Sigma_b \Big).$$

The $\tilde{\Sigma}_a$ are called *Schubert cells*.

Theorem 3.1. *The locally closed subset $\tilde{\Sigma}_a \subset G$ is isomorphic to the affine space $\mathbb{A}^{k(n-k)-|a|}$; in particular it is smooth. The tangent space to $\tilde{\Sigma}_a$ at a point $[\Lambda]$ is the subset of $T_{G,[\Lambda]} = Hom(\Lambda, V/\Lambda)$ consisting of those elements φ that send*

$$V_{n-k+i-a_i} \cap \Lambda \ \subset \ \Lambda$$

into

$$\frac{V_{n-k+i-a_i} + \Lambda}{\Lambda} \ \subset \ V/\Lambda$$

for $i = 1, \ldots, k$.

Proof. Choose a basis $(e_1, \ldots, e_n)$ for V so that

$$V_i = \langle e_1, \ldots, e_i \rangle.$$

Now, suppose $[\Lambda] \in \tilde{\Sigma}_a$, and consider the sequence (3.1) of subspaces of Λ; by definition, the first nonzero subspace in the sequence will be $V_{n-k+1-a_1} \cap \Lambda$, the first of dimension 2 will be $V_{n-k+2-a_2} \cap \Lambda$, and so on. We're going to choose a basis $(v_1, \ldots, v_k)$ for Λ, accordingly, with $v_1 \in V_{n-k+1-a_1}$, $v_2 \in V_{n-k+2-a_2}$, and so on. In terms of this basis, and the basis $(e_1, \ldots, e_n)$ for V, the matrix representative of Λ will look like

$$\begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 \end{pmatrix}$$

(This matrix corresponds to the case $k = 4$, $n = 9$ and $a = (3,2,2,1)$.) Note that if Λ were general in G, and we chose a basis for Λ in this way,

the corresponding matrix would look like

$$\begin{pmatrix} * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & * & * & * & * \end{pmatrix}.$$

Thus the Schubert indices a_i specify the number of "extra zeroes" in the corresponding row.

Next, since $\Lambda \in \tilde{\Sigma}_a$, we know that $v_i \notin V_{n-k+i-a_i-1}$, so the coefficient of $e_{n-k+i-a_i}$ in the expression of v_i as a linear combination of the e_α is nonzero; we can multiply v_i by a scalar to make this coefficient 1, and the matrix representative of Λ will now look like

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 1 & 0 \end{pmatrix}$$

with the 1s appearing in the $(n-k+i-a_i)^{\text{th}}$ column, $i = 1, \ldots, k$.

Finally, we can subtract a linear combination of $v_1, \ldots, v_{i-1}$ from v_i to kill the coefficients of $e_{n-k+j-a_j}$ in the expression of v_i as a linear combination of the e_α for $j < i$, to arrive at the matrix

$$A = \begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 1 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 1 & 0 \end{pmatrix}$$

This is in fact the unique matrix representative for Λ whose $(n-k+1-a_1, \ldots, n-a_k)^{\text{th}}$ submatrix is the identity matrix. Thus, for one of the standard affine open subsets $U \subset G$—specifically, the locus of planes Λ complementary to the span of the remaining $n-k$ basis vectors—we have

$$\tilde{\Sigma}_a \subset U \subset G$$

with the subset $\tilde{\Sigma}_a$ a coordinate subspace of $U \cong \mathbb{A}^{k(n-k)}$.

The number of entries marked with $*$ in the matrix A is

$$\sum_i n-k+i-a_i-i = k(n-k) - \sum_i a_i,$$

proving the dimension statement.

The statement about tangent spaces also follows from the explicit coordinate description of Σ_a, since the tangent space to an affine space may be identified with the corresponding vector space. In our case, the homomorphisms corresponding to tangent directions at Λ (really, the projectivized tangent space) are exactly those that can be lifted to a map $\Lambda \to V$ that has matrix representation in the form of the transpose ${}^tA$ of A. □

Here is a typical and useful application of the description of tangent spaces:

Corollary 3.2.

$$(\sigma_{n-k})^k = (\sigma_{1^k})^{n-k} = \sigma_{(n-k)^k} \in A^{k(n-k)}(G);$$

that is, $(\sigma_{n-k})^k$ and $(\sigma_{1^k})^{n-k}$ are both equal to the class of a point in the Chow ring of $G(k,n)$.

Proof. Note that from the affine stratification, we know that $A_0(G(k,n)) = \mathbb{Z}$, so it suffices to show that the degree of each of $(\sigma_{n-k})^k$ and $(\sigma_{1^k})^{(n-k)}$ is 1. We'll do this by intersecting Schubert cycles associated to general flags, and using the description of tangent spaces to show that the intersection is transverse (note that in characteristic 0, we don't need the the description of tangent spaces—we can just invoke Kleiman's Theorem).

We regard $G(k,n)$ as the variety of k-dimensional subspaces Λ of the n-dimensional vector space V. If $H \subset V$ is a codimension 1 subspace, then

$$\Sigma_{1^k}(H) = \{\Lambda \subset V \mid \Lambda \subset H\}$$

and the tangent space to $\Sigma_{1^k}(H)$ at the point corresponding to Λ is

$$T_{\Sigma_{1^k}(H),[\Lambda]} = \{\varphi \in Hom(\Lambda, V/\Lambda) = T_{G(k,n),[\Lambda]} \mid \varphi(\Lambda) \subset H\}.$$

If $H_1, \ldots, H_{n-k}$ are general codimension 1 subspaces, then there is a unique k-plane Λ in $\cap_{i=1}^k \Sigma_{1^k}(H_i)$, namely the intersection $\Lambda = \cap_{i=1}^k H_i$. Further, the intersection of the tangent spaces is the zero homomorphism, so the intersection is transverse. This proves that $(\sigma_{1^k})^{(n-k)}$ is the class of a point.

To prove the corresponding statement for $(\sigma_{n-k})^k$ we can make an analogous argument (as suggested in Exercise 3.12), or we can simply use duality: the isomorphism $G(k,n) \cong G(n-k,n)$ introduced above carries $\Sigma_{1^k}(H)$ to $\Sigma_{n-k}(H^\perp)$, as we have already remarked, so the transversality of a collection of cycles of the former type is equivalent to the transversality of the corresponding collection of dual cycles. □

By Theorem **??**, the classes σ_a for $a = (a_1, \ldots, a_k)$ with $n-k \geq a_1 \geq \cdots \geq a_k \geq 0$ and $|a| = m$ generate $A^m(G(k,n))$ additively. In particular, $A^{k(n-k)}(G)$ is generated by the class of a point, $[\Sigma_{(n-k)^k}] = \sigma_{(n-k)^k}$, and thus any two points of G are linearly equivalent (as we could see directly from Proposition 5.21). The existence of the degree homomorphism $\deg : A^{k(n-k)} \to \mathbb{Z}$ that counts points shows that $A^{k(n-k)}(G)$ is actually free on the generator $\sigma_{(n-k)^k}$.

Similarly, $A^0 = \mathbb{Z} \cdot [G] = \mathbb{Z} \cdot \sigma_0$ is free on σ_0. We shall soon see that, more generally, the σ_a with $a = a_1, \ldots, a_k$ satisfying $(n-k) \geq a_1 \geq \cdots \geq a_k \geq 0$ form a free basis of $A^*(G)$.

3.2 Intersections in complementary dimension

As in the case of the Grassmannian $\mathbb{G}(1,3)$, we start our description of the Chow ring of G by evaluating intersections of Schubert cycles in complementary codimension. We can do this in characteristic 0 simply by taking Schubert cycles defined relative to two general flags and counting the number of points of intersection; by Kleiman's theorem, this will equal the degree of the intersection. In characteristic $p > 0$, we have to verify directly that the intersections are transverse. In general, we'll omit this verification; the concerned reader can supply it using the description in Theorem 3.1 of the tangent spaces to Schubert cycles.

As before, let $G = G(k, V)$ be the Grassmannian of k-dimensional linear subspaces of an n-dimensional vector space V, and choose two general flags $\mathcal{V}$ and $\mathcal{W}$ on V, consisting of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

and

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset W_n = V.$$

Let $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ be Schubert cycles of complementary codimension, that is, with

$$|a| + |b| = \dim G = k(n-k).$$

Proposition 3.3. *If $|a| + |b| = k(n-k)$ then $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ intersect transversely in a unique point if $a_i + b_{k+1-i} = n-k$ for each $i = 1, \ldots, k$, and are disjoint otherwise. Thus*

$$\deg \sigma_a \sigma_b = \begin{cases} 1 & \text{if } a_i + b_{k-i+1} = n-k \text{ for all } i, \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since the two flags $\mathcal{V}$ and $\mathcal{W}$ are general, the Schubert cycles will meet generically transversely. This is guaranteed by Kleiman's Theorem 5.20 in characteristic zero, or by a computation of tangent spaces in arbitrary characteristic. Thus

$$\begin{aligned} \deg \sigma_a \sigma_b &= \#\big(\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})\big) \\ &= \#\big\{\Lambda : \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i \text{ and } \dim(W_{n-k+i-b_i} \cap \Lambda) \geq i \ \forall i\big\}. \end{aligned}$$

To evaluate the cardinality of this set, consider the conditions in pairs: that is, for each i, consider the i^{th} condition associated to the Schubert cycle $\Sigma_a(\mathcal{V})$:

$$\dim(V_{n-k+i-a_i} \cap \Lambda) \geq i$$

in combination with the $(k-i+1)^{\text{st}}$ condition associated to $\Sigma_b(\mathcal{W})$:

$$\dim(W_{n-i+1-b_{k-i+1}} \cap \Lambda) \geq k-i+1.$$

If these conditions are both satisfied, then the subspaces $V_{n-k+i-a_i} \cap \Lambda$ and $W_{n-i+1-b_{k-i+1}} \cap \Lambda$, having greater than complementary dimension in Λ, must have nonzero intersection; in particular, we must have

$$V_{n-k+i-a_i} \cap W_{n-i+1-b_{k-i+1}} \neq 0,$$

and since the flags $\mathcal{V}$ and $\mathcal{W}$ are general, this in turn says we must have

$$n-k+i-a_i+n-i+1-b_{k-i+1} \geq n+1$$

or in other words

$$a_i + b_{k-i+1} \leq n-k.$$

If we have equality in this last inequality, then the subspaces $V_{n-k+i-a_i}$ and $W_{n-i+1-b_{k-i+1}}$ will meet in a one-dimensional vector space Γ_i, which will necessarily be contained in Λ.

We have thus seen that $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ will be disjoint unless $a_i + b_{k-i+1} \leq n-k$ for all i. But from the equality

$$|a| + |b| = \sum_{i=1}^{k} (a_i + b_{k-i+1}) = k(n-k)$$

we see that if $a_i + b_{k-i+1} \leq n-k$ for all i then we must have $a_i + b_{k-i+1} = n-k$ for all i. Moreover, in this case any Λ in the intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$ must contain each of the k subspaces Γ_i, so there is a unique such Λ, equal to the sum of these one-dimensional spaces, as required. □

Corollary 3.4. *The classes of the Schubert cycles form a free basis for $A^*(G)$. In fact, the intersection forms $A^m(G) \times A^{\dim G - m}(G) \to \mathbb{Z}$ have the Schubert classes as dual bases.* □

Corollary 3.4 suggests a general approach to determining the coefficients in the expression of the class of a cycle as a linear combination of Schubert classes: if $\Gamma \subset G$ is any cycle of pure codimension m, we can write

$$[\Gamma] = \sum_{|a|=m} \gamma_a \sigma_a.$$

To find the coefficient γ_a, we intersect both sides with the Schubert cycle $\Sigma_{n-k-a_k,\ldots,n-k-a_1}(\mathcal{V})$ for a general flag $\mathcal{V}$; we have

$$\begin{aligned} \gamma_a &= \deg([\Gamma] \cdot \sigma_{n-k-a_k,\ldots,n-k-a_1}) \\ &= \#\big(\, \Gamma \cap \Sigma_{n-k-a_k,\ldots,n-k-a_1}(\mathcal{V}) \,\big). \end{aligned}$$

In fact, we have used exactly this approach—called the method of *undetermined coefficients*—in calculating classes of various cycles in $\mathbb{G}(1,3)$ in the preceding chapter; Corollary 3.4 says that it is more generally applicable in any Grassmannian. As an application, we'll use this method to describe the Chow rings of Grassmannians of the form $G(2,n)$ in the following section.

3.2.1 Varieties swept out by linear spaces

Let $C \subset \mathbb{G}(k,n)$ be a curve, and consider the variety $X \subset \mathbb{P}^n$ swept out by the linear spaces corresponding to points of C: that is,

$$X = \bigcup_{[\Lambda] \in C} \Lambda \ \subset \ \mathbb{P}^n.$$

We'd like to relate the geometry of X to that of C; in particular, Keynote Question (b) asks us to find the degree of X in case $C \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ is a twisted cubic curve.

Observe to begin with that X is indeed a closed subvariety of $\mathbb{P}^n$: if

$$\Gamma = \{(\Lambda, p) \mid p \in \Lambda\} \subset \mathbb{G}(k,n) \times \mathbb{P}^n$$

is the universal k-plane over $\mathbb{G}(k,n)$, as described in Exercise 2.16, and $\alpha : \Gamma \to \mathbb{G}(k,n)$ and $\beta : \Gamma \to \mathbb{P}^n$ are the projections, then we can write

$$X = \beta\big(\alpha^{-1}(C)\big).$$

Now, suppose that a general point $x \in X$ lies on a unique k-plane $\Lambda \in C$—that is, the map $\beta : \alpha^{-1}(C) \to X \subset \mathbb{P}^n$ is birational, so that in particular $\dim(X) = k+1$. The degree of X is the number of points of intersection of X with a general $(n-k-1)$-plane $V \subset \mathbb{P}^n$; since these points are general points of X, the number is the number of k-planes Λ that meet V. In other words, we have

$$\begin{aligned} \deg(X) &= \#(X \cap V) \\ &= \#(C \cap \Sigma_1(V)) \\ &= \deg([C] \cdot \sigma_1) \quad \text{(by Kleiman)} \\ &= \deg(C) \end{aligned}$$

where by the degree of C we mean the degree under the Plücker embedding of $\mathbb{G}(k,n)$. Thus, for example, the answer to Keynote Question (b) is that the surface $X \subset \mathbb{P}^3$ swept out by the lines corresponding to a twisted cubic $C \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ is cubic. We will see more about the geometry of X in Exercises 3.16-3.19.

If $Z \subset \mathbb{G}(k,n)$ is a variety of any dimension l, we can similarly form the variety $X \subset \mathbb{P}^n$ swept out by the planes of Z. Its degree in general is expressible in terms of the Schubert coefficients of the class $[Z] \in A_l(\mathbb{G}(k,n))$ (it's not in general equal to the degree of Z); this is the content of Exercise **??**.

3.3 Grassmannians of Lines

Let $G = G(2, V)$ be the Grassmannian of 2-dimensional subspaces of an n-dimensional vector space V (or, equivalently, lines in the projective space $\mathbb{P}V \cong \mathbb{P}^{n-1}$). The Schubert cycles on G are of the form

$$\Sigma_{a_1,a_2}(V) = \{\Lambda : \Lambda \cap V_{n-1-a_1} \neq 0 \text{ and } \Lambda \subset V_{n-a_2}\}.$$

The intersection products of the corresponding Schubert classes $\sigma_{a_1,a_2} \in A(G)$ are given by the following Proposition.

Proposition 3.5.

$$\sigma_{a_1,a_2}\sigma_{b_1,b_2} = \sum_{\substack{|c|=|a|+|b| \\ a_1+b_1 \geq c_1 \geq a_1,b_1}} \sigma_{c_1,c_2}$$

Proof. Let's start with the simplest cases, where the intersection of general Schubert cycles is again a Schubert cycle: if $b_1 = b_2 = b$, then the Schubert cycle

$$\Sigma_{b,b}(\mathcal{W}) = \{\Lambda : \Lambda \subset W_{n-b}\}$$

so that for any a_1, a_2 we have

$$\begin{aligned}
\Sigma_{a_1,a_2}(\mathcal{V}) \cap \Sigma_{b,b}(\mathcal{W}) &= \left\{ \Lambda : \begin{array}{l} \Lambda \cap V_{n-1-a_1} \neq 0; \\ \Lambda \subset V_{n-a_2}; \text{ and} \\ \Lambda \subset W_{n-b} \end{array} \right\} \\
&= \left\{ \Lambda : \begin{array}{l} \Lambda \cap (V_{n-1-a_1} \cap W_{n-b}) \neq 0 \text{ and} \\ \Lambda \subset (V_{n-a_2} \cap W_{n-b}). \end{array} \right\} \\
&= \Sigma_{a_1+b,a_2+b}\big(V_{n-1-a_1} \cap W_{n-b}, V_{n-a_2} \cap W_{n-b}\big).
\end{aligned}$$

Thus, we have

$$\sigma_{a_1,a_2}\sigma_{b,b} = \sigma_{a_1+b,a_2+b}. \tag{3.2}$$

This reduces the general case to the crucial case of the product $\sigma_{a,0}\sigma_{b,0}$: for any a_1, a_2 and b_1, b_2 we can write

$$\begin{aligned}
\sigma_{a_1,a_2}\sigma_{b_1,b_2} &= (\sigma_{a_1-a_2,0}\sigma_{a_2,a_2})(\sigma_{b_1-b_2,0}\sigma_{b_2,b_2}) \\
&= \sigma_{a_1-a_2,0}\,\sigma_{b_1-b_2,0}\,\sigma_{a_2+b_2,a_2+b_2}
\end{aligned}$$

and if we can evaluate the product of the first two terms in the last expression, we can use (3.2) to finish the calculation.

To evaluate the product of a pair of Schubert classes of the form $\sigma_a = \sigma_{a,0}$ and $\sigma_b = \sigma_{b,0}$, we use the approach suggested by Corollary 3.4: that is, we write

$$\sigma_a\sigma_b = \sum \gamma_{a,b;c}\,\sigma_c$$

where the coefficient $\gamma_{a,b;c}$ is equal to the triple intersection

$$\gamma_{a,b;c} = \deg\bigl(\sigma_a\sigma_b\sigma_{n-2-c_2,n-2-c_1}\bigr) \in A^{2n-4}.$$

If $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are general flags this will be equal to the cardinality of the intersection

$$\begin{aligned}\gamma_{a,b;c} &= \#\left(\Sigma_a(\mathcal{U}) \cap \Sigma_b(\mathcal{V}) \cap \Sigma_{n-2-c_2,n-2-c_1}(\mathcal{W})\right)\\ &= \#\left\{\Lambda :\ \begin{array}{l}\Lambda \cap U_{n-1-a} \neq 0;\\ \Lambda \cap V_{n-1-b} \neq 0;\ \text{and}\\ \Lambda \cap W_{c_2+1} \neq 0 \text{ and } \Lambda \subset W_{c_1+2}\end{array}\right\}\end{aligned}$$

Points $[\Lambda]$ in this intersection thus correspond to 2-planes $\Lambda \subset W_{c_1+2}$ meeting each of the three subspaces

$$U_{n-1-a} \cap W_{c_1+2}, \quad V_{n-1-b} \cap W_{c_1+2} \quad \text{and} \quad W_{c_2+1},$$

which are general subspaces of W_{c_1+2} of dimensions c_1+1-a, c_1+1-b and c_2+1 respectively. In order for the intersection to be nonempty, of course, all three must be nonzero, whence the inequalities $c_1 \geq a$ and $c_1 \geq b$ (given that $c_1 + c_2 = a + b$, the inequality $a + b \geq c_1$ is automatic). Conversely, since the sum of the dimensions of these subspaces of W_{c_1+2} is

$$\begin{aligned}(c_1+1-a)+(c_1+1-b)+(c_2+1) &= (c_1+c_2-a-b)+c_1+3\\ &= c_1+3\\ &= \dim W_{c_1+2}+1\end{aligned}$$

there will be exactly one linear relation among them; that is, a linear relation $u+v+w=0$ with $u \in U_{n-1-a} \cap W_{c_1+2}$, $v \in V_{n-1-b} \cap W_{c_1+2}$ and $w \in W_{c_2+1}$. The span of the three vectors u, v and w will then be the unique 2-plane meeting all three, so the cardinality of the intersection is 1. We thus have

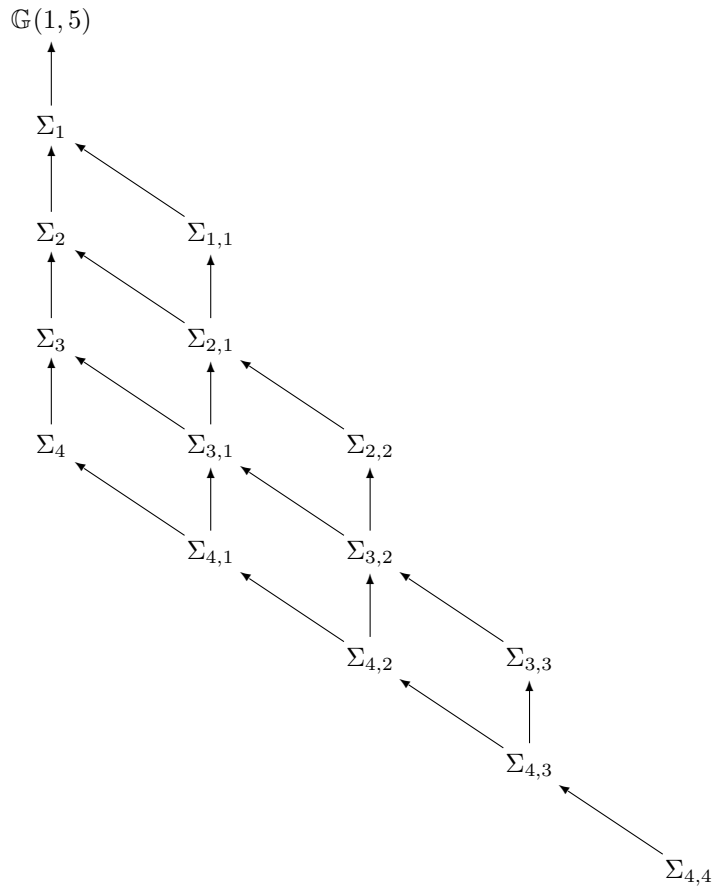
$$\sigma_a\sigma_b = \sum_{\substack{c_1+c_2=a+b\\ a+b\geq c_1\geq a,b}} \sigma_{c_1,c_2}$$

and the general statement follows as indicated above. □

We can use this description of the Chow ring of $\mathbb{G}(1,n)$ (and a little combinatorics) to answer the Keynote Question (d): what is the degree of the Grassmannian $\mathbb{G}(1,n)$ under the Plücker embedding? We observe first that, since the hyperplane class on $\mathbb{P}(\wedge^2 V)$ pulls back to the class $\sigma_1 \in A^1(\mathbb{G}(1,n))$, we have

$$\deg(\mathbb{G}(1,n)) = \deg\left(\sigma_1^{2n-2}\right).$$

To evaluate this product, we make a diagram of the Schubert cycles in $\mathbb{G}(1,n)$ relative to a given flag, with the inclusion relations indicated by vertical or diagonal lines (the diagram shown is the case $n=5$):



In terms of this diagram, the rule expressed in Proposition 3.5 for multiplication by σ_1 is simple: the product of any Schubert cycle $\Sigma_{a,b}$ with Σ_1 is just the sum of the Schubert cycles below it—that is, connected by an inclusion—in the next row. In particular, the degree $\deg(\sigma_1^{2n-2})$ of the Grassmannian is simply the number of paths through this diagram, starting with $\mathbb{G}(1,n)$ and ending with $\Sigma_{n-1,n-1}$. If we designate such a path by a sequence of $n-1$ "v"s and $n-1$ "d"s—corresponding to vertical and diagonal inclusions respectively—these are all the sequences of $n-1$ "v"s and $n-1$ "d"s satisfying the condition that, reading from left to right, there are never more "d"s than "v"s. Equivalently, if we associate to a "v" a left parenthesis and to a "d" a right parenthesis, this is the number of ways in which $n-1$ pairs of parentheses can appear in a grammatically correct sentence. This is called the $(n-1)^{\text{st}}$ *Catalan number*; a standard

combinatorial argument **ref Stanley** gives

$$c_{n-1} = \frac{(2n-2)!}{n!(n-1)!}$$

Note that this number also represents the answer to the enumerative problem: how many lines in $\mathbb{P}^n$ meet each of $2n-2$ general $(n-2)$-planes $V_1, \ldots, V_{2n-2} \subset \mathbb{P}^n$?

We will see in Section 3.6 below how to extend this to calculate the degrees of Grassmannians in general.

We can also use the description of $A(\mathbb{G}(1,n))$ given in Proposition 3.5 to answer the Keynote Question (a): if $V_1, \ldots, V_4 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ are four general n-planes, how many lines $L \subset \mathbb{P}^{2n+1}$ meet all four? The answer is the cardinality of the intersection $\bigcap \Sigma_n(V_i) \subset \mathbb{G}(1, 2n+1)$; given transversality—a consequence of Kleiman's theorem in characteristic 0, and checkable directly in arbitrary characteristic via the description of tangent spaces to Schubert cycles in Proposition **??**—this is just the degree of the product $\sigma_n^4 \in A(\mathbb{G}(1, 2n+1))$. Applying Proposition 3.5, we have

$$\sigma_n^2 = \sigma_{2n} + \sigma_{2n-1,1} + \cdots + \sigma_{n+1,n-1} + \sigma_{n,n};$$

since each term squares to the class of a point and all pairwise products are zero, we have

$$\deg(\sigma_n^4) = n+1,$$

and this is the answer to our question.

We will see in Exercise 3.21 another way to arrive at this number, in a manner analogous to the alternative solution to the four-line problem given in Exercise 2.27; and in Exercise 3.22 a nice geometric consequence.

3.3.1 Dynamic specialization

We now want to revisit the discussion, initiated in Section 2.2.4, of the method of *specialization*, in particular as it may be applied to determine the products of Schubert classes. This time we'll encounter situations where we need a stronger and more broadly applicable version of this technique, called *dynamic specialization*.

To recall: in Section 2.2.4 we described an alternative approach to establishing the relation $\sigma_1^2 = \sigma_{11} + \sigma_2$ in the Chow ring of the Grassmannian $\mathbb{G}(1,3)$. Instead of taking two general translates of the Schubert cycle $\Sigma_1(L) \subset \mathbb{G}(1,3)$—whose intersection was necessarily generically transverse, but the class of whose intersection required additional work to calculate—we considered the intersection $\Sigma_1(L) \cap \Sigma_1(L')$ where L and $L' \subset \mathbb{P}^3$ were not general, but incident lines. The tradeoff here is that now the intersection is visibly a union of Schubert cycles—specifically, if $p = L \cap L'$ is their

point of intersection and $H = \overline{L, L'}$ their span, we have

$$\Sigma_1(L) \cap \Sigma_1(L') = \Sigma_2(p) \cup \Sigma_{1,1}(H)—$$

but we have to work a little to see that the intersection is indeed generically transverse.

Suppose now we're working with the Grassmannian $G = \mathbb{G}(1, 4)$ of lines in $\mathbb{P}^4$ and we try to use an analogous method to determine the product $\sigma_2^2 \in A^4(G)$—that is, the class of the locus of lines meeting each of two given lines in $\mathbb{P}^4$. Specifically, we'd like to find a pair of lines $L, M \subset \mathbb{P}^4$ such that the two cycles

$$\Sigma_2(L) = \{\Lambda : \Lambda \cap L \neq \varnothing\} \quad \text{and} \quad \Sigma_2(M) = \{\Lambda : \Lambda \cap M \neq \varnothing\}$$

representing the class σ_2 are special enough that the class of the intersection is clear, but still sufficiently general that they intersect generically transversely.

Unfortunately, we can't. If the lines L and M are disjoint, they are effectively a general pair, and the intersection is not a union of Schubert cycles. But if L meets M, say at a point p, then the locus of lines through p forms a 3-dimensional component of the intersection $\Sigma_1(L) \cap \Sigma_1(M)$, so the intersection is not even dimensionally transverse.

How do we deal with this? By considering a *family* of lines M_t in $\mathbb{P}^4$, parametrized by $\mathbb{A}^1$, with M_t disjoint from L for $t \neq 0$, and with M_0 meeting L at a point p. We arrive then at a family of intersection cycles $\Sigma_1(L) \cap \Sigma_1(M_t)$: specifically, we consider the subvariety

$$\tilde{\Phi} = \{(t, \Lambda) : t \neq 0 \text{ and } \Lambda \in \Sigma_1(L) \cap \Sigma_1(M_t)\} \subset \mathbb{A}^1 \times G$$

and its closure $\Phi \subset \mathbb{A}^1 \times G$. Since M_t is disjoint from L for $t \neq 0$, the fiber $\Phi_t = \Sigma_1(L) \cap \Sigma_1(M_t)$ of Φ over $t \neq 0$ represents the class σ_2^2, and it follows that Φ_0 does as well. The point is, when we look at the fiber Φ_0 we're looking *not at the intersection* $\Sigma_1(L) \cap \Sigma_1(M_0)$ *of the limiting cycles, but rather at the limit of the intersection cycles*, which is necessarily of the expected dimension.

That said, how do we characterize the fiber Φ_0? Clearly it's contained in the intersection $\Sigma_1(L) \cap \Sigma_1(M_0)$, but as we said it must be properly contained in it. In other words, a line Λ arising as the limit of lines Λ_t meeting both L and M_t must satisfy some additional condition beyond meeting both L and M_0, and we need to say what that condition is.

Fortunately, the answer to that question is reasonably straightforward. For $t \neq 0$, the lines L and M_t together span a hyperplane $H_t = \overline{L, M_t} \cong \mathbb{P}^3 \subset \mathbb{P}^4$, and by the valuative criterion these hyperplanes H_t must have a limit H_0 as $t \to 0$. Moreover, if $\{\Lambda_t\}$ is a family of lines with Λ_t meeting both L and M_t for $t \neq 0$, we see that Λ_t must lie in H_t and hence that *the limiting line* Λ_0 *must be contained in* H_0. Now, if Λ_0 does not pass through

the point $p = L \cap M_0$, then it must be contained in the 2-plane $P = \overline{L, M_0}$ (so that the condition $\Lambda_0 \subset H_0$ is redundant); in sum, we conclude that the support of Φ_0 must be contained in the union of the two 2-dimensional Schubert cycles

$$\begin{aligned}\Phi_0 &\subset \{\Lambda : \Lambda \subset P\} \cup \{\Lambda : p_0 \in \Lambda \subset H_0\} \\ &= \Sigma_{2,2}(P) \cup \Sigma_{3,1}(p_0, H_0).\end{aligned}$$

In fact, we will see in Exercise 3.23 that the support of Φ_0 is all of $\Sigma_{2,2}(P) \cup \Sigma_{3,1}(p_0, H_0)$, and in Exercise 3.24 that Φ_0 is generically reduced. Thus, the cycle associated to the scheme Φ_0 is exactly the sum $\Sigma_{2,2}(P) + \Sigma_{3,1}(p_0, H_0)$, and we can deduce the formula

$$\sigma_2^2 = \sigma_{3,1} + \sigma_{2,2} \in A^4(\mathbb{G}(1,4)).$$

This is a good example of the method of dynamic specialization, in which we consider not a special pair of cycles representing given Chow classes and intersecting generically transversely, but a family of representative pairs specializing from ones that do intersect transversely to one that may not. The goal is then to describe, as we did above, *not the intersection of the limits, but the limit of the intersections.* As we indicated, it's a very powerful and broadly applicable technique, one that is ubiquitous in the subject. For example, it can be used to give a highly effective algorithm for the product of two arbitrary Schubert classes in $G(k,n)$ (or even their generalizations to flag manifolds); the calculation we've just sketched is the starting point for the general algorithms of Coskun [2009] and Vakil [2006a]. For another example of its application, see Griffiths and Harris [1980].

One caveat: a further wrinkle in the technique of dynamic specialization (present in all the examples cited above) is that to carry out the calculation of an intersection of Schubert cycles we may have to specialize in stages. For a relatively elementary example of this, see Exercise 3.25.

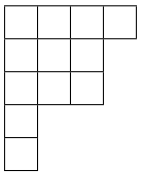
3.4 How to Count Schubert Cycles with Young Diagrams

We can count the Schubert cycles as follows:

Corollary 3.6. *$A(G(k,n)) \cong \mathbb{Z}^{\binom{n}{k}}$ as abelian groups.*

For the proof, and for many other purposes, it is convenient to represent the Schubert class $\sigma_{a_1,\ldots,a_k}$ by a *Young diagram*; that is, as a collection of left-justified rows of boxes, the i-th row having length a_i. For example,

$\sigma_{4,3,3,1,1}$ would be represented by

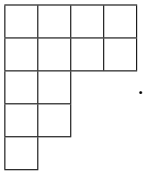
$\sigma_{4,3,3,1,1} \longleftrightarrow$ 

(Warning: there are at least 8 different conventions in use for interpreting the correspondence between Schubert cycles and Young diagrams!) The condition that $n-k \geq a_1 \geq a_k \geq 0$ means that the Young diagram fits into a box with k rows and $n-k$ columns, and the rows of the diagram are weakly decreasing in length from top to bottom. As another example, the relation between a Schubert cycle and the unique Schubert cycle τ such that $\sigma\tau$ is the class of a point, described in Proposition 3.3 could be described by saying that the Young diagrams of σ and τ, after rotating the latter 180°, are complementary in the $k \times (n-k)$ box; if $\sigma = \sigma_{4,3,3,1,1} \in A^*(G(5,10))$, for example, then τ is shown in the picture

σ	σ	σ	σ	τ
σ	σ	σ	τ	τ
σ	σ	σ	τ	τ
σ	τ	τ	τ	τ
σ	τ	τ	τ	τ

;

that is,

$\tau \longleftrightarrow$  .

With this notation, the proof of the Corollary is easy (though clever!):

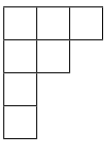
Proof of Corollary 3.6. The number of Schubert cycles is the same as the number of Young diagrams that fit into a $k \times (n-k)$ box of squares B. To count these, we associate to each Young diagram Y in B its "right boundary" L: this is the line, consisting of horizontal and vertical segments of unit length, which starts from the upper right corner of the $k \times (n-k)$ box and ends at the lower left corner of the box, such that the squares in in Y are those to the left of L. (For example, in the case of the Young diagram associated to $\sigma_{4,3,3,1,1} \subset G(5,10)$, illustrated above, we may describe L by the sequence $h, v, h, v, v, h, h, v, v$ where h and v denote horizontal and vertical segments, respectively, and we start from the upper right corner.)

Of course the number of h terms in any such boundary must be $n-k$, the width of the box, and the number of v terms must be k, the height of

the box. Thus the length of the boundary is n, and giving the boundary is equivalent to specifying which k steps will be vertical; that is, the number of Young diagrams in B is $\binom{n}{k}$ as required. □

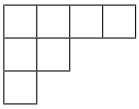
The correspondence between Schubert classes and Young diagrams behaves well with respect to many basic operations on Grassmannians. For example, under the duality $G(k,n) \cong G(n-k,n)$ the Schubert cycle corresponding to the Young diagram Y is taken to the Schubert cycle corresponding to the Young diagram Z that is the *transpose* of Y, that is, the diagram obtained by flipping Y around a 45° line running northwest-southeast. For example if

$$\sigma_{3,2,1,1} \in A^*(G(4,7)) \quad \longleftrightarrow$$



then the corresponding Schubert cycle in $G(3,7)$ is

$$\sigma_{4,2,1} \in A^*(G(3,7)) \quad \longleftrightarrow$$



This is reasonably straightforward to verify, and is the subject of Exercise 3.26.

The combinatorics of Young diagrams is an extremely rich subject with many applications. See for example Fulton [1997] for an introduction.

3.5 Linear spaces on quadrics

Using these tools we can generalize the calculation in Section 2.3.6 of the class of the locus of lines on a quadric surface to a description of the class of the locus of planes of any dimension on a smooth quadric hypersurface of any dimension. We will assume that the characteristic of the ground field is zero, so that we can use Kleiman's Transversality Theorem and so that a nonsingular form $Q(x)$ of degree 2 on $\mathbb{P}(V)$ can be written in the form $Q(x) = q(x,x)$, where $q(x,y)$ is a nonsingular bilinear form $V \times V \to F$. ****should we add something about transversality in positive characteristic? Some information about Char 2?**** A small calculation, using the fact that $2 \neq 0 \in F$, shows that a linear subspace $\mathbb{P}(W) \subset \mathbb{P}(V)$ lies on the quadric $Q(x) = 0$ if and only if W is *isotropic* for q; that is, $q(W,W) = 0$. Thus we want to find the class of the locus $\Phi \subset G = G(k,V)$ of isotropic k-planes for q.

To start, we want to find the dimension of Φ. There are a number of ways to do this; probably the most elementary is to count bases for isotropic subspaces. To find a basis for an isotropic subspace, we can start with any vector v_1 with $q(v_1, v_1) = 0$; then choose $v_2 \in \langle v_1\rangle^\perp \setminus \langle v_1\rangle$ with $q(v_2, v_2) = 0$, $v_3 \in \langle v_1, v_2\rangle^\perp \setminus \langle v_1, v_2\rangle$ with $q(v_3, v_3) = 0$, and so on. Since $\langle v_1, \ldots, v_i\rangle \subset \langle v_1, \ldots, v_i\rangle^\perp$, this necessarily terminates when $i \geq n/2$; in other words, *a nondegenerate quadratic form will have no isotropic subspaces of dimension strictly greater than half the dimension of the ambient space.* (We could also see this by observing that q defines an isomorphism of V with its dual V^* that carries any isotropic subspace $\Lambda \subset V$ into its annihilator $\Lambda^\perp \subset V^*$.)

In this process the allowable choices for v_1 correspond to points on the quadric $Q(x) = 0$; those for v_2 correspond to the points on the quadric $Q|_{v_1^\perp}$, and so forth. In general the v_i form a locally closed subset of V of dimension $n - i$. Thus the space of all bases for isotropic k-planes has dimension

$$(n-1) + \cdots + (n-k) = k(n-k) + \binom{k}{2}.$$

Since there is a k^2-dimensional family of bases for a given isotropic k-plane, the space of such planes has dimension

$$k(n-k) + \binom{k}{2} - k^2 = k(n-k) - \binom{k+1}{2},$$

or in other words the cycle Φ has codimension $\binom{k+1}{2}$ in $G(k, V)$ when $k \leq n/2$, and is empty otherwise.

Having determined the dimension of Φ, we ask now for its class in $A(G(k, V))$. Following our general method for finding the decomposition of a cycle into Schubert cycles, we write

$$[\Phi] = \sum_{|a| = \binom{k+1}{2}} \gamma_a \sigma_a$$

with

$$\begin{aligned} \gamma_a &= \#\left(\ \Phi \cap \Sigma_{n-k-a_k, \ldots, n-k-a_1}(\mathcal{V})\ \right) \\ &= \#\big\{\Lambda : q|_\Lambda \equiv 0 \ \text{ and } \ \dim(\Lambda \cap V_{i+a_i}) \geq i \ \forall i\ \big\}. \end{aligned}$$

To evaluate γ_a, suppose that $\Lambda \subset V$ is a k-plane in this intersection. The subspace $V_{a_i+i} \subset V$ being general, the restriction $q|_{V_{a_i+i}}$ of q to it will again be nondegenerate. Since $q|_{V_{a_i+i}}$ has an isotropic i-plane, we must have $a_i + i \geq 2i$, or in other words

$$a_i \geq i \quad \forall i.$$

But by hypothesis, $\sum a_i = \binom{k+1}{2}$; so in fact we must have equality in each of these inequalities. In other words, $\gamma_a = 0$ *for all* a *except the index* $a = (k, k-1, \ldots, 2, 1)$.

It remains to evaluate the coefficient

$$\gamma_{k,k-1,\dots,2,1} = \#\{\Lambda : q|_\Lambda \equiv 0 \text{ and } \dim(\Lambda \cap V_{2i}) \geq i \ \forall i\ \}; \tag{3.3}$$

we claim that this number is 2^k.

We prove this inductively. To start, note that the restriction $q|_{V_2}$ of q to the 2-dimensional space V_2 has two one-dimensional isotropic spaces, and Λ will necessarily contain exactly one of them: it can't contain both since Φ is disjoint from any Schubert cycle $\Sigma_b(\mathcal{V})$ with $|b| > k(n-k) - \binom{k+1}{2}$.

We may thus suppose that Λ contains the isotropic subspace $W \subset V_2$, so that Λ is contained in $W^\perp$. Now, since $q(W, W) \equiv 0$, q induces a nondegenerate quadratic form q' on the $(n-2)$-dimensional quotient $W' = W^\perp/W$; and the quotient space

$$\Lambda' = \Lambda/W \subset W^\perp/W$$

is a $(k-1)$-dimensional isotropic subspace for q'. Moreover, since the spaces V_{2i} are general subspaces of V containing V_2, the subspaces

$$V'_{2i-2} = (V_{2i} \cap W^\perp)/W \subset W^\perp/W$$

form a general flag in $W' = W^\perp/W$; and we have

$$\dim(\Lambda' \cap V'_{2i-2}) \geq i - 1 \quad \forall i.$$

Inductively, there are 2^{k-1} isotropic $(k-1)$-planes $\Lambda' \subset W'$ satisfying these conditions; and so there are 2^k planes $\Lambda \subset W$ satisfying the conditions of (3.3). We have proven:

Proposition 3.7. *Let q be a nondegenerate quadratic form on the n-dimensional vector space V, and $\Phi \subset G(k, V)$ the variety of isotropic k-planes for q. Assuming $k \leq n/2$, the class of the cycle Φ is*

$$[\Phi] = 2^k \sigma_{k,k-1,\dots,2,1}.$$

****Fig. 4.11****

As an immediate application of this result, we can answer Keynote Question (c). To begin with, we asked how many lines lie on the intersection of two quadrics in $\mathbb{P}^4$. To answer this, let $Q, Q' \subset \mathbb{P}^4$ be two general quadric hypersurfaces, and $X = Q_1 \cap Q_2$. The set of lines on X is just the intersection $\Phi \cap \Phi'$ of the cycles of lines lying on Q and Q'; by Kleiman transversality these are transverse, and so we have

$$\#(\Phi \cap \Phi') = \left(4\sigma_{2,1}\right)^2 = 16.$$

More generally, if Q and $Q' \subset \mathbb{P}^{2n}$ are general quadrics, we ask how many $(n-1)$-planes are contained in their intersection; again, this is the intersection number

$$\#(\Phi \cap \Phi') = \left(2^n \sigma_{n,n-1,\dots,1}\right)^2 = 4^n.$$

3.6 Pieri and Giambelli

3.6.1 Intersecting with σ_1

For any pair of Schubert indices a and b, we can write the product

$$(\sigma_a \cdot \sigma_b) = \sum_{|c|=|a|+|b|} \gamma_{a,b;c}\sigma_c \in A^{|a|+|b|}(G(k,n)). \tag{3.4}$$

Since we can express the coefficients $\gamma_{a,b;c}$ as triple intersections

$$\gamma_{a,b;c} = \deg(\sigma_a \cdot \sigma_b \cdot \sigma_{n-k-c_k,\ldots,n-k-c_1}) \in A^{k(n-k)}(G(k,n)) \cong \mathbb{Z},$$

the $\gamma_{a,b;c}$ are nonnegative integers. If we adopt the convention that $\sigma_a = 0 \in A^{|a|}(G(k,n))$ if a fails to satisfy the conditions $n-k \geq a_1 \geq \cdots \geq a_k \geq 0$ and $a_l = 0\ \forall l > k$, then we can give formulas for coefficients $\gamma_{a,b;c}$ that depend only on the indices a, b and c, and not on k and n, and such that (3.4) holds in all Grassmannians.

The $\gamma_{a,b;c}$ are called *Littlewood-Richardson* coefficients, and they appear in many combinatorial contexts. There are many combinatorial methods for computing them, too, but many qualitative problems about these products remain open; no one knows good criteria, for example, for telling when $\gamma_{a,b;c} \neq 0$, or when $\gamma_{a,b;c} > 1$. (The smallest example of a product of two Schubert classes where another Schubert class appears with multiplicity > 1 is the square $\sigma_{2,1}^2$ in $G(3,6)$, as you'll be asked to work out in Exercise 3.31.)

Nevertheless there is a useful formula, called *Pieri's formula*, in the case where one of the indices, has the form $l, 0, \ldots, 0$. Here is the formula and proof when $l = 1$.

Proposition 3.8. *For any Schubert class $\sigma_a \in A^*(G(k,n))$, we have*

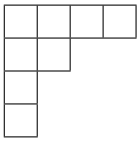
$$\begin{aligned}\sigma_1\sigma_a &= \sum_{\substack{|c|=|a|+1 \\ a_i \leq c_i \leq a_{i-1}\forall i}} \sigma_c \\ &= \sum_{j:a_{j-1}>a_j} \sigma_{a_1,\ldots,a_{j-1},a_j+1,a_{j+1},\ldots,a_k}\end{aligned}$$

We will prove the formula only in characteristic zero, which we use to establish transversality; by considering the tangent spaces explicitly, the same result can be proven in any characteristic.

Pieri's formula can also be described in terms of the Young diagrams: it says that the Schubert cycles appearing in the product $\sigma_1\sigma_a$ (all with coefficient 1) correspond to Young diagrams obtained from the Young diagram of σ_a *by adding one box at the end of any row*, as long as the result is still

a Young diagram: for example, if

$$\sigma_{4,2,1,1} \in A^*(G(4,8)) \quad \longleftrightarrow$$



we can add a box in either the first, second, third or fifth row, to obtain the expression

$$\sigma_1\sigma_{4,2,1,1} = \sigma_{5,2,1,1} + \sigma_{3,3,1,1} + \sigma_{3,2,2,1} + \sigma_{3,2,1,1,1}.$$

Proof of Proposition 3.8 in characteristic zero. Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be general flags on V so that for any Schubert indices a and c with $|c| = |a| + 1$,

$$(\sigma_1 \cdot \sigma_a \cdot \sigma_{n-k-c_k,\ldots,n-k-c_1}) = \#\big(\Sigma_1(\mathcal{U}) \cap \Sigma_a(\mathcal{V}) \cap \Sigma_{n-k-c_k,\ldots,n-k-c_1}(\mathcal{W})\big);$$

writing out explicitly the conditions associated to the Schubert cycles, the Proposition then amounts to the claim that this cardinality is given by

$$\#\left\{\Lambda \subset V : \begin{array}{c} \Lambda \cap U_{n-k} \neq 0; \\ \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i \ \forall i; \text{ and} \\ \dim(\Lambda \cap W_{j+c_{k+1-j}}) \geq j \ \forall j \end{array}\right\}$$
$$= \begin{cases} 1, & \text{if } a_i \leq c_i \leq a_{i-1} \text{ for all } i \\ 0, & \text{otherwise.} \end{cases}$$

As before, we consider the conditions in pairs: that is, we set $j = k + 1 - i$ so that if Λ is a k-plane satisfying the Schubert conditions, then in particular

$$\dim(\Lambda \cap V_{n-k+i-a_i}) \geq i$$

and

$$\dim(\Lambda \cap W_{k+1-i+c_i}) \geq k + 1 - i.$$

We must have

$$V_{n-k+i-a_i} \cap W_{k+1-i+c_i} \neq 0,$$

and hence

$$c_i \geq a_i \quad \text{for all } i.$$

If we have equality in this last inequality, then the subspaces $V_{n-k+i-a_i}$ and $W_{n-i+1-b_{k-i+1}}$ will meet in a one-dimensional vector space Γ_i, which will necessarily be contained in Λ. The difference between this case and the case of complementary dimension is that now, since $|c| = |a| + 1$, we can deduce that if Λ is any k-plane satisfying the conditions above, then for some j,

$$c_i = a_i \ \forall i \neq j; \quad \text{and} \quad c_j = a_j + 1.$$

Moreover, Λ must contain the $(k-1)$-plane Γ spanned by Γ_i for $i \neq j$, and it must also have positive-dimensional intersection with the 2-plane $\Omega = V_{n-k+j-a_j} \cap W_{k+1-j+c_j}$; in particular, it must be contained in the $(k+1)$-plane $\overline{\Gamma, \Omega}$.

Finally, the subspace U_{n-k} will meet $\overline{\Gamma, \Omega}$ in a one-dimensional subspace Υ, and if Λ is to have nonzero intersection with U_{n-k} it must contain this space as well; it follows that Λ must be the k-plane $\overline{\Gamma, \Upsilon}$. □

Pieri can also be proved by the method of specialization; a proof (again in case $l = 1$) is sketched in the solution to Exercise method of specialization.

The formula of Proposition 3.8 gives us the means to answer the generalization of Keynote Question (d) to all Grassmannians: since σ_1 is the class of the hyperplane section of the Grassmannian in its Plücker embedding, the degree of the Grassmannian in that embedding is the degree of $\sigma_1^{k(n-k)}$. This will be worked out (with the aid the of Hook formula from combinatorics) in Exercise 3.33; the answer is that

$$\deg(G(k,n)) = \big(k(n-k)\big)! \prod_{i=0}^{k-1} \frac{i!}{(n-k+i)!},$$

3.6.2 Pieri's formula in general

Pieri's formula is a direct generalization of Proposition 3.8 to intersections with what are called *special Schubert cycles*: these are the cycles of the form

$$\Sigma_b = \Sigma_{b,0,\dots,0} = \{\Lambda : \Lambda \cap V_{n-k+1-b} \neq 0\}.$$

We will not prove this here, but will state the formula:

Proposition 3.9. *For any Schubert class $\sigma_a \in A^*(G(k,n))$ and any integer b, we have*

$$(\sigma_b \cdot \sigma_a) = \sum_{\substack{|c|=|a|+b \\ a_i \leq c_i \leq a_{i-1} \forall i}} \sigma_c$$

For a proof, see Fulton [1997] §2.2.

3.6.3 Giambelli's formula

Pieri's formula tells us how to intersect an arbitrary Schubert cycle with one of the special Schubert cycles $\sigma_b = \sigma_{b,0,\dots,0}$. Giambelli's formula is complementary, in that it tells us how to express an arbitrary Schubert cycle in terms of special ones; the two together give us (in principle) a way of calculating the product of two arbitrary Schubert cycles.

As in the case of the general Pieri's formula, we will state Gimbelli's formula without proof; see Fulton [1997] for a proof in general, and Chapter 14.

Proposition 3.10.

$$\sigma_{a_1,a_2,\ldots,a_k} = \begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+k-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} & \cdots & \sigma_{a_2+k-2} \\ \sigma_{a_3-2} & \sigma_{a_3-1} & \sigma_{a_3} & \cdots & \sigma_{a_3+k-3} \\ \vdots & \vdots & & \ddots & \vdots \\ \sigma_{a_k-k+1} & \sigma_{a_k-k+2} & \sigma_{a_k-k+3} & \cdots & \sigma_{a_k} \end{vmatrix}$$

Thus, for example, we have

$$\sigma_{2,1} = \begin{vmatrix} \sigma_2 & \sigma_3 \\ \sigma_0 & \sigma_1 \end{vmatrix} = \sigma_2\sigma_1 - \sigma_3,$$

which we can then use in conjunction with Pieri to evaluate, for example, $\sigma_{2,1}^2$.

One final word of warning. In theory, Giambelli and Pieri in tandem give us an algorithm for calculating the product of any two Schubert cycles: use Giambelli to express either as a polynomial in the special Schubert cycles, and then use Pieri to evaluate the product of this polynomial with the other. In practice, though, except in low-dimensional examples *this is an absolutely terrible idea*: as you can see from the determinantal form of Giambelli's formula, the number of calculations involved increases extremely rapidly with k and n. Nor can this method be used to prove qualitative results about products of Schubert cycles: for example, it's not even clear from this approach that such a product is necessarily a nonnegative linear combination of Schubert cycles. The algorithms of Coskun and Vakil referred to earlier are far, far better.

3.7 Exercises

Exercise 3.11. Suppose that $\varphi : \Lambda \to V$ is an inclusion of a k-dimensional vector space Λ into an n-dimensional vector space V and let $\psi : V \to \Lambda' = V/\varphi(\Lambda)$ be the quotient map, so that

$$0 \longrightarrow \Lambda \xrightarrow{\varphi} V \xrightarrow{\psi} \Lambda' \longrightarrow 0$$

is exact. Choose bases, so that φ and ψ are represented by matrices. Show that there is a nonzero scalar λ such that the $k \times k$ minor of φ involving rows with indices $i_1 < \cdots < i_k$ is equal to λ times the $(n-k)\times(n-k)$ minor of ψ involving the complementary set of column indices $j_1 < \cdots < j_{n-k}$.

Conclude that the set-theoretic isomorphism $G(k,V) \cong G(n-k,V^*)$ taking a subspace of V to its annihilator in V^* is an algebraic isomorphism as well.

Exercise 3.12. Complete the proof of Corollary 3.2 without invoking duality—that is, prove directly that σ^k_{n-k} is the class of a point in $G(k,n)$.

Exercise 3.13. Use the description of the Chow ring $A(\mathbb{G}(1,4))$ given in Section 3.3 to answer the question: given 6 general 2-planes in $\mathbb{P}^4$, how many lines meet all 6? Note that this number is the same as the degree of the Grassmannian $\mathbb{G}(1,4) = G(2,5)$ in the Plücker embedding.

Exercise 3.14. Let $S \subset \mathbb{P}^4$ be a surface of degree d, and $\Gamma_S \subset \mathbb{G}(1,4)$ the variety of lines meeting S.

(a) Find the class $\gamma_S = [\Gamma_S] \in A^1(\mathbb{G}(1,4))$.

(b) Use this to answer the question: if $S_1, \ldots, S_6 \subset \mathbb{P}^4$ are surfaces of degrees $d_1, \ldots, d_6$, how many lines in $\mathbb{P}^4$ will meet all six?

Exercise 3.15. Let $C \subset \mathbb{P}^4$ be a curve of degree d, and $\Gamma_C \subset \mathbb{G}(1,4)$ the variety of lines meeting C.

(a) Find the class $\gamma_C = [\Gamma_C] \in A^2(\mathbb{G}(1,4))$.

(b) Use this to answer the question: if C_1, C_2 and $C_3 \subset \mathbb{P}^4$ are curves of degrees d_1, d_2 and d_3, how many lines in $\mathbb{P}^4$ will meet all three?

Exercises 3.16-3.19 deal with the geometry of the surface described in Keynote Question (b), whose degree we worked out in Section 3.2.1: the surface $X \subset \mathbb{P}^3$ swept out by the lines corresponding to a general twisted cubic $C \subset \mathbb{G}(1,3)$.

Exercise 3.16. To start, use the fact that the dual of $\mathbb{G}(1,3) \subset \mathbb{P}^5$ has degree 2 to show that a general twisted cubic $C \subset \mathbb{G}(1,3)$ lies on two Schubert cycles $\Sigma_1(L)$ and $\Sigma_1(M)$ for some pair of skew lines $L, M \subset \mathbb{P}^3$

Exercise 3.17. Show that for skew lines L and $M \subset \mathbb{P}^3$, the intersection $\Sigma_1(L) \cap \Sigma_1(M)$ is isomorphic to $L \times M$ via the map sending a point $[\Lambda] \in \Sigma_1(L) \cap \Sigma_1(M)$ to the pair $(\Lambda \cap L, \Lambda \cap M) \in L \times M$, and that it is the intersection of $\mathbb{G}(1,3)$ with the intersection of the hyperplanes spanned by $\Sigma_1(L)$ and $\Sigma_1(M)$

Exercise 3.18. Finally, suppose that $C \subset \Sigma_1(L) \cap \Sigma_1(M)$ is a twisted cubic curve. Using the fact that its bidegree in $\Sigma_1(L) \cap \Sigma_1(M) \cong L \times M \cong \mathbb{P}^1 \times \mathbb{P}^1$ (possibly after switching factors) is $(2,1)$, show that for some degree 2 map $\varphi : L \to M$, the family of lines corresponding to C may be realized as the locus

$$C = \{\overline{p, \varphi(p)} : p \in L\}.$$

Show correspondingly that the surface

$$S = \bigcup_{[L]\in C} L \subset \mathbb{P}^3$$

swept out by the lines of C is a cubic surface double along a line, and that it's the projection of a rational normal surface scroll $X_{1,2} \subset \mathbb{P}^4$.

Exercise 3.19. For what special twisted cubic curves $C \subset \mathbb{G}(1,3)$ will the conclusion of the preceding exercise be false?

Exercise 3.20. Now let $G = \mathbb{G}(1,3) \subset \mathbb{P}^5$ be as in the preceding Exercise, and let $C \subset G$ be a general rational normal quartic curve. Can you describe the surface S swept out by the lines of C? In particular, what is the singular locus of S?

In Section 3.3 we calculated the number of lines meeting four general n-planes in $\mathbb{P}^{2n+1}$. In the following two Exercises, we'll see another way to do this (analogous to the alternative count of lines meeting four lines in $\mathbb{P}^3$ given in Exercise 2.27), and a nice geometric sidelight.

Exercise 3.21. Let $\Lambda_1, \ldots, \Lambda_4 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ be four general n-planes. Calculate the number of lines meeting all four by showing that the union of the lines meeting Λ_1, Λ_2 and Λ_3 is a Segre variety $S_{1,n} = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ and using the calculation above **ref Chapter 1** for the degree of $S_{1,n}$.

Exercise 3.22. By the preceding exercise, we can associate to a general configuration $\Lambda_1, \ldots, \Lambda_4$ of k-planes in $\mathbb{P}^{2k+1}$ an unordered set of $k+1$ cross-ratios. Show that two such configurations $\{\Lambda_i\}$ and $\{\Lambda_i'\}$ are projectively equivalent if and only if the corresponding sets of cross-ratios coincide.

The next two Exercises deal with the example of dynamic specialization given in Section 3.3.1, and specifically with the family Φ of cycles described there.

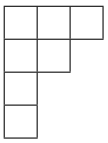
Exercise 3.23. Show that the support of Φ_0 is all of $\Sigma_{2,2}(P) \cup \Sigma_{3,1}(p_0, H_0)$.

Exercise 3.24. Verify the last assertion made in the calculation of σ_2^2; that is, show that Φ_0 has multiplicity 1 along each component. [Hint: argue that by applying a family of automorphisms of $\mathbb{P}^4$ we can assume that the plane H_t is constant and use the calculation of the preceding chapter.]

Exercise 3.25. A further wrinkle in the technique of dynamic specialization is that to carry out the calculation of an intersection of Schubert cycles we may have to specialize in stages. To see an example of this, use dynamic specialization to calculate the intersection σ_2^2 in the Grassmannian $\mathbb{G}(1,5)$. [Hint: you have to let the two 2-planes specialize first the a pair intersecting in a point, then to a pair intersecting in a line.]

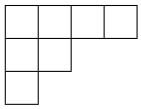
Exercise 3.26. Suppose that the Schubert class $\sigma_a = \sigma_{a_1,\ldots,a_k} \in A^*(G(k,n))$ corresponds to the Young diagram Y in a $k \times (n-k)$ box B. Show that under the duality $G(k,n) \cong G(n-k,n)$, the class σ_a is taken to the Schubert class $\sigma_b = \sigma_{b_1,\ldots,\beta_{n-k}}$ corresponding to the Young diagram Z that is the *transpose* of Y, that is, the diagram obtained by flipping Y around a 45° line running northwest-southeast. For example if

$$\sigma_{3,2,1,1} \in A^*(G(4,7)) \quad \longleftrightarrow$$



then the corresponding Schubert cycle in $G(3,7)$ is

$$\sigma_{4,2,1} \in A^*(G(3,7)) \quad \longleftrightarrow$$



Exercise 3.27. Show that the map $i^* : A^d(G,(k,n+1)) \to A^d(G(k,n))$ is a monomorphism if and only if $n-k \geq d$, and that $i^* : A^d(G,(k+1,n+1)) \to A^d(G(k,n))$ is a monomorphism if and only if $k \geq d$. (Thus, for example, the formula

$$\sigma_1^2 = \sigma_2 + \sigma_{11},$$

which we established in $A^*(\mathbb{G}(1,3))$, *holds true in every Grassmannian*).

Exercise 3.28. Let $C \subset \mathbb{P}^r$ be a smooth, irreducible, nondegenerate curve of degree d and genus g, and let $S_1(C) \subset \mathbb{G}(1,r)$ be the variety of chords to C, as defined in Section 2.3.4 above. Find the class $[S_1(C)] \in A_2(\mathbb{G}(1,r))$.

Exercise 3.29. Prove that the Schubert cycle

$$\Sigma := \Sigma_{a_1,\ldots,a_k}(\mathcal{V}) \subset G(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$$

is defined scheme-theoretically by the polynomials vanishing on $G(k,n)$ set of Plücker coordinates $p_{i_1,\ldots,i_k}$ with $i_j > n-k-a_j$ for some j. (These equations even generate the homogenous ideal of the Schubert cycle; see for example **?**].) To show that these equations vanish on Σ you can follow the ideas of Section 2.2.2. To show that they generate the ideal of the Grassmannian, pass to an open affine set on which the Plücker coordinates look like determinants of various sizes of matrices in the variables.)

Exercise 3.30. Let Q, Q' and Q'' be three general quadrics in $\mathbb{P}^8$. How many 2-planes lie on all three? [Warning: this involves a substantially harder calculation of intersections of Schubert classes. The reader is encouraged to try it now, but should be aware that more tools for answering this will be introduced in Section 3.6 below.]

Exercise 3.31. Find the decomposition of $\sigma_{2,1}^2$ in $G(3,6)$. This is the smallest example of a product of two Schubert classes where another Schubert class appears with multiplicity > 1.

Exercise 3.32. Give an alternative proof of Pieri's formula for intersections with σ_1 using the method of specialization.

Exercise 3.33. More generally, use Pieri to identify the degree of $\sigma_1^{k(n-k)}$ with the number of standard tableaux: that is, ways of filling in $k \times (n-k)$ matrix with the integers $1, \ldots, k(n-k)$ in such a way that every row and column is strictly increasing. Then use the "hook formula" (see for example, Fulton [1997]) to show that this number is

$$\big(k(n-k)\big)! \prod_{i=0}^{k-1} \frac{i!}{(n-k+i)!},$$

Exercise 3.34. Using Pieri's formula, determine all products of Schubert classes in the Chow ring of the Grassmannian $\mathbb{G}(1,4)$, and compare this with the result of Proposition 3.5.

Exercise 3.35. Using Pieri's formula, determine all products of Schubert classes in the Chow ring of the Grassmannian $\mathbb{G}(2,5)$. [Note: an extra step will be required, specifically to find the square $(\sigma_{2,1})^2$.]

Exercise 3.36. Give a statement of Pieri's formula in terms of Young diagrams.

4

Chow Groups

Keynote Questions:.

(a) Let $S \subset \mathbb{P}^3$ be a surface, and $\pi : S \to \mathbb{P}^1$ a morphism. Is it possible that one fiber of π is a line in $\mathbb{P}^3$ while another fiber is a conic?

(b) Let $L, Q \subset \mathbb{P}^3$ be a line and a nonsingular conic in $\mathbb{P}^3$. Is $(\mathbb{P}^3 \setminus L) \cong (\mathbb{P}^3 \setminus Q)$ as schemes?

We now start over. We will refer to Chapter **??** for a few definitions and examples, but by and large the development given in this chapter and the following ones is independent of what has come before.

In this Chapter we will define the basic objects of Intersection Theory, the *Chow groups* of rational equivalence classes of cycles on a scheme. In this generality examples such as Example 5.23 suggest that there is no useful intersection theory, but the Chow groups are still interesting.

The most important and useful results of this Chapter, Theorems 4.14 and 4.16, say that the Chow groups are functorial for projective morphisms, and allow us to compute the maps on Chow groups in the finite case. These depend in turn on a beautiful and elementary result in commutative algebra, the *Determinant Lemma*, Theorem 4.18.

We also show that the Chow groups are contravariant functors with respect to flat maps. The important result that they are also contravariant functors for projective morphism among smooth quasiprojective varieties will be treated in Chapter **??**.

4.1 Chow groups and basic operations

One of the areas where algebraic geometry is most successful is the study of subvarieties of codimension 1—the theory of divisors. (The fact that there are *only* subvarieties of codimension 1 on an algebraic curve is one of the things that makes the theory of curves so tractable.) The idea of cycles, and the rational equivalence of cycles is the very useful extension of the notion of divisors that underlies all of intersection theory.

Let X be any algebraic variety or scheme. The *group of cycles* on X, denoted $Z_*(X)$, is the free abelian group generated by the symbols $\langle V \rangle$ where V is a subvariety (reduced irreducible subscheme) of X (most of the time we will write V in place of $\langle V \rangle$, since there will be no danger of confusion.) The group $Z_*(X)$ is graded by dimension: we write $Z_d(X)$ for the group of cycles that are formal linear combinations of subvarieties of dimension d. The elements of $Z_d(X)$ will be called *d-cycles* of X. When X is smooth and *equidimensional* (that is, every irreducible component of X has the same dimension) then, since we have assumed that X is of finite type, any subvariety $V \subset X$ satisfies $\operatorname{codim} V = \dim X - \dim V$, so the Chow groups are also graded by codimension. We can write $Z^c(X) := Z_{\dim X - c}$ for the group of codimension c cycles. ****We said we weren't going to use the notation A^c or Z^c unless X was smooth (so that Chow cohomology coincides with Chow), but I don't know how to explain at this point why we require X to be smooth.****

A cycle $Z = \sum_i n_i Y_i$ is called *effective* if the coefficients n_i are all non-negative. A *divisor* is a cycle whose components all have codimension 1. Note that $Z_*(X) = Z_*(X_{\mathrm{red}})$; that is, $Z_*(X)$ is insensitive to whatever non-reduced structure X may have.

We define an effective cycle Z associated with any closed subscheme $Y \subset X$ by summing the irreducible components of Y, each with a multiplicity defined using the following algebraic idea:

If A is any commutative ring then we say that an A-module M has finite length if it has a finite *composition series*; that is, a sequence of submodules $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_l = 0$ such that each factor module M_i/M_{i+1} is a simple A-module. The Jordan-Hölder Theorem asserts that the number of times a given simple A-module appears as a factor M_i/M_{i+1} does not depend on the composition series chosen; in particular, the length l of a composition series for M, written $\operatorname{length}_A(M)$, depends only on A and M.

Consider again a subscheme $Y \subset X$. Since our schemes are assumed to be of finite type over a field they are Noetherian. In particular, Y has finitely many (reduced) irreducible components $Y_1, \ldots, Y_s$, and each local ring $\mathcal{O}_{Y,Y_i}$ has finite length, say l_i. We define the cycle $\langle Y \rangle$ associated to Y to be the formal combination $\sum_i l_i Y_i$.

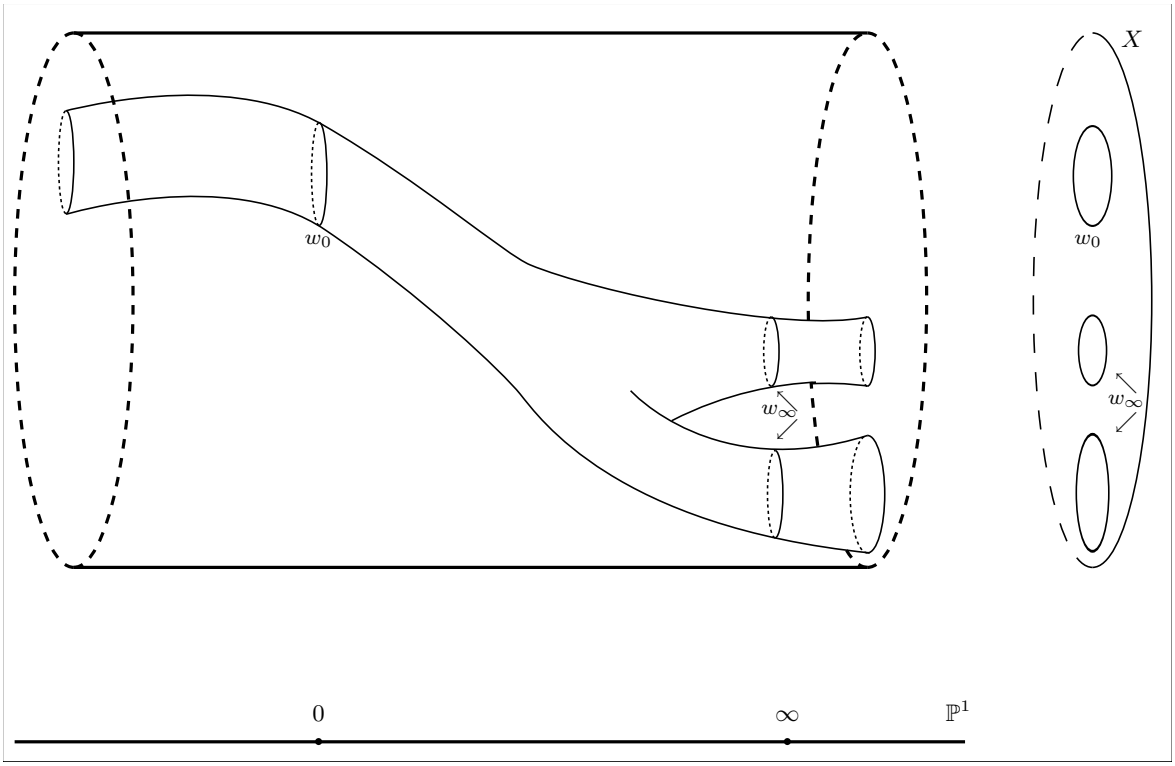


FIGURE 1.1. Rational equivalence between two cycles ω_0 and ω_∞ on X

We will next define the relation of *rational equivalence* between cycles. Our goal is to make two subvarieties rationally equivalent if there is rationally parametrized "family" of subvarieties of which they are both members. In Exercise 4.30 the reader will see that the relation we define restricts to the classical notion of linear equivalence in the case of divisors on a smooth variety.

To emphasize the analogy with ordinary cohomology theory we introduce a "boundary" map $\partial_X : Z_*(X \times \mathbb{P}^1) \to Z_*(X)$ defined on free generators as follows: Let W be a subvariety of $X \times \mathbb{P}^1$. If the projection $\pi : W \to \mathbb{P}^1$ on the second factor is not dominant—that is, if $W \subset X \times \{t\}$ for some $t \in \mathbb{P}^1$—we set $\partial_X(W) = 0$. On the other hand, if $W \to \mathbb{P}^1$ is dominant then we set $W_0 = \pi^{-1}(0) \subset X \times \{0\} = X$ and $W_\infty = \pi^{-1}(\infty) \subset X \times \{\infty\} = X$, and define $\partial_X(W) = \langle W_0 \rangle - \langle W_\infty \rangle$. We write $Rat_*(X) \subset Z_*(X)$ for the image $\partial(Z_*(X \times \mathbb{P}^1))$, the subgroup generated by all the cycles of the form $\langle W_0 \rangle - \langle W_\infty \rangle$. We will sometimes write $\langle W_0 \rangle \sim \langle W_\infty \rangle$ to indicate that the difference is in $Rat_*(X)$.

Definition 4.1. Two cycles A and B are rationally equivalent, written $A \sim B$, if their difference lies in $Rat_*(X)$.

A *rational equivalence* from A to B is an element $Z \in Z(\mathbb{P}^1 \times X)$ such that $\partial Z = A - B$.

The *Chow group* $A_*(X)$ is the group of rational equivalence classes,

$$\begin{aligned} A_*(X) :&= Z_*(X)/Rat_*(X) \\ &= \operatorname{coker}\big(\partial_X : Z_*(X \times \mathbb{P}^1) \to Z_*(X)\big). \end{aligned}$$

With this terminology, $Rat_d(X)$ is the group of d-cycles rationally equivalent to 0. When X is equidimensional the groups $Rat_*(X)$ and $A_*(X)$ may be graded by codimension instead of dimension, and we can write $A^c(X) = Z^c(X)/Rat^c(X)$ for the corresponding summands. When X is smooth, we will see in Chapter **??** that, with the grading by codimension, $A^*(X)$ is be a graded ring. (We won't use the notation $A^*(X)$ when X is singular since it is used for a different ring—the "operator Chow ring"—in Fulton [1984].)

Let $W \subset \mathbb{P}^1 \times X$ be a subvariety dominating $\mathbb{P}^1$, as above. If the map $W \to X$ is not generically finite then, since each fiber is a subset of $\mathbb{P}^1$, a dense set of fibers would be equal to $\mathbb{P}^1$. By semicontinuity of fiber dimension, every fiber would be equal to $\mathbb{P}^1$, whence $W = W_0 \times \mathbb{P}^1$, and $W_0 = W_\infty$. Thus we may restrict our attention to the case where the map $W \to X$ is generically finite.

If $W \subset \mathbb{P}^1 \times X$ has dimension $d+1$, then by Krull's Principal Ideal Theorem each W_i is either empty or is a subscheme of dimension d. Thus $Rat_*(X)$ is generated by homogeneous elements in the grading by dimension, and it follows that the Chow group $A_*(X)$ is also graded by dimension, that is, $A_*(X) = \oplus_d A_d(X)$, with

$$A_d(X) = Z_d(X)/\partial_X(Z_{d+1}(X \times \mathbb{P}^1)).$$

We will use the notation $[Y]$ for the class modulo rational equivalence of the cycle Y; likewise, if $Y \subset X$ is any subscheme we will write $[Y]$ instead of $[\langle Y \rangle]$ for the class of the cycle associated to the subscheme Y. Also, we will write ∂ in place of ∂_X when X is clear from the context.

It is often convenient to look for rational equivalences using a subvariety W' of $U \times X$ where U is $\mathbb{A}^1$ or some other open subset of $\mathbb{P}^1$. In this setting, it's still true that any two fibers $W'_p = W' \cap \{p\} \times X$ and $W_q = W' \cap \{q\} \times X$ are rationally equivalent: in order to apply the definition above, we could replace W' by its closure in $\mathbb{P}^1 \times X$, and move the points p, q to $0, \infty$ by an automorphism of $\mathbb{P}^1$.

A hidden complexity in the definition of a rational equivalence Z from A to B is that, writing $Z = \sum m_i Z_i$ with irreducible Z_i, we may have

cancellation among the components of the various $Z_i \cap \{0\} \times X$ or $Z_i \cap \{\infty\} \times X$. Thus it is hard to conclude anything about the varieties Z_i from a knowledge of A and B. However, if Z is itself an irreducible variety then a lot is preserved. Here is the case of rational equivalence on projective space:

Proposition 4.2. *If $Z \subset \mathbb{P}^1 \times X$ is a closed subvariety dominating $\mathbb{P}^1$, then the fibers of Z at the points of $\mathbb{P}^1$ all have the same Hilbert polynomial; that is, $\chi(\mathcal{O}_{Z_p}(l)) = \chi(\mathcal{O}_{Z_q}(l))$ for any points $p, q \in \mathbb{P}^1$ and any integer l.*

Proof. Let (t_0, t_1) be coordinates on $\mathbb{P}^1$. We may assume that $p = 0$ and $q = \infty$. Consider the sequences

$$0 \longrightarrow \mathcal{O}_W(0,-1) \xrightarrow{t_0} \mathcal{O}_W \longrightarrow \mathcal{O}_{Z_\infty} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_W(0,-1) \xrightarrow{t_1} \mathcal{O}_W \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0,$$

where $\mathcal{O}_W(0,-1)$ denotes the pullback to $W \subset \mathbb{P}^1 \times \mathbb{P}^n$ of $\mathcal{O}_{\mathbb{P}^1}(-1)$. The sequences are exact because W is irreducible.

Twisting by the pullback to $\mathbb{P}^1 \times \mathbb{P}^n$ of any line bundle $\mathcal{O}_{\mathbb{P}^n}(l)$ on $\mathbb{P}^n$, and using the additivity of the Euler characteristic, we see that

$$\chi(\mathcal{O}_{Z_\infty}(l)) = \chi(\mathcal{O}_{Z_0}(l)) \text{ for every integer } l,$$

and in particular the Hilbert polynomials of Z_0 and Z_∞ are the same. □

For any subscheme $Y \subset \mathbb{P}^n$ we define the *degree of Y* to be the leading coefficient of the Hilbert polynomial $l \mapsto \chi(\mathcal{O}_Y(l))$, multiplied by $(\dim Y)!$. This is of course related to the degree map $\deg A_0(\mathbb{P}^n) \to \mathbb{Z}$ by the formula $\text{degree } Y = \deg(L \cap Y)$ where L is a linear space of dimension $n - \dim Y$ that meets Y in a zero-scheme. We can extend this definition to any cycle $Z = \sum_i m_i \langle X_i \rangle$ of pure dimension m by setting

$$\text{degree } Z = \sum_{\{i | \dim X_i = m\}} m_i \text{ degree } X_i$$

If Y is any scheme, then from the definition and the additivity of the Euler characteristic we see that $\text{degree } Y = \text{degree } \langle Y \rangle$. We can now prove Theorem **??**:

Theorem 4.3. *The m-th Chow group of $\mathbb{P}^n$ is*

$$A_m(\mathbb{P}^n) \cong \mathbb{Z},$$

generated by the class of a plane L of dimension m. The class of any cycle of pure dimension m and degree d is $d[L]$.

Proof. We already know from Proposition **??** that the class $[L]$ of a plane L of dimension m generates $A_m(\mathbb{P}^n)$ (see the first part of the proof of Theorem 1.21). If X, X' are cycles that are rationally equivalent by a rational equivalence $Z = \sum m_i \langle Z_i \rangle$, where the $Z_i \subset \mathbb{P}^1 \times \mathbb{P}^n$ are irreducible subvarieties dominating $\mathbb{P}^1$, then for each i the fibers $(Z_i)_0$ and $(Z_i)_\infty$ have the same Hilbert polynomial, and thus the same degree. Since degree is additive, and defined on the associated cycle, we get degree $X =$ degree X'. We must have $[X] = d[L]$ and $[X'] = e[L]$ for some d, e and this shows that $d = e$. In particular, if $d[L] = 0$, then $d = 0$. □

Given the constancy of Hilbert polynomials in Proposition 4.2, it may seem odd that the *only* properties of a variety in $\mathbb{P}^n$ preserved by rational equivalence are its dimension and degree—after all, the Hilbert polynomial encodes the arithmetic genus, and more information as well. The reason is two schemes defining the same cycle can have very different embedded components. We will see this in detail in Example 4.7.

4.1.1 Examples of Rational Equivalences

We now turn to some useful examples of rational equivalences.

Example 4.4 (Lines)**.** To start with the simplest of all such equivalences, let $L, L' \subset \mathbb{P}^2$ be two lines, intersecting at a point p. The set of lines in $\mathbb{P}^2$ passing through p is itself a line in the dual plane: explicitly, if we take $p = [0, 0, 1]$, $L = V(X_0)$ and $L' = V(X_1)$, we can write the family of lines L_t in $\mathbb{P}^2$ through p as

$$\{L_t = V(t_0 X_0 - t_1 X_1)\}_{t=[t_0,t_1]\in\mathbb{P}^1}.$$

If we then set

$$W = \{(X, t) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid t_0 X_0 - t_1 X_1\},$$

the subvariety $W \subset \mathbb{P}^2 \times \mathbb{P}^1$ is a rational equivalence between $W_0 = L$ and $W_\infty = L'$. Note that by taking chains of such equivalences, we see more generally any two lines in $\mathbb{P}^n$ are rationally equivalent.

Similarly, any two hypersurfaces $F = 0$ and $G = 0$ of the same degree in $\mathbb{P}^n$ are rationally equivalent via the scheme $\mathcal{W} \subset \mathbb{P}^n \times \mathbb{P}^1$ defined by the bihomogeneous equation $t_0 F(x) - t_1 G(x) = 0$.

********the following (up to the end of the proof of Proposition 4.5) is a rewrite; the original is still in the text, commented out********

Dynamic projection. Since the ideals of varieties of higher codimension are typically generated by many forms, it is not clear how to interpolate between them in general. In projective space, however, we can make use of the large group of automorphisms of $\mathbb{P}^n$ and construct families made from one-parameter subgroups of PGL_{n+1}. Since every representation of the multiplicative group can be diagonalized over an algebraically closed field, one can always write such an action, in suitable coordinates, in the form

$$G_m \ni t : (x_0, \ldots, x_n) \mapsto (t^{w_0} x_0, \ldots, t^{w_n} x_n)$$

for some integral weights w_n. For simplicity we will consider a special case:

Fix two disjoint planes A (the "attractor") and R (the "repellor") that are disjoint and span $\mathbb{P}^n$. Choose coordinates $x_0, \ldots, x_r, y_0, \ldots, y_a$ on $\mathbb{P}^n$ so that the equations of A are $\{x_i = 0\}$ and the equations of R are $\{y_i = 0\}$, and consider the action Ψ of the multiplicative group G_m on $\mathbb{P}^n$ given by

$$\begin{aligned}\psi_t : (x_0, \ldots, x_a, y_0, \ldots, y_r) \mapsto &(tx_0, \ldots, tx_a, y_0, \ldots, y_r) \\ = &(x_0, \ldots, x_a, t^{-1}y_0, \ldots, t^{-1}y_r).\end{aligned}$$

(In what follows, we'll abbreviate $(x_0, \ldots, x_a, y_0, \ldots, y_r)$ to (x, y); for example, we'll write $\psi_t(x, y) = (tx, y)$.) It is clear that the points of A and R remain fixed under the action of G_m. On the other hand we can say intuitively that a point not in A or R will "flow toward A" as t approaches zero, and will "flow toward R" as t approaches ∞. More precisely, note that any point $p \notin A \cup R$ lies on a unique line that meets both A and R. (The span $\overline{p, A}$, being an $(a+1)$-plane, must meet R. Since A and R are disjoint, $\overline{p, A}$ can meet R only in a point $q \in R$; the line $\overline{p, q}$ is then the unique line containing p and meeting A and R.) This line is the closure of the orbit of p under the given action of G_m. In particular, any point in $\mathbb{P}^n \setminus R$ has a well-defined limit in A as t approaches zero.

Now suppose $X \subset \mathbb{P}^n$ is any variety. We consider the images of X under the automorphisms ψ_t, and in particular their limit as $t \to 0$: that is, we set

$$\tilde{Z} = \{(t, p) : p \in \psi_t(X)\} \subset G_m \times \mathbb{P}^n;$$

we let $Z \subset \mathbb{A}^1 \times \mathbb{P}^n$ be the closure of $\tilde{Z}$ and look at the fiber X_0 of Z over $t = 0$. We think of X_0 as the limit of the varieties $\psi_t(X)$ as t approaches 0. This way of thinking is justified by the fact that Z is flat over $\mathbb{A}^1$, and there are unique flat limits of one-parameter flat families of schemes. See for example Eisenbud and Harris [2000] Chapter ****. ****insert picture(s)****

The following properties of the limit X_0 make it easy to analyze some interesting cases:

Proposition 4.5. *With notation as above:*

(a) $X_0 \subset \mathbb{P}^n$ *is stable under the action of* G_m.

(b) $X_0 \cap R = X \cap R$

(c) $(X_0)_{\text{red}}$ *is contained in the cone over* $X_0 \cap R$ *with base* A *(in case* $X_0 \cap R = \varnothing$*, we take this to mean* $(X_0)_{\text{red}} \subset A$*.)*

In addition, we know (as for any rational equivalence) that X_0 is equidimensional, and $\dim X_0 = \dim X$; and, by Proposition 4.2, that the Hilbert polynomial of X_0 is the same as the Hilbert polynomial of X. See Exercise 4.39 for the general case of a G_m action.

Proof. For the first part, consider the action of G_m on the product $\mathbb{A}^1 \times \mathbb{P}^n$ given as the product of the standard action of G_m on $\mathbb{A}^1$ and the action Ψ above of G_m on $\mathbb{P}^n$; that is,

$$\varphi_t : (s, p) \to (ts, \psi_t(p)).$$

This carries Z to itself and the fiber $\{0\} \times \mathbb{P}^n$ to itself, so it carries X_0 to itself. But it acts on the fiber $\{0\} \times \mathbb{P}^n$ via the action Ψ above; thus X_0 is invariant under Ψ.

The second point is more subtle. (In particular, it's asymmetric: the same statement, with R replaced by A, would be false.) It is not, however, intuitively unreasonable: since points in $\mathbb{P}^n \setminus R$ flow away from R as $t \to 0$, the only way a point $p \in R$ can be a limit of points $\psi_t(p_t)$ is if it's there all along, that is, $p \in X \cap R$.

In any case, note first that one inclusion is immediate: since R is fixed pointwise by the automorphisms ψ_t, we have

$$\mathbb{P}^1 \times (X \cap R) \subset Z$$

and hence $X \cap R \subset X_0 \cap R$. To see the other inclusion, we want to show that the ideal $I(X \cap R)$ is contained in $I(X_0 \cap R)$. Let $f(x) \in I(X \cap R)$. We can then write

$$f(x) = g(x, 0) \quad \text{for some } g(x, y) \in I(X).$$

Now observe that

$$I(Z) \supset \{f(x, ty) : f \in I(X)\}$$

so $h(t, x, y) = g(x, ty) \in I(Z)$. Setting $t = 0$, we see that $g(x, 0) \in I(X_0)$, and hence $f \in I(X_0 \cap R)$.

To prove the third assertion, note that the G_m orbit of any point not contained in $R \cup A$ is a straight line joining a point of R to a point of A. Since X_0 is stable under G_m, it is the union of such lines, together with any points of $A \cup R$ it contains. □

We sometimes call this construction *dynamic projection*: we are realizing the projection map

$$\begin{aligned}\pi_R : \mathbb{P}^n \setminus R &\to A \\ : (x, y) &\mapsto (0, y)\end{aligned}$$

as the limit of a family of automorphisms ψ_t of $\mathbb{P}^n$. As we'll see, though, considering the limit of the images $\psi_t(X)$ yields more information than simply taking the projection $\pi_R(X)$.

Example 4.6. Let $D \subset \mathbb{P}^n$ be a subvariety of dimension m and degree d. We will construct an irreducible subvariety of $\mathbb{P}^n \times \mathbb{P}^1$ that gives a rational equivalence between the cycle $\langle D \rangle$ and the d-fold m-plane $d\langle A \rangle$, and another that gives a rational equivalence between D and the reduced union of d distinct m-planes containing a fixed $(m-1)$-plane.

Of course Theorem 4.3 already implies that these varieties are rationally equivalent, but knowing that there is an irreducible rational equivalence is stronger; we will make use of the extra strength in our proof of the Moving Lemma below.

To make the first construction, let $A \subset \mathbb{P}^n$ be any m-dimensional subspace, and choose an $n-m-1$-plane $R \subset \mathbb{P}^n$ disjoint from D and from A, to which to apply Proposition 4.5. Since $X \cap R = \varnothing$ we see that $X_0 \cap R = \varnothing$ as well, and it follows that $(X_0)_{\text{red}} \subset A$. Since $\dim X_0 = \dim X = \dim A$, we see that the support of X_0 is exactly A, and computing the degree we have $\langle X_0 \rangle = d\langle A \rangle$, as claimed.

To make the second construction, choose the repelling subspace R to be a general plane of dimension $n-m$, so that $R \cap X$ consists of degree X distinct points, and take A to be an $m-1$-plane disjoint from R. We see that $(X_0)_{\text{red}}$ is contained in the union of the m-planes that are the cones over the degree X points of $X \cap R = X_0 \cap R$. Also, X_0 must contain all these points. Since X_0 is equidimensional of degree degree X, it follows that X_0 is the generically reduced union of these distinct planes, as required.

Example 4.7 (Twisted cubics)**.** Let $X \subset \mathbb{P}^3$ be a twisted cubic curve. Let A be any 2-plane in $\mathbb{P}^3$, and let $R \notin A \cup X$ be a point on some tangent line to X. We will show that applying the construction of Proposition 4.5, the scheme X_0 will be generically reduced, with support equal to the cubic curve with a cusp that is the projection of X from R to $\mathbb{P}^2$. Since this cuspidal curve has genus 1, while X has genus 0, the scheme X_0 cannot be reduced. In fact, it has an embedded point not contained in the plane, located at the cusp of $(X_0)_{\text{red}}$. In this case, we can compute everything:

We may take the equations of X to be the three 2×2 minors of the matrix

$$M := \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

We choose the repelling plane R to be the point $x_0 = x_1 = x_3 = 0$ and the attracting plane A to be the 2-plane $x_2 = 0$, so that the equations of tX, for $t \in G_m$, are the minors of

$$M_t := \begin{pmatrix} tx_0 & tx_1 & x_2 \\ tx_1 & x_2 & tx_3 \end{pmatrix}.$$

As usual we write $Z \subset \mathbb{A}^1 \times \mathbb{P}^3$ for the corresponding rational equivalence.

Dividing each of the three 2×2 minors of M_t by the biggest possible power of t, we see that the ideal of W contains the polynomials

$$\begin{aligned} f_1 =& x_0x_2 - tx_1^2, \\ f_2 =& tx_0x_3 - x_1x_2, \\ f_3 =& x_2^2 - t^2x_1x_3. \end{aligned}$$

Since $I(Z)$ contains the polynomial $x_1f_1 + x_2f_2 = t(x_0^2x_3 - x_1^3)$, it also contains the polynomial

$$g = x_0^2x_3 - x_1^3.$$

(A Gröbner basis computation shows that the polynomials f_i and g generate the ideal of W—see Eisenbud [1995], Chapter 15—but we will not need this.)

By setting $t = 0$ we see that the fiber X_0 of Z over $0 \in \mathbb{A}^1$ is contained in the subscheme $X_0' \subset \mathbb{P}^3$ defined by the ideal

$$(x_0x_2,\ x_1x_2,\ x_2^2,\ g).$$

The form g alone defines the cone in $\mathbb{P}^3$ over a reduced irreducible cubic curve $D = \{g = x_2 = 0\}$, lying in the plane $x_2 = 0$ and having a cusp at the point $x_0 = x_1 = x_2 = 0$. The rest of the equations

$$(x_0x_2, x_1x_2, x_2^2) = x_2(x_0, x_1, x_2) = (x_2) \cap (x_0, x_1, x_2^2)$$

define the union of the plane $x_2 = 0$ and an embedded point at $x_0 = x_1 = x_2 = 0$. Thus X_0' is the non-reduced scheme consisting of the reduced, irreducible cuspidal plane cubic curve together with an embedded point at the cusp, not contained in the plane.

Now the fiber X_0 of Z must be one-dimensional, and it follows that it must at least contain the cuspidal cubic D and be contained in X_0'. Since the arithmetic genus of X_0 must agree with that of X, which is 0, while the arithmetic genus of D is 1, we must have $D \subsetneq X_0 \subset X_0'$. But since the multiplicity of the embedded point of X_0' is only 1, it follows that $X_0 = X_0'$.

On the other hand, since the cycle associated to X_0 is insensitive to the embedded points, we have

$$[X] \sim [X_0] = [D],$$

as claimed.

4.2 Rational Equivalence through Divisors

Another important way to construct rational equivalences is by using rational functions on subvarieties of X (in fact, Corollary 4.17 shows that this method can be used to construct all rational equivalences, and thus gives an alternate presentation of $A_*(X)$.) ****insert picture**** To set the stage, we review the definition of the divisor associated to a nonzero rational function on an arbitrary variety.

Consider first the affine case, and the divisor associated to a nonzero principal ideal in a domain. Let U be an affine variety with coordinate ring A, and let $f \in A$ be a nonzero function. Krull's Principle Ideal Theorem asserts that each irreducible component Y of the subscheme $V(f)$ defined by f is of codimension 1 in X. Thus the local ring $R = \mathcal{O}_{U,Y}$ of the generic point of Y is 1-dimensional, so, as above, the factor ring $R/(f)$ is of finite length. We define the *order* of f at Y to be this length, for which we will write $\mathrm{ord}_R(f)$ or $\mathrm{ord}_{Y/U}(f)$, so that

$$\mathrm{ord}_R(f) = \mathrm{ord}_{Y/U}(f) = \text{length } \mathcal{O}_{U,Y}/f\mathcal{O}_{U,Y}.$$

We define the divisor of f to be the divisor of the subscheme defined by f,

$$Div(f) := \sum_{\substack{\{Y | Y \text{ is a} \\ \text{component of } V(f)\}}} \mathrm{ord}_Y(f) \cdot Y.$$

Note that, by Krull's Principal Ideal Theorem all the components of $V(f)$ have codimension exactly 1 in U.

What makes the case of a principal ideal special is that the association $f \mapsto Div(f)$ turns multiplication into addition: that is,

Proposition 4.8. *If f, g are nonzerdivisors in a local Noetherian ring R of dimension 1, then*

$$Div_X(fg) = Div(f) + Div(g).$$

Proof. Since g is a nonzerodivisor, multiplication by g defines an isomorphism $R/(f) \to (g)/(fg)$. The resulting exact sequence

$$0 \longrightarrow R/(f) \xrightarrow{g} R/(fg) \longrightarrow R/(g) \longrightarrow 0,$$

gives $\mathrm{ord}_R(fg) = \mathrm{ord}_R(f) + \mathrm{ord}_R(g)$. □

This additivity has a crucially important consequence: we can define the divisor not just of a regular function, but also of a rational function. Suppose that $\alpha \in K(X)$ is a rational function on a (not necessarily affine) variety X, and Y is any codimension 1 subvariety in X, then we can choose nonzero $f, g \in R := \mathcal{O}_{X,Y}$ such that $\alpha = f/g$. The number $\mathrm{ord}_R(f) -$

$\mathrm{ord}_R(g)$ does not depend on the choice of f and g, since if $f/g = f'/g'$ we will have $fg' = f'g$, whence by additivity $ord_R(f) + ord_R(g') = ord_R(f') + ord_R(g)$, so that $\mathrm{ord}_R(f) - \mathrm{ord}_R(g) = \mathrm{ord}_R(f') - \mathrm{ord}_R(g')$. Thus we can unambiguously define $\mathrm{ord}_Y(\alpha) = \mathrm{ord}_R(f) - \mathrm{ord}_R(g)$.

We claim that, for any nonzero rational function α, there are only finitely many codimension 1 subvarieties $Y \subset X$ such that $\mathrm{ord}_Y(\alpha) \neq 0$. To see this, take a finite covering by affine open subsets U_i. For each i, write $\alpha|_{U_i} = f_i/g_i$ with $f_i, g_i \in \mathcal{O}_X(U_i)$. If Y appears with a nonzero coefficient in $Div(\alpha)$, and U_i is one of the open sets such that $Y \cap U_i$ is dense, then $Y \cap U_i$ must be among the finitely many irreducible components of $V(f_i)$ or $V(g_i)$. This leaves only finitely many possibilities for Y.

Proposition 4.9. *Let X be a variety of dimension n. There is a homomorphism of groups*

$$Div : K(X)^* \to Rat_{n-1}(X) \subset Z_{n-1}(X)$$

taking $\alpha \in K(X)^*$ to

$$Div(\alpha) = \sum_{\substack{\{Y | Y \text{ is a} \\ \text{codimension 1} \\ \text{subvariety of } X.\}}} \mathrm{ord}_Y(\alpha) Y. \quad \square$$

Proof. The argument above shows that the map $Div : K(X)^* \to Z_{n-1}(X)$ is well-defined. To see that its image is in $Rat_{n-1}(X)$, let α be a rational function on X. The element α defines a rational map $\varphi : X \to \mathbb{P}^1$ and the graph of φ is a subset of $X \times \mathbb{P}^1$ birational to X. What we have called $Div(\alpha)$ is, under this isomorphism, precisely $\varphi^{-1}(0) - \varphi^{-1}(\infty)$. $\square$

Example 4.10. Let $P \subset \mathbb{P}^3$ be a plane. Any two curves $D, D' \subset P$, both of degree d, are rationally equivalent in P and thus in $\mathbb{P}^3$: for if $g = 0$ and $g' = 0$ are the equations of D and D' respectively, then the ratio $\alpha = g/g'$ is a rational function on P because g and g' are homogeneous of the same degree. Further, $Div(\alpha) = [D] - [D']$.

Here are some observations that will help us with further examples:

Proposition 4.11. *Let X be a scheme.*

(a) $A_*(X) = A_*(X_{\mathrm{red}})$.

(b) *If X is a variety of dimension d, then $A_d(X) \cong \mathbb{Z}$, with generator $[X]$, called the* fundamental class *of X. More generally, if X is reduced, then $A_d(X)$ is the free abelian group on the classes $[X_i]$ of the irreducible components of X that have dimension d.*

(c) *If $Y \subset X$ is a closed subscheme and $U = X \setminus Y$ is the complementary open subscheme, then there is a right-exact sequence*

$$A_*(Y) \xrightarrow{\beta} A_*(X) \xrightarrow{\alpha} A_*(U) \longrightarrow 0.$$

(d) *If X_1, X_2 are closed subschemes of X, then there is a right exact sequence*

$$A_*(X_1 \cap X_2) \longrightarrow A_*(X_1) \oplus A_*(X_2) \longrightarrow A_*(X_1 \cup X_2) \longrightarrow 0.$$

Proof of Proposition 4.11. (a) By definition the generators of $Z_*(X)$ are varieties in X, and since they are reduced they are contained in X_{red}. The same argument shows that $Z_*(X \times \mathbb{P}^1) = Z_*(X_{\mathrm{red}} \times \mathbb{P}^1)$, so the cokernel of ∂ is unchanged if we replace X by X_{red}.

(b) Since $X \times \mathbb{P}^1$ is the only subvariety of $X \times \mathbb{P}^1$ of dimension $d+1$, the map $\partial_X : Z_{d+1}(X \times \mathbb{P}^1) \to Z_d(X)$ is zero, so $A_d(X) = Z_d(X) = \mathbb{Z} \cdot [X]$.

For the second statement, let $X_1, \ldots, X_m$ be the distinct irreducible components of X of dimension d. By definition, $A_d(X)$ is generated by $[X_1], \ldots, [X_m]$. Any relation among the $[X_i]$ comes from a variety $W \subset X \times \mathbb{P}^1$ dominating $\mathbb{P}^1$ and containing $X_i \times \{0\}$.

Since $W = \bigcup_i (W \cap (X_i \times \mathbb{P}^1))$ and W is irreducible, we must have $W \subset X_i \times \mathbb{P}^1$. Since W dominates $\mathbb{P}^1$, it is strictly larger than $X_i \times 0$, so $\dim W \geq 1 + \dim X_i = \dim(X_i \times \mathbb{P}^1)$. Thus $W = X_i \times \mathbb{P}^1$. Thus the relation on $A_d(X)$ provided by W is of the form $[X_i] = [X_i]$—the trivial relation.

(c) There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_*(Y \times \mathbb{P}^1) & \longrightarrow & Z_*(X \times \mathbb{P}^1) & \longrightarrow & Z_*(U \times \mathbb{P}^1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_Y} & & \downarrow{\scriptstyle \partial_X} & & \downarrow{\scriptstyle \partial_U} & & \\
0 & \longrightarrow & Z_*(Y) & \longrightarrow & Z_*(X) & \longrightarrow & Z_*(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A_*(Y) & \longrightarrow & A_*(X) & \longrightarrow & A_*(U) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the map $Z_*(Y) \to Z_*(X)$ takes the class $[A] \in Z_*(Y)$, where A is a subvariety of Y, to $[A]$ itself, considered as a class in X, and similarly for $Z_*(Y \times \mathbb{P}^1) \to Z_*(X \times \mathbb{P}^1)$. The map $Z_*(X) \to Z_*(U)$ takes each free generator $[A]$ to the generator $[A \cap U]$, and similarly for $Z_*(X \times \mathbb{P}^1) \to Z_*(U \times \mathbb{P}^1)$. The two middle rows and all three columns are evidently exact. A diagram chase shows that the map $A_*(X) \to A_*(U)$ is surjective, and the bottom row of the diagram above is right exact, yielding part (c).

(d) Let $Y = X_1 \cap X_2$. We may assume $X = X_1 \cup X_2$. We may argue exactly as before from the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_*(Y \times \mathbb{P}^1) & \longrightarrow & Z_*(X_1 \times \mathbb{P}^1) \oplus Z_*(X_2 \times \mathbb{P}^1) & \longrightarrow & Z_*(X \times \mathbb{P}^1) & \longrightarrow & 0 \\
 & & \partial \downarrow & & \partial \oplus \partial \downarrow & & \partial \downarrow & & \\
0 & \longrightarrow & Z_*(Y) & \longrightarrow & Z_*(X_1) \oplus Z_*(X_2) & \longrightarrow & Z_*(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A_*(Y) & \longrightarrow & A_*(X_1) \oplus A_*(X_2) & \longrightarrow & A_*(X) & \longrightarrow & 0
\end{array}$$

where, for example, the map $Z_*(Y) \longrightarrow Z_*(X_1) \oplus Z_*(X_2)$ takes a generator $[A] \in Z_*(Y)$ to $([A], -[A]) \in Z_*(X_1) \oplus Z_*(X_2)$ and the map $Z_*(X_1) \oplus Z_*(X_2) \longrightarrow Z_*(X)$ is addition. □

Proposition 4.11 allows us to answer Keynote Question (b) in the negative. Indeed, we can generalize as follows.

Corollary 4.12. *If $X \subset \mathbb{P}^n$ is a variety of dimension m and degree d, then $A_m(\mathbb{P}^n \setminus X) \cong \mathbb{Z}/(d)$, while if $m < m' \leq n$ then $A_{m'}(\mathbb{P}^n \setminus X) = \mathbb{Z}$. In particular, m and d are determined by the isomorphism class of $\mathbb{P}^n \setminus X$.* □

Proof. Part (c) of Proposition 4.11 shows that there are exact sequences $A_i(X) \to A_i(\mathbb{P}^n \to A_i(\mathbb{P}^n \setminus X \to 0$. Furthermore $A_m(X) = \mathbb{Z}$ by Part **??** of Proposition 4.11, while $A_{m'}(X) = 0$ for $m < m' \leq n$. By Theorem 1.21 we have $A_i(\mathbb{P}^n) = \mathbb{Z}$ for $0 \leq i \leq n$, and the image of the generator of $A_m(X)$ in $A_m(\mathbb{P}^n)$ is d times the generator of $A_i(\mathbb{P}^n)$. The results of the Corollary follow. □

4.3 Proper push-forward

To get much further we need stronger tools. One of them is very compactly expressed below by saying that $A(X)$ behaves like a covariant functor with respect to proper maps, and this preserves the grading by dimension. This can be used to prove, for example, that the class of a given cycle is nonzero—we just exhibit a proper map that takes it to a cycle whose class we already know to be nonzero. We will see examples of this below.

If $f : X \to X'$ is a proper map of schemes, then the image of a subvariety $Y \subset X$ is a subvariety $f(Y) \subset X'$. This would make it possible to define a push-forward map f_* on cycles by taking any subvariety to its image. However, this map gives no information about the Chow groups because it does not preserve rational equivalence (see Exercise **??**). But if we introduce

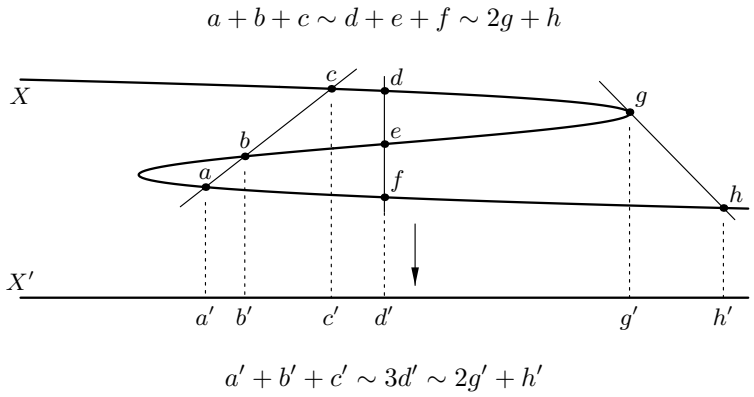


FIGURE 1.2. Pushforwards of equivalent cycles are equivalent

certain multiplicities, rational equivalence will be preserved, and we get a map on Chow groups. Here is how this is done:

Definition 4.13. Let $f : X \to X'$ be a proper map of schemes, and let $Y \subset X$ be a subvariety.

(a) If $f(Y)$ has lower dimension than Y, then we define $f_*(\langle Y \rangle) = 0$.

(b) If $\dim f(Y) = \dim Y$, then the map $f|_Y : Y \to f(Y)$ is generically finite. If $n := [K(Y) : K(f(Y))]$ is the degree of the extension of fields of rational functions, we say that f is *generically a cover of degree* n, and we define $f_*\langle Y \rangle = n\langle f(Y) \rangle$.

(c) If $z = \sum n_i \langle Y_i \rangle \in Z(X)$ is a formal $\mathbb{Z}$-linear combination of subvarieties, then we define $f_*(z) = \sum n_i f_* \langle Y_i \rangle$.

Theorem 4.14. *If $f : X \to X'$ is a proper map of schemes, then the map $f_* : Z_*(X) \to Z_*(X')$ defined above preserves rational equivalence, and thus induces a map $f_* : A_*(X) \to A_*(X')$.*

We will prove Theorem 4.14 and the closely related Theorem 4.16 below after developing an important algebraic tool in Lemma 4.18.

Before starting the proof of Theorem 4.14 we discuss some geometric consequences. A first hint of the importance of the apparently bland statement of the Theorem is that it makes possible the existence of a degree function $A_0(X) \to \mathbb{Z}$ when X is projective over a field. In particular, it allows us to prove that $A_0(\mathbb{P}^n_K) = \mathbb{Z} \cdot [p]$ for any closed K-rational point $p \in \mathbb{P}^n$, something we could not prove in the last section.

Corollary 4.15. *Let K be a field, and let X be a scheme that is proper over* $\operatorname{Spec} K$. *There is a map $A_0(X) \to \mathbb{Z}$ that sends the class $[Y]$ of a*

zero-dimensional subscheme $Y \subset X$ to $\deg Y := \dim_k \mathcal{O}_Y$. *In particular* $A_0(X) \neq 0$.

Proof. First, the definition of $\deg : Z_0(X) \to \mathbb{Z}$ makes sense because any zero-dimensional scheme, proper over $\operatorname{Spec} K$, is finite over $\operatorname{Spec} K$. Let $f : X \to \operatorname{Spec} K$ be the constant map. If p is a closed point of X then, by definition, $f_*(\langle p \rangle) = \dim_K \kappa(p) \cdot \langle \operatorname{Spec} K \rangle$, so $\deg(\langle p \rangle) = \deg f_*(\langle p \rangle) = \dim_k \kappa(p)$. By Theorem 4.14, the map $\deg \circ f_*$ is well-defined on $A_0(X)$. □

The idea of this proof also shows that we cannot drop the assumption of properness in Theorem 4.14: for example, if we could define f_* on $A_0(\mathbb{A}^n_K)$ when $f : \mathbb{A}^n_K \to \operatorname{Spec} K$, we would deduce, as above, that $A_0(\mathbb{A}^n_k) \neq 0$ and we have already seen that $A_0(\mathbb{A}^n_K) = 0$. See Exercises 4.25 and 4.26 for further examples.

The statement of Theorem 4.14 is obvious when f is a closed immersion since in this case any rational equivalence in $Z(X)$ is also a rational equivalence in $Z(X')$. But already in this case, we can use the result to answer Keynote Question a in the negative: If $X \subset \mathbb{P}^3$ is a surface with a map $\pi : X \to \mathbb{P}^1$ then the fibers of π are rationally equivalent in X. By Theorem 4.14 they are also rationally equivalent in $\mathbb{P}^3$. By Theorem 4.3 this implies that they have the same degree as projective curves.

We can sharpen Theorem 4.14, in the generically finite case, by giving an explicit formula for the push-forward of the divisor of a rational function.

Theorem 4.16. *Let $f : X \to Y$ be a proper, generically finite morphism of varieties, and let α be a rational function on X. If $N : K(X) \to K(Y)$ denotes the norm function,*

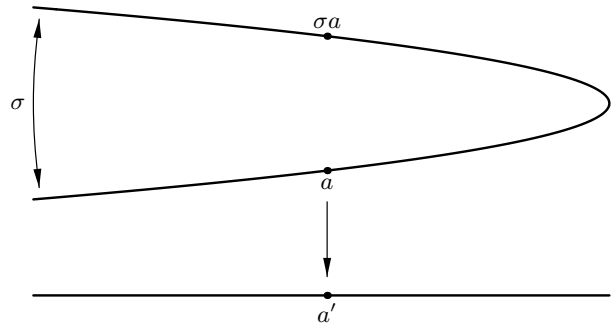
$$N(\alpha) := \det{}_{K(Y)}\big[K(X) \xrightarrow{\alpha} K(X)\big]$$

then $f_*(Div(\alpha)) = Div(N(\alpha)) \in Z_*(Y)$. In particular, the push-forward of the divisor of a rational function under a generically finite morphism is again the divisor of a rational function.

We saw (just before Proposition 4.11) that if W is a subvariety of X and α is a rational function on W, then $Div(\alpha)$, regarded as a cycle on X, is rationally equivalent to zero. As a consequence of Theorem 4.16 we can show that the divisors of rational functions suffice to generate the relation of rational equivalence:

Corollary 4.17. *For any scheme X, the kernel of the map $Z_*(X) \to A_*(X)$ is generated by the divisors of the form $Div(\alpha)$ where α ranges over rational functions on subvarieties of X.*

Proof of Corollary 4.17. We already know that such $Div(\alpha)$ are in the kernel. On the other hand, the kernel is generated by elements of the form



$$\langle a' \rangle = \pi_* \langle a \rangle$$

$$\langle a \rangle + \langle \sigma a \rangle = \pi^* N \langle a \rangle$$

FIGURE 1.3. The pushforward of the divisor of a function is the divisor of its norm

$\langle V_0 \rangle - \langle V_\infty \rangle$ where the V_i are subschemes of X such that there exists a variety $V \subset X \times \mathbb{P}^1$ generically finite over its image in X, and dominating $\mathbb{P}^1$, with $V \cap X \times \{0\} = V_0$ and $V \cap X \times \{\infty\} = V_\infty$. As a cycle on V we have $\langle V_0 \times \{0\} \rangle - \langle V_\infty \times \{\infty\} \rangle = Div(\alpha)$, where α is the rational function corresponding to the projection map $V \to \mathbb{P}^1$.

Let V' be the image of V in X, and note that V' is a subvariety. As cycles on X we have $V_0 = f_*(\langle V_0 \times \{0\} \rangle)$, and similarly for V_∞. Thus $\langle V_0 \rangle - \langle V_\infty \rangle = f_*(Div(\alpha)) = Div(N(\alpha))$, the divisor of a rational function on V', completing the argument. □

4.3.1 The Determinant of a Homomorphism

The proof of Theorems 4.14 and 4.16 rely on a fundamental result in commutative algebra relating a homomorphism to its determinant.

Suppose that R is a domain with quotient field K. If M is a finitely-generated torsion-free R-module and $\varphi : M \to M$ is an endomorphism, then $\varphi \otimes_R K$ is an endomorphism of the finite dimensional vector space $M \otimes_R K$. We define the determinant $\det(\varphi)$ of φ to be the usual determinant of $\varphi \otimes_R K$, an element of K.

For example, if $r \in R$ and φ is multiplication by r, then $\det \varphi = r^{\operatorname{rank} M}$; this is because the endomorphism $\varphi \otimes_R K$ of $M \otimes_R K = K^{\operatorname{rank} M}$ is also

multiplication by r, represented by a diagonal matrix

$$\begin{pmatrix} r & 0 & \dots & 0 \\ 0 & r & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & r \end{pmatrix}$$

Lemma 4.18 (Determinant Lemma)**.** *Suppose that R is a 1-dimensional local Noetherian domain, and let $\varphi : M \to M$ be an endomorphism of a finitely generated torsion-free R-module M. If $\det \varphi$ is nonzero, then* coker *φ is a module of finite length, and its length is equal to $\operatorname{ord}_R(\det \varphi)$. In particular, if $r \in R$ and M has rank n, then*

$$\operatorname{length}(M/rM) = n \cdot \operatorname{length}(R/rR).$$

The Lemma implies, in particular, that $\operatorname{ord}_R \det \varphi \geq 0$, even when $\det \varphi$ is not in R. This comes about because $\det \varphi$ is always in the integral closure of R (use, for example, Eisenbud [1995], Corollary 4.6).

When M is free and the endomorphism is defined by a diagonal matrix, the result says that

$$\text{length } R/(\prod_i x_i) = \sum_i \text{length } R/(x_i),$$

an easy special case of Proposition 4.8. The idea of the proof we will give is to reduce to this situation by moving to the integral closure of R and then localizing so that R becomes a discrete valuation ring. We can then use the structure of matrices over such rings.

Proof. The second statement follows from the first because, taking φ to be multiplication by r on M, the remark just before the Lemma shows that $\det \varphi = r^n$.

In the remainder of the proof we will assume that the domain R is a localization of an algebra finitely generated over a field, so that the integral closure R' of R is a finite R-module. This suffices for the applications we will make. See the remarks after the proof for various ways of removing this restriction and generalizing the lemma further.

Let $\varphi : M \to M$ be an endomorphism with nonzero determinant D. Since R has dimension 1, the cokernel of φ is a module of dimension zero, and thus finite length. Let R' be the integral closure of R in its quotient field K, and let $M' = M \otimes_R R'/(\text{torsion})$, so that $M \subset M' \subset K \otimes_R M$ (in fact, M' is the image of $M \otimes_R R'$ in $M \otimes_R K$.)

We will use three properties of this construction. First, the endomorphism φ extends to $\varphi \otimes 1 : M \otimes_R R' \to M \otimes_R R'$, and preserves the torsion submodule, so it induces a map $\varphi' : M' \to M'$ extending the map φ on M.

Second, since $M' \subset M \otimes_R K$, the quotient M'/M is torsion, and thus of finite length. Of course φ' induces an endomorphism of M'/M, which we will call $\overline{\varphi'}$. Finally, since $M \otimes_R K = M' \otimes_R K = M \otimes_{R'} K$, the maps φ and φ' have the same determinant $D = \det \varphi = \det \varphi'$. We will reduce to the case where $R = R'$, and then use the structure theorem for modules over a principal ideal domain.

With these points in mind, consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & \operatorname{Ker}\overline{\varphi'} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & M'/M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \overline{\varphi'}} & & \\
0 & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & M'/M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker}\ \varphi & \longrightarrow & \operatorname{coker}\ \varphi' & \longrightarrow & \operatorname{coker}\ \overline{\varphi'} & \longrightarrow & 0.
\end{array}$$

A diagram chase (the "Snake Lemma") yields an exact sequence

$$0 \to \operatorname{Ker}\ \overline{\varphi'} \to \operatorname{coker}\ \varphi \to \operatorname{coker}\ \varphi' \to \operatorname{coker}\ \overline{\varphi'} \to 0,$$

from which we deduce that

$$\begin{aligned}
&\text{length coker } \varphi' - \text{length coker } \varphi \\
&\qquad = \text{length coker } \overline{\varphi'} - \text{length Ker}\,\overline{\varphi'}.
\end{aligned}$$

But from the exact sequence

$$0 \to \operatorname{Ker}\ \overline{\varphi'} \to M'/M \xrightarrow{\overline{\varphi}'} M'/M \to \operatorname{coker}\ \overline{\varphi'} \to 0,$$

we similarly deduce that

$$\begin{aligned}
0 = \text{length } M/M' &- \text{length } M/M' \\
&= \text{length coker } \overline{\varphi'} - \text{length Ker } \overline{\varphi'},
\end{aligned}$$

proving that length coker φ = length coker φ'.

As a special case, we may take $M = R$, and we see that length $R/D =$ length R'/D. Thus it suffices to prove the Lemma in the case $R = R'$; that is, we may assume that the 1-dimensional ring R is integrally closed. By the Chinese Remainder Theorem, any R-module of finite length is the direct sum of its localizations at various maximal ideals, so we may further assume that R is a discrete valuation ring, with parameter π, say.

Any finitely-generated torsion-free module over such a ring R is free, so the endomorphism φ is defined by a square matrix of elements of R. After

row and column operations, we can reduce this matrix to a diagonal matrix with powers of π on the diagonal,

$$\varphi = \begin{pmatrix} \pi^{a_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi^{a_m} \end{pmatrix}.$$

Since $\text{length}(R/\pi^a) = a$, we see that

$$\text{length coker } \varphi = \text{length } \bigoplus_{i=1}^{m}(R/\pi^{a_i}) = \sum_{i=1}^{m} a_i$$

while

$$\text{length } R/(\det \varphi) = \text{length } R/(\prod_{i=1}^{m} \pi^{a_i}) = \sum_{i=1}^{m} a_i$$

as required. □

As we remarked in the beginning of the proof, the restriction to algebras "essentially of finite type over a field," made in order to guarantee the finiteness of the integral closure, is unnecessary. One way to avoid it is to pass to the completion, and use the result that the integral closure of any complete local Noetherian ring is finite. This is the path taken by Grothendieck [1967] 21.10.17.2. There is an attractive direct proof along very different lines in Fulton [1984], Appendix A1–A3. But the result given here and in these references is actually a special case of a far more general story involving multiplicity: the essential point is that two modules with the same multiplicity will have the same length after factoring out a regular sequence modulo which they have finite length. This idea can be used to show, for example that if $\varphi : R^p \to R^q$ is a map over a Cohen-Macaulay ring R of dimension $p - q + 1$ and the cokernel of φ has finite length, then its length is the same as that of R/I, where I is the ideal generated by all the $q \times q$ minors of φ; our Lemma 4.18 contains the case $p = q$. This result is attributed in Bruns and Vetter [1986] to Angeniol and Giusti, though the paper by Angeniol and Giusti seems never to have appeared. In any case the result is generalized in the cited paper. The proof by multiplicities and specialization was given by Hochster and Huneke (also unpublished).

Example 4.19. The second statement of Lemma 4.18 is already interesting when the rank n is 1; a good example is the case where $M = K[t]$, $R = K[t^2, t^5]$ and $r = t^5$. In this case the length is simply the dimension as a K-vector space. It is easy to see that the length of

$$M/rM = K[t]/ < t^5, t^6, t^7, \cdots >$$

is 5. The reader may check that the quotient

$$R/rR = K[t^2, t^4, t^5, t^6, \dots]/ < t^5, t^7, t^9, t^{10}, t^{11}, \cdots > .$$

is generated as a vector space by $1, t^2, t^4, t^6$, and t^8, so the length is 5 in this case too, though the two vector spaces are not related in a very obvious way.

Proof of Theorem 4.14. It suffices to show that the diagram

$$\begin{array}{ccc} Z_*(X \times \mathbb{P}^1) & \xrightarrow{(f \times 1)_*} & Z_*(X' \times \mathbb{P}^1) \\ \partial_X \downarrow & & \partial_{X'} \downarrow \\ Z_*(X) & \xrightarrow{f_*} & Z_*(X') \end{array}$$

commutes. To this end, suppose $W \subset X \times \mathbb{P}^1$ is a subvariety dominating $\mathbb{P}^1$, and let $W' = (f \times 1)(W)$. Since f is proper, W' is a subvariety of $X' \times \mathbb{P}^1$. Of course the projection $W \to \mathbb{P}^1$ factors through W', so W' dominates $\mathbb{P}^1$ if and only if W does. Write W_0 and W_0' for the preimages of $0 \in \mathbb{P}^1$ in the varieties W and W'. We may think of W_0 as a subvariety of $X = (X \times \mathbb{P}^1)_0$ and similarly for W_0' and X', and from this point of view, $W_0 = f^{-1}(W_0')$ as schemes.

****insert picture****

Suppose first that W' has strictly smaller dimension than W, so that $(f \times 1)_*\langle W \rangle = 0$. By the Principal Ideal Theorem, every component of W_0 or of W_∞ has dimension one less than that of W, and similarly the components of W_0' and W_∞' have dimension one less than that of W', so $f_*(\langle W_0 \rangle - \langle W_\infty \rangle) = 0$ as well.

Thus we may assume that $\dim W' = \dim W$, so that $W \to W'$ is generically finite. Write n for its degree, so that $(f \times 1)_*\langle W \rangle = n\langle W' \rangle$.

Let V' be an irreducible component of W_0', and suppose that $\langle V' \rangle$ appears in the cycle associated to W_0' with multiplicity p. We will show that $\langle V' \rangle$ appears in the cycle $f_*(\langle W_0 \rangle)$ with multiplicity np. Summing over the contributions of all the components of W_0' we see that $f_*(\langle W_0 \rangle) = n\langle W_0' \rangle$. The same argument will of course apply to W_∞, proving the Theorem.

Since f induces a surjective map $W \to W'$, and $V' \subset W'$ has codimension 1, the map $W \to W'$ must be finite over the generic point of V' (otherwise the preimage of V' would have dimension at least that of W, so W would be reducible.) Because the map f is proper, the induced injection

$$f^* : R' := \mathcal{O}_{W',V'} \to f_*\mathcal{O}_W \otimes_{\mathcal{O}_{W'}} \mathcal{O}_{W',V'} = \mathcal{O}_{W,f^{-1}V'} =: R'$$

is finite.

Both R' and R have dimension 1. Saying that V' occurs with multiplicity p in W_0' means that $p = \text{length}_{R'}(R'/xR')$, where x is a coordinate on $\mathbb{P}^1$ that vanishes at 0. Since $f \times 1 : W \to W'$ has degree n, the ring R has

rank n as an R'-module. The Determinant Lemma tells us that

$$\text{length}_{R'}(R/xR) = n \cdot \text{length}_{R'}(R'/xR') = np.$$

The maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ of R correspond precisely to the subvarieties $V_1, \ldots, V_s \subset W_0$. If the multiplicity of x at $\mathfrak{m}_i$ is e_i, then the Chinese Remainder Theorem shows that $\text{length}_{R'} R/xR = \sum_i e_i n_i$, where n_i is the degree of the field extension $K(V_i) : K(V')$. From the definition of f_* we see that $f_*(\langle V_i \rangle) = n_i \langle V' \rangle$. Putting this together we compute that the multiplicity of $\langle V' \rangle$ in $f_*(\langle W_0 \rangle)$ is

$$f_*(\sum_i e_i \langle V_i \rangle) = \sum_i e_i n_i \langle V' \rangle = np \langle V' \rangle$$

as required. □

Proof of Theorem 4.16. Write $Div(\alpha) = \sum a_i \langle V_i \rangle$, where the V_i are subvarieties. The cycle $f_*(Div(\alpha))$ is the sum of the terms $a_i f_*(\langle V_i \rangle)$. By definition, such a term is zero unless $\dim f(V_i) = \dim(V_i)$, in which case the restriction $f|_{V_i}$ is generically finite. Since the dimension of X is the same as that of Y, the subset of X over which f has finite fibers must contain an open set of any codimension 1 component of Y. Thus we may restrict to this set and assume, without loss of generality, that f is finite.

The components of $f_*(Div(\alpha))$ and $Div(N(\alpha))$ are all of codimension 1, so it suffices to prove the equality of the theorem at the generic point of a codimension 1 subvariety $V' \subset Y$. That is, we may suppose that Y is the spectrum of a one-dimensional local ring $R = \mathcal{O}_{Y,V'}$ with quotient field $K(Y)$. Since X is finite over $\text{Spec}(R)$ it is also affine, say $X = \text{Spec}(S)$, where $S = \mathcal{O}_{X, f^{-1}(V')}$ is a one-dimensional ring with quotient field $K(X)$, finite over R.

We may write $\alpha = r/s$ for some elements $r, s \in S$. Since the norm is multiplicative, $N(\alpha) = N(r)/N(s)$. Further, since Div takes multiplication to addition, it suffices to show that $f_*(Div(x)) = Div(N(x))$ for any element $x \in S$.

The components V_i of $Div(x)$ that map onto V' correspond to the maximal ideals P_i of S that contain x, and $f_* \langle V_i \rangle = [K(V_i) : K(V')] \langle V' \rangle$. By the Chinese remainder theorem, $S/xS = \oplus_i S_{P_i}/xS_{P_i}$, so $\text{length}_R(S/xS) = \sum \text{ord}_{V_i}(x)'[K(V_i) : K(V')]$. Thus $f_* Div(x) = \text{length}_R(S/xS) \langle V' \rangle$.

We can now apply Lemma 4.18 to the endomorphism of S given by multiplication by x. The determinant of this map is $N(x)$, and we see that $\text{length}_R(S/xS) = \text{length}_R(R/N(x))$, which is the coefficient of V' in $\langle Div(N(x)) \rangle$, as required. □

4.4 Flat Pullback

In this section we will study and apply a different sort of functoriality of the Chow groups:

Theorem 4.20 (Flat Pullback)**.** *If $f : Y \to X$ be a flat map of equidimensional schemes, then there is a unique group homomorphism $f^* : A(X) \to A(Y)$ satisfying*

$$f^*([A]) = [f^{-1}(A)]$$

for any variety $A \subset X$. This homomorphism preserves codimension.

In the next Chapter we will define an intersection product on the Chow ring of a smooth quasiprojective variety, and it will follow from the methods there that flat pullback between such varieties preserves products (Corollary 5.11).

We will use Theorem 4.20 in the next subsections to analyze affine and projective bundles. It will also be useful as a step in the proof of existence of pullbacks along arbitrary proper morphisms of smooth quasiprojective varieties, a topic of Chapter 5.

Proof of Theorem 4.20. We first prove that the flat pullback preserves codimension. Suppose that $W \subset X$ is a subvariety, and that V is an irreducible component of $f^{-1}(W)$.

Suppose that $y \in V$ is a closed point not belonging to any other component of $f^{-1}(W)$, and $x = f(y)$, which is again a closed point by the Nullstellensatz. Because $f|_V : V \to W$ is flat in a neighborhood of y, the dimension of V at y is equal to the dimension of the image of V plus the dimension at y of the fiber of f through y; that is

$$\dim V = \dim W + \dim \mathcal{O}_{f^{-1}(x),y}$$

(see for example Eisenbud [1995] Theorem 10.10). Since the same is true of the map $f : Y \to X$ itself, we have,

$$\dim V - \dim W = \dim \mathcal{O}_{f^{-1}(x),y} = \dim Y - \dim X,$$

as required.

We next show that f^{-1} preserves rational equivalence. By Corollary 4.17 the relation of rational equivalence is generated by the divisors of rational functions on irreducible subvarieties. Thus it suffices to show that if $X' \subset X$ is a subvariety and α is a rational function on X' then $f^*(Div(\alpha))$ is the divisor of the rational function $\alpha \circ f$ on $f^{-1}(X')$. The flatness of f implies the flatness of $f|_{f^{-1}(X')} : f^{-1}(X') \to X'$, so we may begin by replacing X with the variety X' and Y with $f^{-1}(X')$, and assume that X is irreducible and α is a rational function on X. Of course we can no longer assume that

Y is irreducible, or even reduced, but because f is flat, the function $\alpha \circ f$ can be written, locally near any point $y \in Y$, in the form a/b where a and b are nonzerodivisors on $\mathcal{O}_{Y,y}$, so $[Div(\alpha \circ f)] \in A(Y)$. Thus proving that

$$f^*(Div(\alpha)) = Div(\alpha \circ f)$$

will show that f^* preserves rational equivalence.

Let $V \subset Y$ be any codimension 1 subvariety, and let $W \subset X$ be the closure of $f(V)$. We must show that if W has codimension 1 in X, then the coefficient of $[V]$ in $Div(\alpha \circ f)$ is equal to the coefficient of $[W]$ in in $Div(\alpha)$ times the coefficient of $[V]$ in $[f^{-1}(W)]$. This can be checked after localizing at V and W; that is, we may replace the original f by its restriction $f : \mathrm{Spec}(\mathcal{O}_{Y,V}) \to \mathrm{Spec}(\mathcal{O}_{X,W})$ corresponding to the flat, local map of 1-dimensional local rings

$$S := \mathcal{O}_{X,W} \xrightarrow{-\circ f} \mathcal{O}_{Y,V} := R.$$

To simplify the notation we set $S := \mathcal{O}_{X,W}$ and $R := \mathcal{O}_{Y,V}$. Let $\mathfrak{m}$ be the maximal ideal of S, which corresponds to the subvariety W.

Since X is a variety, S is a domain. Since R is flat over S the map $S \to R$ is an inclusion, and we will identify S with its image in R. Writing $\alpha = a/b$ for $a, b \in S$, we have $Div(\alpha) = Div(a) - Div(b)$, so it suffices to treat the case $\alpha = a \in S$.

We now write:

- p for the coefficient of $[W]$ in $Div(a)$, so $p := \mathrm{length}(S/aS)$;
- q for the coefficient of $[V]$ in $f^*[W] = [f^{-1}(W)]$, so $q := \mathrm{length}(R/\mathfrak{m}R)$;
- r for the coefficient of $[V]$ in $Div(a \circ f)$, so $r := \mathrm{length}(R/aR)$.

We must show that $pq = r$. Let

$$aS \subset M_1 \subset \cdots \subset M_p = S$$

be a chain of inclusions, where each M_{i+1}/M_i a simple module, witnessing the fact that $p := \mathrm{length}(S/aS)$. Since S is local, it has only one simple module, namely $M_{i+1}/M_i \cong S/\mathfrak{m}$. Tensoring this module with R we get the R-module $R/\mathfrak{m}R$, whose length is q. Since R is flat over S, the chain of inclusions above gives rise to a chain of inclusions

$$aR \subset R \otimes_S M_1 \subset \cdots \subset R \otimes_S M_p = R$$

Since each quotient $R \otimes M_{i+1}/R \otimes M_i$ has length q, we deduce that $r = pq$ as required. □

Remark. The proof of Theorem 4.20 extends easily to a slightly more general class of flat maps: we can allow X to be any scheme, but assume that there is a fixed integer n such that for each irreducible component $X' \subset X$ the preimage $f^{-1}(X')$ is equidimensional of dimension $n + \dim X'$. When

this condition is satisfied, we will say that f has constant relative dimension n. Each part of the proof of Theorem 4.20 reduced to the consideration of what happened over some subvariety of X, and thus it extends at once to the more general case.

4.4.1 Affine Space Bundles

We have seen in Proposition **??** that the Chow groups of an affine space are trivial: $A(\mathbb{A}^n_K) = \mathbb{Z} \cdot [\mathbb{A}^n]$. One way to interpret this formula is to say that the flat pullback map $f^* : A^{\operatorname{Spec} K} \to A(\mathbb{A}^n_K)$ is surjective. We can extend this result to any $\mathbb{A}^n$ *bundle*; that is, to a map such that X is covered by open affine subsets U_i such that $f^{-1}(U_i) \cong \mathbb{A}^n \times U_i$, and $f|_{f^{-1}(U_i)}$ corresponds under this isomorphism to the projection onto the second factor. Since flatness is a local property, any affine space bundle is flat, so we can use Theorem 4.20. (By the remark following the proof of that Theorem, this would work over any base scheme.)

Theorem 4.21. *If X is a variety and $f : Y \to X$ is an affine space bundle, then the map $f^* : A(X) \to A(Y)$ is surjective. In particular, if the relative dimension of f is d, then $A_m(Y) = 0$ for $m < d$.*

Proof. We may begin by replacing X with the subvariety that is the closure of $f(V)$, and Y with $f^{-1}f(V)$, and thus assume that X is a variety and f is dominant. We do induction on the dimension of X; the case $\dim X = 0$ is precisely Proposition **??**.

Now suppose that $\dim X > 0$ and let U be an open affine subset of X such that $f^{-1}(U) \cong \mathbb{A}^n \times U$. We have a commutative diagram with exact rows given by Proposition 4.11:

$$\begin{array}{ccccccc} A(Y \setminus f^{-1}(U)) & \longrightarrow & A(Y) & \longrightarrow & A(f^{-1}U) & \longrightarrow & 0 \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \\ A(X \setminus U) & \longrightarrow & A(X) & \longrightarrow & A(U) & \longrightarrow & 0. \end{array}$$

By induction the right hand vertical map is onto, so by a diagram chase (the "5-lemma") it suffices to show that the left-hand vertical map is onto. Thus we are reduced to the case where X is an affine variety, $Y \cong \mathbb{A}^n \times X$, and f is the projection. Since the projection $f : \mathbb{A}^n \times X \to X$ factors through projections

$$\mathbb{A}^n \times X \to \mathbb{A}^{n-1} \times X \to \cdots \to X,$$

we can also assume that $n = 1$.

Our situation is now that $X = \operatorname{Spec} S$ for some affine domain S and $Y = \operatorname{Spec} S[t]$. The variety $V \subset Y$ corresponds to a prime ideal $P \subset S[t]$.

Because $f|_V$ is dominant we have $P \cap S = 0$. Let K be the field of fractions of S. Since $K[t]$ is a principal ideal domain we may write $PK[t] = (a)$ for some polynomial a, and clearing denominators we may assume $a \in S[t]$.

If $a = 0$ then $V = Y$, so $[V] = f^*[X]$ as required. Else we may consider the divisor of a, and we have $Div(a) = [V] + [V']$, where V' is not dominant. Since the closure of the image of V' has lower dimension than X, we see by induction that $[V']$ is in the image of f^*, and we are done. (The reader may also see directly that V' has the form $\mathbb{A}^1 \times V''$, so that $[V'] = f^*([V''])$.) □

As a taste of what is to come, we can strengthen Theorem 4.21 in the case when Y is the total space of a vector bundle over X, or more generally when f admits a section, simply by using the existence of an intersection product (Theorem 5.4).

Corollary 4.22. *Suppose that X is a smooth quasiprojective variety. If $f : Y \to X$ is an affine space bundle, and f admits a section $\sigma : X \to Y$, then the map $f^* : A(X) \to A(Y)$ is an isomorphism.*

The result holds even without the hypothesis on X; this follows from the existence of a pullback along the "regular embedding" σ. See Fulton [1984] Chapter 6.

Proof. Suppose that $V \subset X$ is a subvariety, so that $f^*[V] = [f^{-1}(V)]$. By Theorem 5.4 there is an intersection product that preserves rational equivalence such that $[f^{-1}(V)][\sigma(X)] = [f^{-1}(V) \cap \sigma(X)]$. But the map f restricted to $\sigma(X)$ is an isomorphism, so $f_*([f^{-1}(V) \cap \sigma(X)]) = [V]$. Thus the composite

$$A(X) \xrightarrow{f^*} A(Y) \xrightarrow{f_*([\sigma(X)] \cdot -)} A(X)$$

is the identity. By Theorem 4.21, the left hand map is a surjection, so both maps are isomorphisms. □

4.5 The connection between A_{n-1} and Pic.

Let X be a scheme of pure dimension n. The Picard group of X, written $Pic(X)$, is the group whose elements are isomorphism classes of locally free sheaves of rank 1 (also called *invertible sheaves* or *line bundles*), with group law given by tensor product. The Picard group can be computed through cohomology: the fact that any line bundle becomes trivial on an affine open cover leads to the identification $Pic(X) = H^1(\mathcal{O}_X^*)$. This makes the Picard group relatively easy to compute. Exercises **??-??** compare this group with the Chow group of divisor classes, $A_{n-1}(X)$.

4.6 Exercises

Exercise 4.23. Let X be a variety, and let $f : Y \to X$ be a finite flat map, so that both f^* and f_* are well defined. Let $d := (\kappa(Y) : \kappa(X))$ be the degree of f. Show that the composition $f_* \circ f^* : A_*(X) \to A_*(X)$ is multiplication by d.

Exercise 4.24. Show that rational equivalence is the finest equivalence relation that is defined by graded subgroups $R_*(Y) \subset Z_*(Y)$ with the following properties:

(a) $R_*(Y) \subset Z_*(Y)$ is defined for all schemes algebraic over a field k.

(b) $R_*(Y)$ is preserved by proper push-forward and flat pull-back.

(c) $R_*(\mathbb{P}^1)$ contains $\langle 0 \rangle - \langle \infty \rangle$.

Exercise 4.25. The following example shows that the assertion of Theorem 4.14 can fail if f is an open immersion: Let C be a conic in $\mathbb{P}^2$, and let U be the open set $\mathbb{P}^2 \setminus C$ in $\mathbb{P}^2$. Show that U is an affine variety, with Chow groups $A_0(U) = 0,\ A_1(U) = \mathbb{Z}/(2),\ A_2(U) = \mathbb{Z}$. There is a map on cycles $\alpha : Z_*(U) \to Z_*(\mathbb{P}^2)$ taking the class of a curve in U to the class of its closure in $\mathbb{P}^2$. If this induced a map $\alpha : A_1(U) \to A_1(\mathbb{P}^2)$, it would split the surjection $A_*(\mathbb{P}^2) \to A_*(U)$ defined in Proposition 4.11(c). Give an example of two rationally equivalent cycles Z_1, Z_2 on U such that $\alpha(Z_1)$ is not rationally equivalent to $\alpha(Z_2)$.

Exercise 4.26. Show by example that Theorem 4.14 is false without the hypothesis "proper" even when f is a birational map between smooth curves.

Exercise 4.27. Show that if L is a line bundle on a quasi-projective scheme X, then L may be embedded as a subsheaf of the total quotient sheaf $\kappa(X)$ of X as a subsheaf locally generated by one non-zerodivisor. (The result is true much more generally, but there are subtleties in the most general setting—see Kleiman [1979].)

Exercise 4.28. Suppose that X is quasi-projective and reduced. Show that if L and M are two line bundles on X embedded in $\kappa(X)$ as above, and $\varphi : L \to M$ is an isomorphism, then φ extends to an isomorphism $\kappa(X) \to \kappa(X)$, and that such an isomorphism is given by multiplication with a rational function. Conclude that $Pic(X)$ is the same as the group of Cartier divisors modulo "principal" Cartier divisors—those defined by invertible rational functions on X.

Exercise 4.29. Suppose that X is quasi-projective and reduced. Krull's Principal Ideal Theorem implies that in a Noetherian ring every prime ideal minimal over the ideal generated by a non-zerodivisor has codimension 1. Extend this idea to define a map from the multiplicative group of Cartier

divisors on X to the group of codimension 1 cycles $Z^1(X)$, to make a commutative diagram

$$\begin{array}{ccccccccc} \mathcal{O}_X^* & \longrightarrow & \kappa(X)^* & \longrightarrow & \{\text{Cartier Divisors on } X\} & \longrightarrow & Pic(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & \kappa(X)^* & \longrightarrow & Z_{n-1}(X) & \longrightarrow & A_{n-1}(X) & \longrightarrow & 0 \end{array}$$

with exact rows. Conclude that there is a natural map $Pic(X) \to A_{n-1}(X)$.

Exercise 4.30. Suppose that X is quasi-projective and normal. Using the result from commutative algebra that a normal domain is the intersection of discrete valuation rings (see for example Eisenbud [1995] §11.2), show that the map from the group of Cartier divisors to the additive group of one-cycles on X is a monomorphism. Conclude that the map $Pic(X) \to A_{n-1}(X)$ is a monomorphism in this case.

********Is the following exercise true if X is factorial? Isn't the quadric cone in $\mathbb{P}^3$ factorial?********

Exercise 4.31. Suppose that X is quasi-projective and regular—that is, that every local ring of X is a regular local ring (it would be enough that every local ring of X is factorial.) Show that the map $Pic(X) \to A_{n-1}(X)$ is an isomorphism.

********The following exercise is actually false: $Pic(X)$ is positive-dimensional. Do you want to make X the union of the three coordinate axes in $\mathbb{P}^3$?********

Exercise 4.32. Let X consist of three concurrent lines in $\mathbb{P}^2$. Show that $A_1(X) = \mathbb{Z}$ but $Pic(X) = \mathbb{Z}^3$. (Hint: Use the isomorphism $PicX = H^1(\mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the sheaf of invertible functions on X.)

Exercise 4.33. Let X be the cone over a smooth plane conic C. Prove that the natural comparison map $Pic(X) = \mathbb{Z} \to \mathbb{Z} = A_1(X)$ is multiplication by 2.

Exercise 4.34. (For those who know what a Gorenstein ring is). Extend Exercises 4.28 and 4.29 to the case where X is quasiprojective and generically Gorenstein. Let X be the triple line in $\mathbb{P}^2_k$, where k is a field. Show that $A_1(X) = \mathbb{Z}$, but that $Pic(X) = \mathbb{Z} \oplus k^+$, the additive group of k, and that the map $Pic(X) \to A_*(X)$ is the projection on $\mathbb{Z}$ followed by multiplication by 3. (Hint: Use $Pic(X) = H^1(\mathcal{O}_X^*)$.)

********Are we assuming $k = \overline{k}$ here? Do we need to?********

The following exercises deal with some examples of the construction of dynamic projection. In each, we suppose $X \subset \mathbb{P}^n$ is a variety; we let A and $R \subset \mathbb{P}^n$ be complementary linear subspaces general with respect to X, and Ψ the action on the multiplicative group G_m on $\mathbb{P}^n$ described in

Section 4.1.1, with attractor A and repellor R. In each case we let X_0 be the limit, as $t \to 0$, of the images $\psi_t(X)$.

Exercise 4.35. Let $X \subset \mathbb{P}^3$ be a twisted cubic curve, and consider the dynamic projection of X with attractor a point $p \in \mathbb{P}^3$ and repellor a plane. Show that X_0 is the reduced union of three noncoplanar lines through p.

Exercise 4.36. Again let $X \subset \mathbb{P}^3$ be a twisted cubic curve, and now consider the dynamic projection of X with attractor a point $L \subset \mathbb{P}^3$ and repellor another line. Show that $X_0 = V(I_L^2)$ is the scheme defined by the square of the ideal of L.

Exercise 4.37. Now let $X \subset \mathbb{P}^3$ be a nondegenerate quartic curve, and consider the dynamic projection of X with attractor a point $p \in \mathbb{P}^3$ and repellor a plane.

(a) If X has genus 1, show that X_0 is the reduced union of four noncoplanar lines through p.

(b) If X has genus 0, show that X_0 is the union of four noncoplanar lines through p, with an embedded point of multiplicity 1 at p.

Exercise 4.38. There are two isomorphism classes of subschemes $X \subset \mathbb{P}^3$ with Hilbert polynomial $p_X(m) = 3m + 1$ consisting of a planar triple line and an embedded point p: one where the embedded point is spatial (that is, $\dim T_p X = 3$), such as

$$X = V(x_0^2, x_0x_1, x_0x_2, x_1^3);$$

and one where the embedded point is planar (that is, $\dim T_p X = 2$), such as

$$X = V(x_0, x_1^3, x_1^4).$$

Show that the former is a flat limit of twisted cubics, and that the latter is not.

Exercise 4.39. Consider the G_m action on $\mathbb{P}^n$ given by

$$\varphi_t : (x_0, \ldots, x_n) \mapsto (t^{w_1}x_1, \ldots, t^{w_n}x_n)$$

for $t \in G_m$, with weights $w_0 > w_1 > \cdots > w_n \in \mathbb{Z}$. Let $X \subset \mathbb{P}^n$ be a subvariety. Prove analogs for each part of Proposition 4.5 for each pair of spaces $R = \{x_0 = \cdots = x_i = 0\}$ and $A = \{x_{i+1} = \cdots = x_n = 0\}$. If X has dimension m, and the coordinate system is chosen general with respect to X, then show that the limit of tX as t approaches 0 is supported on the m-plane $x_{m+1} = \cdots x_n = 0$.

Hint for Exercise 4.39: Show as in Proposition 4.5 that for any point not in R or A has G_m orbit closure meeting both of these.

5

Intersection Products and Pullbacks

5.1 The Chow Ring

In this chapter we will work with quasi-projective varieties over an algebraically closed field. This setting makes possible a direct and intuitive approach to intersection theory. The reader who needs a more general case will find pointers below and a full treatment in Fulton [1984].

We wish to define a ring structure on the Chow group of a variety X. To give back geometric information about transverse intersections, such a product should satisfy the requirement that the intersection product be "geometric" in the simplest case:

(*) If subvarieties A, B of X intersect transversely at a general point of each component of $A \cap B$, then the product of their classes should be the class of their intersection,

$$[A][B] = [A \cap B].$$

This is not so simple: in fact, no ring structure with property (*) can exist on the Chow groups of an arbitrary projective variety X! This problem remains even if we replace the hypothesis in (*) with the hypothesis that A and B meet everywhere transversely, as the reader will see in Example 5.23. The goal of this chapter is to show the existence and uniqueness of the ring structure when X is smooth, and to prove certain functoriality properties that make it even more useful.

The situation is analogous to that in algebraic topology. The Chow groups are like the homology groups of a space. For singular spaces the cohomology groups form a ring and homology groups do not. However, for smooth orientable manifolds, Poincaré duality allows one to identify homology and cohomology (interchanging dimension and codimension) and thus transport the ring structure to homology. Even for singular varieties one can define a ring (from Fulton-Macpherson's "bivariant theory" Fulton [1984] Chapter 17; see also Totaro [to appear]) that acts on the Chow groups much as the cohomology ring of a topological space acts on the homology via cap product, but we will not pursue this construction here.

5.1.1 Products via the Moving Lemma

To describe the path we will take, we use the following terminology:

Definition 5.1. (a) Irreducible subschemes A and B of a variety X are *dimensionally transverse* if for every component C of $A \cap B$ we have $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$.

(b) Subvarieties A and B of a variety X are *transverse* at a point $p \in X$ if X, A and B are smooth at p and their tangent spaces at p satisfy $T_pA + T_pB = T_pX$.

(c) Subvarieties A and B of a variety X are *generically transverse* if every irreducible component of $A \cap B$ contains a point p at which A and B are transverse.

(d) Two cycles $\sum m_i A_i$ and $\sum n_j Bj$ are dimensionally (respectively generically) transverse if A_i, B_j are dimensionally (respectively generically) transverse for every i, j.

The meaning of generic transversality may be clarified by the following easy result:

Proposition 5.2. *Subvarieties A and B of a variety X are generically transverse if and only if they are dimensionally transverse and each isolated irreducible component of $A \cap B$ is reduced and contains a smooth point of X.*

********The term "isolated irreducible component" here is confusing to me (probably a function of my background). Can we just say "each irreducible component of $A \cap B$ is generically reduced and contains a smooth point of X"?********

Proof. First suppose that A and B are generically transverse, so that for each isolated component C of $A \cap B$ there is a smooth point $p \in X$ such

that A and B are smooth and transverse at p. It follows that C is smooth at p, and thus is generically reduced.

To prove the converse, let C be an irreducible component of $A \cap B$. Since C is reduced there is a smooth point $p \in C$. The Zariski tangent space to C at p is the intersection of T_pA and T_pB in T_pX. By hypothesis, $\dim C = \dim A + \dim B - n$, and since $\dim T_pC = \dim C$, it follows that $\dim T_pA = \dim A$ and $\dim T_pB = \dim B$, proving that A and B are smooth at p as well. □

We have mentioned the problem of intersections having the "wrong codimension," but it is important to note that for subvarieties of a smooth variety, the codimension of a component of an intersection can only be too small, not too large. We will use the following important extension of the Principal Ideal Theorem many times:

Theorem 5.3. *If X is a smooth variety and $f : Y \to X$ is a morphism of schemes then for any subvariety $A \subset X$,*

$$\operatorname{codim} f^{-1}A \leq \operatorname{codim} A$$

In particular, if A, B are subvarieties of X, and C is an irreducible component of $A \cap B$, then $\operatorname{codim} C \leq \operatorname{codim} A + \operatorname{codim} B$. □

The proof of this result can be reduced to the case where one of the varieties is locally a complete intersection by the technique of "reduction to the diagonal," explained below; and in this case, it follows from the Principal Ideal Theorem. It is proven in greater generality (by the same sort of techniques) in Serre [2000] Theorem V.3. Smoothness is necessary for this; see Exercise **??**

Here is the main result of the chapter:

Theorem 5.4. *If X is a smooth quasiprojective variety then there is a unique bilinear product structure on $A(X)$ that satisfies*

$$[A][B] = [A \cap B]$$

whenever A and B are subvarieties that are generically transverse. This structure makes $A(X)$ into an associative, commutative ring graded by codimension.

We will call $A(X)$ with its ring structure the *Chow ring* of X.

As a first, easy example, we deduce that the degree of a projective variety is an invariant of its rational equivalence class.

Corollary 5.5. *$A(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/(\zeta^{n+1})$ as rings, where ζ is the class of a hyperplane. The class $[X] \in A^c(\mathbb{P}^n)$ of a projective variety $X \subset \mathbb{P}^n$ of degree d and codimension c is $d\zeta^c$.*

Proof. This is Theorem 1.21, and with Theorem 5.4 we can verify the proof given in Chapter 1. In detail: we already know a projective variety $X \subset \mathbb{P}^n$ can be written as $d[L]$ for some d, where L is a plane of dimension $n - c$ (Example 4.6 or Proposition 1.20). Since any two hyperplanes are rationally equivalent (Example 4.4) and L is the transverse intersection of c hyperplanes, we see that $[X] = d\zeta^c$ for some d. Both X and L meet a general c-plane transversely in $\deg X$ and in 1 point, respectively, and we know that the degree of a zero-dimensional scheme is well defined on $A_0(\mathbb{P}^n)$. Thus, using the ring structure of $A(\mathbb{P}^n)$, the degree of X is $\deg([X][M]) = \deg(d[L][M]) = d$. The same computation shows that $d[L] \neq 0$ for all $d \neq 0$, completing the proof. □

To show that the Chow groups form a ring we must understand how to define the product of two cycles that intersect in an arbitrary way. Since the structure is to be bilinear, the problem reduces immediately to constructing a cycle representing the class of a product $[A][B]$ where A and B are subvarieties of X, of dimensions a and b respectively. Since the Chow ring is to be graded by codimension, this should be a cycle of codimension

$$\operatorname{codim} A + \operatorname{codim} B$$

We cannot define $[A][B]$ to be $[A \cap B]$: this may not have the right codimension, and even when it has the right codimension, $[A \cap B]$ is not always determined by the rational equivalence classes $[A]$ and $[B]$ (see Example 5.27.)

Instead we will define $[A][B]$ via a strong *Moving Lemma*:

Theorem 5.6 (Moving Lemma)**.** *Let X be a smooth quasiprojective variety.*

(a) *If $\alpha \in A(X)$ is any Chow class and $\{B_j \subset X\}$ any finite collection of subvarieties, then there exists a cycle $A = \sum m_i A_i$ such that $[A] = \alpha$ and A_i intersects B_j generically transversely for every i and j.*

(b) *For each j, the class*

$$\sum_i m_i [A_i \cap B_j] \in A(X)$$

is independent of the choice of such A.

For other treatments of various versions of the Moving Lemma, see Roberts [1972a] and Hoyt [1971].

Proof of Theorem 5.4. For any $\alpha, \beta \in A(X)$ we define the product $\alpha\beta$ to be the class of $\sum_{i,j} m_i n_j [A_i \cap B_j] \in A(X)$, where $A = \sum m_i A_i$ and $B = \sum n_j B_j$ are any cycles representing the classes α and β and intersecting generically transversely . The first part of the Moving Lemma (Lemma 5.6) implies that we can find such cycles, while the second part

of the Moving Lemma assures us that this is well-defined. The product is obviously commutative.

To prove that the product is associative, let $\alpha, \beta, \gamma \in A(X)$. We claim that there are cycles A, B and C representing α, β and γ, and such that

$$(\alpha\beta)\gamma = [A \cap B \cap C] = \alpha(\beta\gamma).$$

First, we may choose cycles $B = \sum n_j B_j$ and $C = \sum o_l C_l$ representing β and γ and such that the B_j are generically transverse to the C_l; and then choose $A = \sum m_i A_i$ representing α so that the A_i are generically transverse to the $B_j \cap C_l$ *and* to all the B_j. Since A_i is generically transverse to $B_j \cap C_l$, it follows that $A_i \cap B_j \cap C_l$ is generically reduced and of codimension equal to the sum of the codimensions of A_i, B_j and C_l, so $A_i \cap B_j$ is generically transverse to C_l as well. Associativity of the intersection product now follows from that of ordinary intersections. Thus the product we have defined makes

$$A(X) = \bigoplus_k A^k(X)$$

into a commutative graded ring. □

Even if α and β are the classes of effective cycles, we can't necessarily choose A and B effective in the Moving Lemma. For instance, in Example **??** the class e represents an effective divisor, but $e^2 = -\eta$ is the negative of an effective class, and thus cannot be represented in the form $\sum m_i n_j [A_i \cap B_j]$ with the m_i and n_j non-negative.

The restriction to smooth quasiprojective varieties in Theorems 5.4 and 5.6 cannot be avoided; see Section 5.4 for a discussion.

The second part of the statement of Theorem 5.6 may seem at first to be the more mysterious, but it will follow easily from the first part using the following Lemma.

Lemma 5.7. *Let X be a quasiprojective variety, and let A and A' be pure dimensional cycles on X. Suppose that $Z = \sum m_i Z_i \in Z(\mathbb{P}^1 \times X)$ is a linear combination of subvarieties Z_i giving a rational equivalence between A and A'; that is, such that $Z|_0 = A$ and $Z|_\infty = A'$. If B is a subvariety of X that is generically transverse to $Z_i|_0$ and $Z_i|_\infty$ for all i, then $A \cap B$ is rationally equivalent to $A' \cap B$ as cycles on B (and hence also as cycles on X).*

Proof of Lemma 5.7. We may harmlessly discard all the Z_i except those of dimension $\dim Z_i = 1 + \dim A$. The scheme $Z_i|_0 \cap B$ is a Cartier divisor on $Z_i \cap (\mathbb{P}^1 \times B)$, and is reduced, except possibly for some components lying in fibers over points of $\mathbb{P}^1$. It follows that there is an open set $U \subset \mathbb{P}^1$ containing $0, \infty$ such that $Z_i \cap (U \times B)$ is generically reduced and of

dimension $1+\dim A\cap B$. Thus $\sum m_i(Z_i\cap(U\times B))$ is a rational equivalence between $A\cap B$ and $A'\cap B$ as cycles on B. □

5.1.2 From Intersections to Pullbacks

Theorem 5.4 can be made much more powerful by recasting it as a functoriality statement. The idea is that intersections are the simplest sorts of pull-backs. Thus we are led to try to generalize the inclusion $B\subset X$ to be a morphism $f: Y\to X$, and to generalize the intersection product $[A][B]$ to be a class, which we will call $f^*([A])$, in $A(Y)$.

Even in the case where f is the inclusion map of $Y=B$ in X, this gives something new, since the class $[A][B]$ was defined above in $A(X)$, not in $A(B)$. The two classes will be related by the formula

$$f_*(f^*([A])=[A][B]\in A(X).$$

Fulton's theory of refined intersection products (Fulton [1984], Chapter ****) gives more: it yields a well defined class in $A(A\cap B)$, but we will not need this extra information.

A naive attempt at a definition of the pullback would be to take $f^*([A])$ to be the class $[f^{-1}(A)]$ of the preimage of A in Y. As in the simpler case of intersections, this can't work all the time: given a subvariety $A\subset X$, the preimage $f^{-1}(A)$ need not have the expected codimension. But it does work in two cases: when the map f is flat (in this case the preimage of a subvariety $A\subset X$ necessarily has the expected codimension, as has been shown in 4.20), and also when both X and Y are smooth and the map f is projective (in this case a variant of the Moving Lemma assures us that we can replace A with a rationally equivalent and better-behaved cycle).

The following notion generalized the definition of generic transversality for a subscheme $A\subset X$ and an inclusion $Y\subset X$.

Definition 5.8. Let $f: Y\to X$ be a projective map of quasiprojective varieties. A subvariety A of X of codimension c is said to be *generically transverse* to $f: Y\to X$ if:

(a) $f^{-1}(A)$ is generically reduced of codimension c.

(b) Y is smooth at a general point q of each irreducible component of $f^{-1}(A)$ and X and A are smooth at $f(q)$.

A cycle $A=\sum m_iA_i$ on X is generically transverse to f if every component A_i is.

The following result, in conjunction with the Moving Lemma, shows that we can "move" cycles to be generically transverse to maps, at least in the

presence of separability. Recall that a morphism $f : Y \to X$ is *generically separable* if the field extension $K(Y)/K(X)$ is separable; in the infinite case this means that there is a transendence base $\{x_i\}_{i\in I}$ for $K(Y)$ over $K(X)$ such that $K(Y)$ is finite and separable over $K(X)(\{x_i\}_{i\in I})$. Every extension of a perfect field is generically separable; see for example Eisenbud [1995] Appendix ****.

Lemma 5.9. *Let $f : Y \to X$ be a generically separable morphism of quasiprojective varieties. There is a finite collection of subvarieties B_i of X such that a cycle A on X is generically transverse to f if A is generically transverse to each B_i.*

Suppose that $\alpha \in A(X)$ and A is a cycle representing α and generically transverse to f. We will show that the class $[f^{-1}(A)] \in A(Y)$ is independent of the choice of A. This allows us to define $f^*\alpha$ to be the class $[f^{-1}(A)]$ for such an A. Here is the main result:

Theorem 5.10. *Let $f : Y \to X$ be a generically separable map of smooth quasiprojective varieties, and assume that X is projective.*

(a) *There exists a unique ring homomorphism $f^* : A(X) \to A(Y)$ with the property that if A is a subvariety of X generically transverse to f then $f^*([A]) = [f^{-1}(A)]$.*

(b) *(Push-Pull Formula) If we make $A(Y)$ into an $A(X)$ module via the ring homomorphism f^*, then the map $f_* : A(Y) \to A(X)$ is a map of modules, in the sense that if $\alpha \in A(X)$ and $\beta \in A(Y)$ then*

$$f_*\big(\alpha\beta\big) = \alpha f_*(\beta),$$

or, as it is often written,

$$f_*\big(f^*(\alpha)\beta\big) = \alpha f_*(\beta).$$

By introducing multiplicities, one can drop the assumption that f is generically separable, and f^* can be defined as a homomorphism of Chow groups even when Y is singular and X is only quasiprojective (but still smooth). For these refinements see Fulton [1984].

In the case of a flat morphism f we have already defined a group homomorphism $f^* : A(X) \to A(Y)$ by $f^*([A]) = [f^{-1}(A)]$ (Theorem 4.20). This definition clearly satisfies the condition for uniqueness in Theorem 5.10 so Theorem 4.20 implies that f^* is a ring homomorphism in this case. Using the Moving Lemma (and, again under the unnecessary hypothesis of generical separability) we will show directly that, if f is flat, then the map f^* is a ring homomorphism in the quasiprojective case:

Proposition 5.11. *If $f : Y \to X$ is a generically separable flat morphism of smooth quasi-projective varieties, then the pullback $f^* : A(X) \to A(Y)$ is a ring homomorphism.*

The assertions above will be proven in Section 5.7.

We can also show that the inclusion of a closed subvariety of a projective variety defines an ideal in the Chow ring:

Proposition 5.12. *Let $\iota : X' \subset X$ be the inclusion of a closed subset. If X is a smooth projective variety, then the subgroup $\iota_*(A(X')) \subset A(X)$ is an ideal in the Chow ring.*

Proof. Set $U = X \setminus X'$. By Proposition 4.11, Part c there is a right exact sequence of Chow groups

$$A(X') \xrightarrow{\iota_*} A(X) \xrightarrow{k_U} A(U) \longrightarrow 0.$$

The scheme U is a smooth quasiprojective variety, and the restriction map $k_U : A(X) \to A(U)$ is the same as flat pullback; thus it is a ring homomorphism. This implies that the image of $(\iota_X)_*$ is an ideal. □

5.2 Proof of the Moving Lemma

To prove the first part of the Moving Lemma, it will turn out to suffice to start with an arbitrary pair of subvarieties $A, B \subset X$ and construct a cycle A' that intersects B generically transversely and is rationally equivalent to A. (As we'll see in the course of the proof, if we can construct an A' generically transverse to one subvariety B, we can find one generically transverse to any finite collection of subvarieties.) In this construction, we will make use of a class of cycles on X that are easy to move using the idea of Example 4.6:

Definition 5.13. Let $X \subset \mathbb{P}^N$ be a smooth quasiprojective variety. An *ambient cycle* on X is a cycle D that can be written in the form $D = \langle \tilde{D} \cap X \rangle \in Z^a(X)$ where $\tilde{D}$ is a subvariety of $\mathbb{P}^N$ that meets X generically transversely.

The essential ingredient in our proof of the Moving Lemma is a process, called the *cone construction*, that produces, from an $N - n - 1$-plane not meeting X, a scheme $\tilde{D}_\Gamma = X \cap \overline{\Gamma, A}$ containing A so that

$$D_\Gamma = \tilde{D}_\Gamma \cap X$$

is generically reduced, and is thus an ambient cycle. We can also express D_Γ in terms of the projection from Γ,

$$D_\Gamma = \pi_\Gamma^{-1}(\pi_\Gamma(A)),$$

where π_Γ denotes the projection with center Γ. We write $A'_\Gamma := D_\Gamma - A$ for the residual cycle; it is also generically reduced with all components of

dimension equal to the dimension of A. The key here is to show that the residual cycle A'_Γ can be made "more transverse than A" to a given cycle B on X by choosing Γ general; this is the content of Lemma 5.18 below. Further, the ambient cycle D_Γ can be moved so as to become transverse to B, using the next Lemma.

Lemma 5.14. *Let $X \subset \mathbb{P}^N$ be a quasiprojective variety. If $\tilde{D} \subset \mathbb{P}^N$ is a subvariety of dimension m and degree d that is generically transverse to X, then the ambient cycle $D := \tilde{D} \cap X$ is rationally equivalent in X to a cycle of the form $\sum_{i=1}^d \langle L_i \rangle$ where the L_i are the intersections of X with distinct planes $\tilde{L}_i$, and $\cup_{i=1}^d \tilde{L}_i$ is generically transvserse to X.*

Proof. If $n + m < N$ then, since $\tilde{D}$ is generically transverse to X we will have $\tilde{D} \cap X = \varnothing$. In this case the result is trivial, so we will assume that $n + m \geq N$.

Let $A, R \subset \mathbb{P}^N$ be general planes of dimension $m-1$ and $N-m$ respectively. The second construction in Example 4.6 produces an irreducible family $Z \subset \mathbb{P}^1 \times \mathbb{P}^N$ that is a rational equivalence between $\tilde{D}$ and $\tilde{L} := \cup_{i=1}^d \tilde{L}_i$, where the $\tilde{L}_i$ are the spans of A with the points of the intersection $R \cap \tilde{D}$. We will show that $\tilde{L}$ is generically transverse to X. Given this, Lemma 5.7 implies that D is rationally equivalent to the cycle $\langle \tilde{L} \cap X \rangle = \sum_{i=1}^d \langle L_i \rangle$.

It remains to show that $\tilde{L}$ and X are generically transverse when $A \in \mathbb{G}(m-1, \mathbb{P}^N)$ and R are chosen generally. To this end let $\tilde{D}^\circ = \tilde{D} \setminus (\tilde{D} \cap X)$ and consider the incidence correspondence Λ defined by

$$\Lambda = \{(A, p, x) \in \mathbb{G}(m-1, \mathbb{P}^N) \times \tilde{D}^\circ \times X \mid q \in \overline{A, p}\}$$

The fiber of Λ over a point $(p, x) \in \tilde{D}^\circ \times X$ is the set of $(m-1)$-planes meeting the line $\overline{p, x}$, an irreducible subvariety of the Grassmannian, so Λ is irreducible.

Since $\tilde{D}$ is generically transverse to X, a general point of $\tilde{D}$ (and thus also of $\tilde{D}^\circ$) is not contained in the tangent plane to a general point x of X. Thus for such a point p, and a general choice of $(m-1)$-plane A meeting the line $\overline{p, x}$, the span $\overline{A, p}$ is transverse to X at q. It follows that the subset

$$\Lambda' = \{(A, p, x) \in \Lambda \mid \overline{A, p} \text{ is not generically transverse to } X \text{ at } x\}$$

has smaller dimension than Λ. Let V_A be the set of points $p \in \tilde{D}^\circ$ such that the fiber of Λ over (A, p) is contained in Λ'. The inequality of dimensions shows that for general A the set V_A is contained in a proper closed subset of $\tilde{D}^\circ$. If we choose the repelling plane R generally then it will meet $\tilde{D}$ in only finitely many points, and will miss both $\tilde{D} \cap X$ and V_A. Thus for every point $p_i \in \tilde{D} \cap R$ the plane $\tilde{L}_i = \overline{A, p_i}$ is generically transverse to X, as required. □

In the course of the proof of Part (a) of the Moving Lemma we will actually prove a slightly stronger statement. We record it here:

Theorem 5.15 (Sharp Moving Lemma, Part (a))**.** *If A, B are subvarieties of a smooth quasiprojective variety X, then $\langle A\rangle$ can be written as $\langle D\rangle - \langle A'\rangle$ where D is an ambient cycle (for some embedding of X in projective space) and where A' is a scheme that is generically transverse to B such that $A' \cap B$ does not contain any component of $A \cap B$.*

As we said, our tool for proving Theorem 5.15 is the *cone construction*, so-called because D will be constructed as the intersection of X with a cone over A. This idea seems to have been introduced by Severi, and is the basis for all the proofs (and attempted proofs) of the Moving Lemma that we know.

It will be convenient to begin by making sure that X is embedded in a good way in projective space, using the next Lemma. See Exercise 5.31 for a natural generalization.

Lemma 5.16. *Let $\nu_3 : \mathbb{P}^N \to \mathbb{P}^M$ be the third Veronese map. If $X \subset \mathbb{P}^n$ is a smooth variety, then no three points of $\nu_3(X)$ are collinear, and the tangent spaces to X at distinct points of $\nu_3(X)$ are disjoint.*

Proof. We may as well assume that $X = \mathbb{P}^n$. We claim first that any finite subscheme Y of degree $m \leq 4$ in $\mathbb{P}^N$ imposes independent conditions on cubics. In terms of the homogeneous coordinate ring S_Y of Y, this means that the component $(S_Y)_3$ of degree 3 has dimension m. To see this, note that if l is a linear form on $\mathbb{P}^N$ not vanishing on Y, then $R := S_Y/(l)$ is a homogeneous algebra of vector space dimension

$$\sum_{d=0}^{\infty} \dim_K R_d = m \leq 4.$$

If $R_d = 0$ for some particular $d \geq 1$, then $R_e = 0$ for all $e \geq d$, because R is generated as an algebra in degree 1. Thus the infinite sum above can be truncated after the 4-th term,

$$\sum_{d=0}^{3} \dim_K R_d = m.$$

From the exact sequence

$$0 \to S_Y(-1) \to S_Y \to R \to 0$$

we see by an induction starting from $e = 0$ that

$$\dim_K (S_Y)_e = \sum_{d \leq e} \dim_K R_d.$$

In particular, $\dim_K (S_Y)_3 = m$ as required.

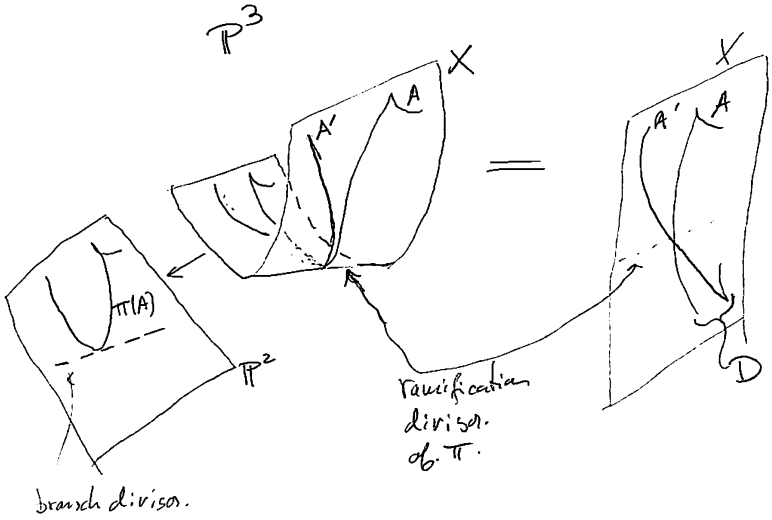


FIGURE 2.2a. Cone Construction (not showing the cone)

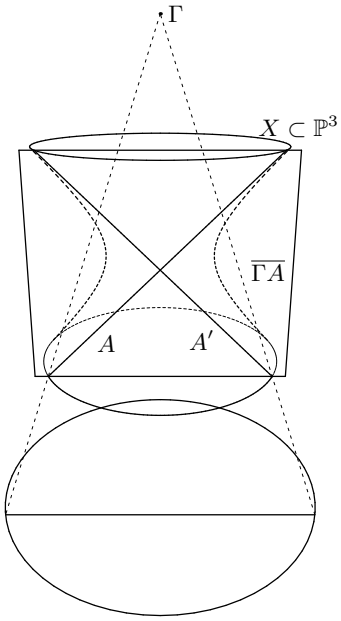
Since linear forms on $\mathbb{P}^M$ pull back to cubics on $\mathbb{P}^N$, it follows that subschemes of length $m \leq 4$ of $\nu_3(\mathbb{P}^N)$ impose independent conditions on linear forms. In particular, no subscheme of $\nu_3(\mathbb{P}^N)$ of length ≥ 3 can be contained in a line, and no subscheme of length ≥ 4 can be contained in a 2-plane.

If the tangent spaces to $\nu_3(\mathbb{P}^N)$ at points p, q met in some point r, then the lines $L_1 = \overline{p,r}$ and $L_2 = \overline{q,r}$ would be contained in a 2-plane, and this plane would contain the subscheme $(L_1 \cap Y) \cup (L_2 \cap Y)$. This scheme has length at least 4, a contradiction. □

Standing Hypothesis: *Henceforth in this section we will assume that $X \subset \mathbb{P}^N$ is a smooth, quasiprojective variety of dimension n, that no three points of X are colinear in $\mathbb{P}^N$, and that any two tangent planes to X at distinct points, regarded as n-dimensional projective subspaces of $\mathbb{P}^N$, are disjoint.*

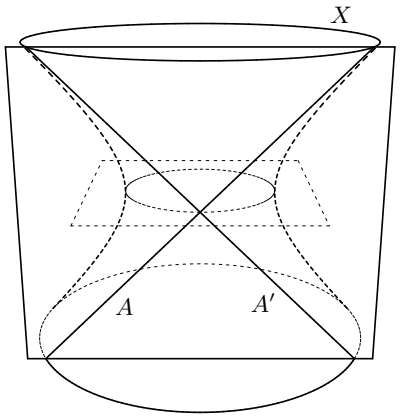
Most of the salient points in the arguments below are already present in the very simplest case, with $n = 2$ and $N = 3$ (that is, $X \subset \mathbb{P}^3$ a smooth surface), and $A = B \subset X$ a (possibly) singular curve. The reader may want to picture this case in going through the argument—see Figures 2.2a and 2.2.

We fix the following notation: Let $A \subset X \subset \mathbb{P}^N$ be a proper subvariety of dimension a. Let $\mathbb{G} = \mathbb{G}(N-n-1, \mathbb{P}^N)$ be the Grassmannian of $N-n-1$-planes in $\mathbb{P}^N$, and let $\mathbb{G}^* \subset \mathbb{G}$ be the set of $(N-n-1)$-planes in $\mathbb{P}^N$ that do



$\overline{\Gamma A} \cap X = A + A'$

FIGURE 2.2. The cone construction (in this case the cone is a plane.) ****Silvio: Equation should be $D = \overline{\Gamma A} \cap X = A + A'$****



$A + A' \sim E$

FIGURE 2.3. Illustration of the Moving Lemma ****the dotted conic in the middle should be labeled E AND this should be amalgamated with Fig 2.2****

not meet X. For $\Gamma \in \mathbb{G}^*$ we let $\pi_\Gamma : X \to \mathbb{P}^n$ be the linear projection with projection center Γ. Set $D_\Gamma := \pi_\Gamma^{-1}(\pi_\Gamma(A) = X \cap \overline{\Gamma, A}$. The next Lemma shows that,when Γ is general, D_Γ is generically reduced, and thus defines an ambient cycle on X.

Lemma 5.17. *There is a dense open set of* $\Gamma \in \mathbb{G}^*$ *such that* $D_\Gamma := \pi_\Gamma^{-1}(\pi_\Gamma(A))$ *is generically reduced of pure dimension* a.

Proof. By Theorem 5.3, the components of D_Γ have codimension at most the codimension of $\pi_\Gamma(A)$. Since π_Γ is quasifinite, every irreducible component of D_Γ is of pure dimension $a = \dim A$.

It suffices to show that there exists a plane Γ disjoint from X such that D_Γ is generically reduced. To find such a Γ, we first choose a general point p of A; then we choose a general plane Σ of dimension $N - n$ containing p; and finally we take Γ to be a general hyperplane of Σ, so that $\Sigma = \overline{\Gamma, p}$.

Being a general $(N - n)$-plane through p, Σ meets X transversely at p; and as it's the intersection of n general hyperplanes through p, Bertini's Theorem implies that Σ meets X transversely away from p. Thus Σ meets X transversely and $\pi_\Gamma^{-1}(\pi_\Gamma(p)) = \Sigma \cap X$ is a finite collection of reduced points.

Since $\dim A < \operatorname{codim} \Sigma$ and Σ is a general plane through p we see that $\Sigma \cap A = \{p\}$, and $T_p A \cap T_p \Sigma = 0$. It follows that the map $\pi_\Gamma|_A : A \to \mathbb{P}^n$ is one-to-one over $\pi_\Gamma(p)$ and that $\pi_\Gamma|_A$ has injective differential at p. Thus $\pi_\Gamma(p)$ is a smooth point of $\pi_\Gamma(A)$, and the tangent space to $\pi_\Gamma(A)$ at $\pi_\Gamma(p)$ is the image of $T_p A$ under π_Γ.

Since $\pi_\Gamma(A)$ is smooth at $\pi_\Gamma(p)$ and the fiber $\pi_\Gamma^{-1}(\pi_\Gamma(p)) = \Sigma \cap X$ is reduced, $D_\Gamma = \pi_\Gamma^{-1}(\pi_\Gamma(A))$ is smooth at each point q of $\pi_\Gamma^{-1}(\pi_\Gamma(p))$. We claim that there is such a point on each component of D_Γ. Since π_Γ is quasifinite and every component A_i of D_Γ has the same dimension as A, every component of D_Γ dominates $\pi_\Gamma(A)$. Since $\pi_\Gamma(p)$ is a general point of $\pi_\Gamma(A)$, every component of D_Γ must contain a point of $\pi_\Gamma^{-1}(\pi_\Gamma(p))$ as claimed. Thus $\pi_\Gamma^{-1}(\pi_\Gamma(A))$ is generically reduced. □

Lemma 5.17 implies that

$$A = D_\Gamma - A'_\Gamma$$

as cycles, where A'_Γ is a generically reduced scheme of pure dimension a that does not contain A, and D_Γ is an ambient cycle.

Since the cycle D_Γ produced in the cone construction is ambient, it is rationally equivalent to a cycle of the form $d[L]$, where L is a general plane of appropriate dimension. By Bertini's Theorem, L can be chosen transverse to any finite collection of subvarieties of X.

Now let $B \subset X$ be a fixed subvariety, and again suppose that A'_Γ, D_Γ are produced from $A \subset X \subset \mathbb{P}^N$ by the cone construction with respect to a plane $\Gamma \in \mathbb{G}^*$. The central argument of the proof of the Moving Lemma is the next Lemma, which shows that, for general Γ, the scheme A'_Γ meets B "more transversely" than does A: although $A'_\Gamma \cap B$ may still have components where the intersection is non-transverse or of the wrong dimension, they will at least be smaller than those of $A \cap B$.

Lemma 5.18. *With notation and hypotheses as above there is a dense open set of $\Gamma \in \mathbb{G}^*$ such that:*

(a) *The irreducible components of $A'_\Gamma \cap B$ that are contained in A have dimension $< \dim(A \cap B)$. In particular, if A is dimensionally transverse to B then no component of $A'_\Gamma \cap B$ is contained in A.*

(b) *The irreducible components of $A'_\Gamma \cap B$ that are not contained in A are generically transverse intersections of A'_Γ and B.*

Proof. **(a):** By choosing Γ not to meet the tangent planes of X at one chosen point of each component of $A \cap B$, we can ensure that the map $\pi_\Gamma : X \to \mathbb{P}^n$ is nonsingular at those points, and thus those points do not lie in $A \cap A'_\Gamma$. It follows that $A'_\Gamma \cap A \cap B$ has codimension strictly greater than $A \cap B$, proving the first statement of part (a).

If A and B are dimensionally transverse, then $\operatorname{codim}(A \cap B) = \operatorname{codim} A + \operatorname{codim} B$. Since X is smooth, every component of $A'_\Gamma \cap B$ has codimension $\leq \operatorname{codim} A + \operatorname{codim} B$, proving the second statement of part (a).

(b): By Lemma 5.17, there is a dense open subset $\mathbb{G}' \subset \mathbb{G}^*$ such that the scheme A'_Γ is generically reduced whenever $\Gamma \in \mathbb{G}'$.

We will prove that for general Γ the scheme D_Γ, or equivalently the scheme A'_Γ, is generically transvserse to the scheme $B^* := B \setminus A \cap B$. We can study the intersection $\mathbb{A}'_\Gamma \cap B^*$ through the incidence correspondence that measures the set of pairs mapping to the same point under projection from Γ:

$$\begin{aligned} \Xi :&= \left\{ (\Gamma, p, q) \in \mathbb{G}' \times A \times B^* \mid \pi_\Gamma(p) = \pi_\Gamma(q) \right\} \\ &= \left\{ (\Gamma, p, q) \in \mathbb{G}' \times A \times B^* \mid \Gamma \cap \overline{p, q} \neq \varnothing \right\}. \end{aligned}$$

The fiber of Ξ over any point $\Gamma \in \mathbb{G}'$ is

$$F_\Gamma := \{(p, q) \in A \times B^* \mid \pi_\Gamma(p) = \pi_\Gamma(q)\}.$$

Since π_Γ is a finite map, the projection $(p, q) \mapsto q$ maps F_Γ finite-to-one onto $A'_\Gamma \cap B^*$. Since every component of A'_Γ has dimension equal to the dimension of A, and X is smooth, Theorem **??** shows that every component of $A'_\Gamma \cap B^*$ has dimension $\geq \dim A + \dim B - n$, so the same goes for the components of F_Γ.

Of course if the image of Ξ in $\mathbb{G}'$ were not dense, then the generic fiber would be empty, so $A'_\Gamma \cap B^*$ would be empty for generic Γ, and there would be nothing further to prove. To understand this case, we will compute the dimension of Ξ and of its fibers over $\mathbb{G}'$. We will need is to know the codimensions of various strata of the Grassmannian that were computed in Theorem 3.1.

The variety Ξ surjects to $A^* \times B$ with fiber over p, q equal to $\Sigma_n(\overline{p,q})$. By Theorem 3.1, $\Sigma_n(\overline{p,q})$ has codimension n in the Grassmannian, so $\dim \Xi = \dim A + \dim B + \dim G - n$. It follows that the projection $\Xi \to \mathbb{G}'$ has dense image if and only if $\dim F_\Gamma \leq \dim A + \dim B - n$. Putting this computation together with the discussion before the Lemma, we are done unless $\dim A + \dim B \geq n$ and there is a dense open set $\mathbb{G}'' \subset \mathbb{G}'$ such that for $\Gamma \in \mathbb{G}''$, every component of $D_\Gamma \cap B* = A'_\Gamma \cap B^*$ has dimension exactly $\dim A + \dim B - n$; that is, the intersections are dimensionally transverse, and we may henceforward assume that these additional hypotheses are satisfied.

Lemma 5.19. *With $\pi_\Gamma X \to \mathbb{P}^n$ and $A, B \subset X$ as above, suppose that $p \in A$ and $q \in \pi^{-1}\pi(p) \subset D_\Gamma$, with $q \notin A$. If:*

1) *B is smooth at p;*
2) *$\pi(A)$ is smooth at $\pi(p)$; and*
3) *The three linear subspaces Γ, $\mathbb{T}_p A$ and $\mathbb{T}_q B$ together span $\mathbb{P}^N$,*

then the subvarieties D_Γ and B meet transversely at q.

Proof. ****to be filled in**** □

To prove that for generic Γ the intersections are in fact generically transverse, we introduce the incidence correspondence of triples consisting of a "bad" subspace Γ together with points $p \in A$ and $q \in B^*$ that constitute a "reason" why Γ is "bad". That is,

$$\Psi := \{(\Gamma, p, q) \in \mathbb{G}'' \times A \times B^* \mid \pi_\Gamma(p) = \pi_\Gamma(q) \text{ and } D_\Gamma \text{ is not transverse to } B \text{ at } q\}.$$

We must show that the image of Ψ in $\mathbb{G}''$ is not dense, which we will do by proving that $\dim \Psi < \dim \mathbb{G}''$. We will describe three sets Ψ_1, Ψ_2, Ψ_3 corresponding to the conditions of Lemma 5.19, so that $\Psi \subset \bigcup_i \Psi_i$ and prove that each of them has dimension $< \dim \mathbb{G}''$.

****The following is only hastily revised. Check for consistency!****

Note that $\Psi \subset \Xi$, and that if Γ is in the image of Ψ, then the fiber of Ψ over Γ is equal to the corresponding fiber F_Γ of Ξ. It thus suffices to show that $\dim \Psi_i < \dim \Xi$.

To say that a pair of points (p, q) project to the same point under π_Γ means that Γ meets $\overline{p, q}$. For each of the conditions 1)–3) of Lemma 5.19, we will show that the set of triples (Γ, p, q) not satisfying condition i) with respect to these points is a proper closed subset of $\mathbb{G}''$, and thus a general $\Gamma \in \mathbb{G}'$ will satisfy all three conditions.

To this end, we let $\Psi_i \subset \mathbb{G}''$, for $i = 1, 2, 3$, be the set of planes *not* satisfying condition i, so that $\Psi \subset \Psi_1 \cup \Psi_2 \cup \Psi_3$. Thus, to start with,

$$\Psi_1 = \{(\Gamma, p, q) \in \Xi \mid q \in B_{\text{sing}}\} \subset \Xi.$$

The condition for a point $q =$ is a singular point of D_Γ, and for this to happen $\pi_\Gamma(p)$ would have to be a singular point of $\pi_\Gamma(A)$. This, in turn, can happen only if at least one of the following three things occurs:

(2a) $q \in \overline{\Gamma, p}$ with p a singular point of A;

(2b) $q \in \overline{\Gamma, p}$ for two or more distinct points $p \in A$; or

(2c) $q \in \overline{\Gamma, p}$, $p \in A_{sm}$ and $\Gamma \cap \mathbb{T}_p A \neq \varnothing$.

Accordingly we set

$$\Psi_{2a} = \{(\Gamma, p, q) \in \Xi \mid p \in A_{\text{sing}}\} \subset \Xi.$$

$$\Psi_{2b} = \{(\Gamma, p, q) \in \Xi \mid \exists p' \neq p \in A \text{ with } \Gamma \cap \overline{p', q} \neq \varnothing\} \subset \Xi.$$

and

$$\Psi_{2c} = \{(\Gamma, p, q) \in \Xi \mid p \in A_{\text{sm}} \text{ and } \Gamma \cap \mathbb{T}_p A \neq \varnothing\} \subset \Xi.$$

Even if both D_Γ and B are smooth at q, the intersection may still fail to be transverse. When D_Γ is smooth at q the tangent plane to the cone $\overline{A, \Gamma}$ at q is the span of Γ and the tangent space $\mathbb{T}_p A$. This span fails to intersect B transversely at q, only if the three linear spaces Γ, $\mathbb{T}_p A$ and $\mathbb{T}_q B$ fail to span all of $\mathbb{P}^N$. From our hypothesis on the embedding of X it follows that $\mathbb{T}_p A$ and $\mathbb{T}_q B$ are disjoint. Thus a necessary condition for non-transversality at q in the case where D_Γ and B are both smooth at q is that

(3) Γ is not transverse to $\overline{\mathbb{T}_p A, \mathbb{T}_q B}$.

Since $\dim \overline{\mathbb{T}_p A, \mathbb{T}_q B} = a + b + 1$ and $\dim \Gamma = N - n$, the relevant set is

$$\Psi_3 = \{(\Gamma, p, q) \in \Xi \mid p \in A_{\text{sm}},\ q \in B_{\text{sm}};\ \text{and} \\ \dim(\Gamma \cap \overline{\mathbb{T}_p A, \mathbb{T}_q B}) > a + b - n \geq 0\}.$$

We can compute the dimension of Ψ_1 and Ψ_{2a} just as we computed the dimension of Ξ itself. Since A and B are reduced, the sets A_{sing} and B^*_{sing} have strictly smaller dimension than A and B^*. Since the fiber over (p, q) of the projection $\Xi \to A \times B^*$ is an open subset of a Schubert cycle $\Sigma_n(\overline{p, q})$,

it has codimension n in $\mathbb{G}$ so Ψ_1 and Ψ_{2a} have strictly smaller dimension than Ξ.

The sets Ψ_{2c} and Ψ_5 dominate $A_{\mathrm{sm}} \times B^*$, but with strictly smaller fibers than Ξ: By our hypothesis on the embedding of X in $\mathbb{P}^N$ we have $q \notin \mathbb{T}_p A$. Also $p \notin \Gamma$. If $\Gamma \cap \mathbb{T}_p A \neq \varnothing$ then, in addition to meeting the line $\overline{pq}$ in at least a point, Γ must intersect the $(a+1)$-plane $\overline{\mathbb{T}_p A, q}$ in at least a line. Similarly, a plane Γ in the fiber of Ψ_3 over a point $(p,q) \in A_{\mathrm{sm}} \times B^*_{\mathrm{sm}}$ not only belongs to $\Sigma_n(\overline{p,q})$ but also meets $\overline{\mathbb{T}_p A, \mathbb{T}_q B}$ in at least a line. By Theorem 3.1, this implies in either case that that Γ belongs to a proper closed subvariety of $\Sigma_n(\overline{pq})$, as required.

To handle Ψ_{2b} we introduce one more incidence correspondence. Set

$$\tilde{\Psi}_{2b} = \{(\Gamma, p, p', q) \in \mathbb{G}' \times A \times A \times B^* \mid p \neq p',\ \Gamma \cap \overline{p,q} \neq \varnothing \text{ and } \Gamma \cap \overline{p',q} \neq \varnothing\},$$

so that Ψ_{2b} is the image of $\tilde{\Psi}_{2b}$ under projection to $\mathbb{G}' \times A \times B^*$. It suffices to show that $\dim \tilde{\Psi}_{2b} < \dim \Xi$.

To estimate the dimension of $\tilde{\Psi}_{2b}$, consider the projection to $A \times A \times B^*$. By our hypothesis on the embedding of X in $\mathbb{P}^N$, three distinct points $p, p', q \in X$ span a 2-plane containing $\overline{p,q}$. The intersection of Γ with $\overline{p,q}$ cannot occur at q, so Γ must meet this 2-plane in at least a line. Thus the fiber over (p,p',q) consists of planes Γ contained in $\Sigma_{n,n}(\overline{p,p',q})$, which has codimension n in $\Sigma_n(\overline{p,q})$. Comparing this situation with that in the computation of the dimension of Ξ, we see that $\dim \Psi_3 \leq \dim \Xi + a - n < \dim \Xi$, completing the proof. □

Proof of Theorem 5.15. We may immediately reduce to the case where A is reduced and irreducible. By Theorem 5.3 each $A \cap B_i$ has dimension at least $\dim A + \dim B - n$.

We do induction on the maximum of the "excess" dimensions,

$$m := \max_i \{\dim(A \cap B_i) - (\dim A + \dim B_i - n)\} \geq 0.$$

The cone construction, with general Γ, yields an expression

$$A = D_\Gamma - A'_\Gamma,$$

where D_Γ is an ambient cycle. By Lemma 5.14, $D_\Gamma \sim d[L \cap X]$, where L is a generic plane; in particular $L \cap X$ is generically transverse to B.

Lemma 5.18 tells us that if $m = 0$ then A'_Γ is generically transverse to each B_i. Further, if $m > 0$, then the maximum of the excess dimensions for the intersections of A'_Γ with the B_i,

$$m' := \max_i \{\dim(A'_\Gamma \cap B_i) - (\dim A'_\Gamma + \dim B - n)\} < m,$$

so we are done. □

Proof of Theorem 5.6 Part (b). We must show that if A and A' are rationally equivalent cycles on X and B is a cycle generically transverse to both, then $A \cap B$ is rationally equivalent to $A' \cap B$.

We can now finish the proof of Theorem 5.6, part (b): By definition, there exists a rational equivalence $Z = \sum m_i Z_i \in Z(\mathbb{P}^1 \times X)$, a linear combination of subvarieties Z_i dominating $\mathbb{P}^1$, such that $Z|_0 = A$ and $Z|_\infty = A'$. By Lemma 5.18, we may apply the cone construction to B to get an ambient cycle D_Γ and a cycle B'_Γ with $[B] = [D_\Gamma] - [B'_\Gamma]$ and such that B'_Γ is generically transverse to $Z_i|_0$ and $Z_i|_\infty$ for all i. By Lemma 5.14, D_Γ is equivalent to a cycle $d\langle L\rangle$, where L is a general linear section of X and is thus generically transverse to $Z_i|_0$ and $Z_i|_\infty$ for all i. Applying Lemma 5.7 twice, we get

$$\begin{aligned}[A \cap B] &= [A \cap D_\Gamma] - [A \cap B'_\Gamma] \\ &= d[A' \cap L] - [A' \cap B'_\Gamma] \\ &= [A' \cap B].\end{aligned}$$

This completes the proof of the Moving Lemma. □

There is a different construction, called *linkage,* that is sometimes useful for "moving" a cycle $A \subset X$ in the special case that it is *regularly embedded*, which means that each component of A is locally a complete intersection in X. Suppose that $A \subset X$ is a regularly embedded subvariety, and let B be any cycle in X. Choose $n - a$ general hypersurfaces $Z_1, \ldots, Z_{n-a} \subset \mathbb{P}^N$ containing A, and write

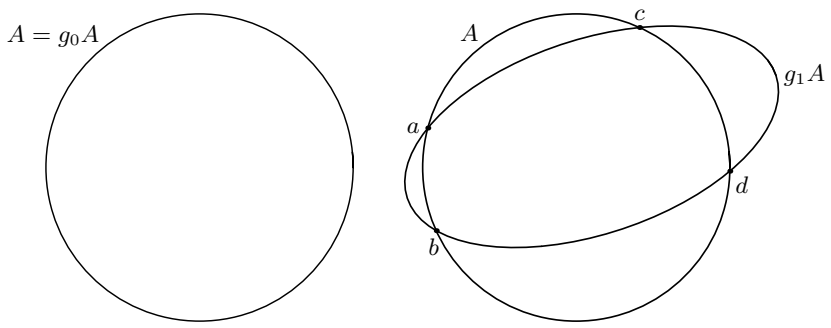
$$Z_1 \cap \cdots \cap Z_{n-a} \cap X = A \cup A'.$$

By Lemma 5.14, the intersection $Z_1 \cap \cdots \cap Z_{n-a} \cap X$ is rationally equivalent to a multiple of a general linear space section E of X, so we can write

$$A \sim d \cdot E - A'.$$

The reader should be warned that, in general, $A' \subset X$ may not be a regular embedding, so continuing the linkage process becomes complicated.

While the Moving Lemma provides us with assurance that intersection products are well defined in $A(X)$, it is rarely used in practice to calculate products in the Chow ring. Exercises 5.28–5.30 give examples of this: in each, we have to find cycles representing a given class and intersecting another cycle transversely; but in each the cone construction is not the easiest way to do it.



$$[A]^2 = [A]\,[g_1 A] = [a + b + c + d]$$

FIGURE 2.4. The cycle A meets a general translate of itself generically transversely

5.3 Kleiman's Transversality Theorem: The Case of a Transitive Group Action

There is one circumstance in which it is easy to prove a Moving Lemma, at least in characteristic 0: that is, when there is an affine algebraic group G acting transitively on X:

Theorem 5.20 (Kleiman [1974])**.** *Let X be a quasiprojective variety over an algebraically closed field, and let G be an algebraic group acting transitively on X. If A and $B \subset X$ are two subvarieties, then the intersection of the general translate gA of A with B is dimensionally transverse; that is,*

$$\operatorname{codim}(gA \cap B) = \operatorname{codim} A + \operatorname{codim} B.$$

If the characteristic is zero, then this intersection is generically transverse as well. If the group G is affine, then $A \sim gA$ for any $g \in G$.

********To be consistent with the text, in Figure 2.4 delete "$= g_0 A$" in upper left, and replace g_1 by g below figure.********

Proof. Let the dimensions of X, A, B and G be n, a, b and m respectively. Set

$$\Gamma = \{(x, y, g) : x \in A, y \in B \text{ and } gx = y\} \subset G \times X \times X.$$

Because the group action is transitive, the projection $\pi : \Gamma \to A \times B$ is surjective. Its fibers are the cosets of stabilizers of points in X, and hence have dimension $m - n$. It follows that Γ has dimension

$$\dim \Gamma = a + b + m - n.$$

On the other hand, the fiber over g of the projection $\Gamma \to G$ is isomorphic to $gA \cap B$. Thus either this intersection is empty for general g, or else it has dimension $a + b - n$, as required.

Now suppose that X is smooth and the ground field has characteristic zero. Since any affine group in characteristic zero is smooth, the fibers of the projection to $A \times B$ are also smooth, so Γ itself is smooth over $A_{\text{sm}} \times B_{\text{sm}}$. Sard's Theorem then tells us that the general fiber of the projection of Γ to G is smooth outside the singular locus Γ_{sing} of Γ. If the projection of Γ_{sing} to G is not dominant, then $gA \cap B$ is smooth for general g. Since we already know that $gA \cap B$ has codimension equal to $\operatorname{codim} A + \operatorname{codim} B$, the smoothness of $gA \cap B$ at a point p implies that both gA and B are smooth at p, so gA is transverse to B at p.

To complete the proof of generic transversality, we may assume that the projection $\Gamma_{\text{sing}} \to G$ is dominant, so that for general g the intersection $gA \cap B$ is non-empty and singular. Since algebraic groups in characteristic zero are smooth, each point of G is locally a complete intersection. By the Principal Ideal Theorem, every component of every fiber of $\Gamma \to G$ has codimension $\leq \dim G$, and thus every component of the general fiber has codimension exactly $\dim G$ in Γ. Since $\Gamma_{\text{sing}} \to G$ is dominant, its general fiber has dimension $\dim \Gamma_{\text{sing}} - \dim G < \dim \Gamma - \dim G$, so no component of a general fiber can be contained in Γ_{sing}. Thus $gA \cap B$ is generically smooth for general $g \in G$. By the argument above, this implies that gA and B are generically transverse.

When the group G is affine it is rationally connected in the sense that any point of G can be connected to the origin by a rational curve C (see Borel [1991], Theorem 18.2). It follows that the translate gA of A is rationally equivalent to A by the family $\{(g, gA) \mid g \in C\} \subset C \times X$. □

In Kleiman [1974] the main point is a result extending this Theorem to positive characteristic, under the stronger hypothesis that the group action is transitive on both points and nonzero tangent vectors. This stronger hypothesis is necessary, even in the case of the Grassmannian (where the automorphism group acts transitively on points, but preserves the *rank* of a tangent vector; see Exercise 2.22); examples can be found in Kleiman [1974] and Roberts [1972b]. Kleiman's theorem may seem to be applicable only in relatively special circumstance. But in fact we will use it quite often.

Theorem 5.20 invokes the general fact that a connected, affine algebraic group is rationally connected. This is not an easy statement to prove in general; however, the following special case will suffice for all the applications we will make in this book.

Proposition 5.21. *Suppose that a group of the form $G = \mathrm{GL}_{n_1} \times \cdots \mathrm{GL}_{n_p}$ acts algebraically on a scheme X, and let $g \subset G$. If $V \subset X$ is a subvariety then gV is rationally equivalent to V.*

Proof. We consider G as an open subset of the affine space of p-tuples of matrices. As such we may connect g to the identity $e \in G$ by a $\mathbb{P}^1$ of tuples of matrices $g_{(s,t)}se + tg$ with $(s,t) \in \mathbb{P}^1$. The element $g_{(s,t)}$ is in G for $(s,t) \in U$, a dense open subset of $\mathbb{P}^1$. Let $\Gamma \subset U \times X$ be the closure of the variety consisting of pairs $((s,t),x) \mid x \in g_{(s,t)}(V)$. Evidently, Γ dominates U and $\Gamma_{(1,0)} = V$ while $\Gamma_{(0,1)} = gV$. □

5.4 Intersections on singular varieties

In this section we discuss the problems of defining intersection products on singular varieties.

To begin with, the Moving Lemma (5.6) may fail if X is even mildly singular:

Example 5.22. Let $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ be a smooth conic and let $X = \overline{p, C} \subset \mathbb{P}^3$ be the cone with vertex $p \notin \mathbb{P}^2$. Let $L \subset X$ be a line (which necessarily contains p). ****picture of cone over conic, showing two lines of the ruling****. We claim that every cycle on X that is rationally equivalent to L has support containing p, and thus the conclusion of the first part of the Moving Lemma does not hold for X.

To see this, note first that the degree function is well defined on rational equivalence classes of curves in X: for if $i : X \to \mathbb{P}^3$ is the inclusion, then for any curve D on X we have

$$\deg D = \deg(\zeta \cdot i_*([D]))$$

where ζ is the class of a hyperplane in $\mathbb{P}^3$. It follows that we have a function $\deg : A_1(X) \to \mathbb{Z}$ that takes the class of each irreducible curve to its degree.

If $D \subset X$ is a curve that does not pass through p, then the projection from p is a finite map from D to C, and it follows that the degree of D is even. Thus any cycle of dimension 1 on X whose support does not contain p has even degree, and cannot be rationally equivalent to L.

Continuing with the notation of Example 5.22, one might hope to define an intersection product on $A(X)$ anyway. It seems natural to think that since two distinct lines L, L' through p meet in the reduced point p, we would have $[L][L'] = [p]$. However, if ζ is the class of a general hyperplane section $H \cap X$ of X through p, then (since such a hyperplane meets each L'

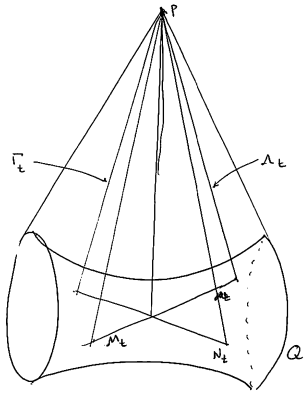


FIGURE 3.27. Cycles whose intersection product cannot be defined

transversely in one point) we might also expect $\zeta[L'] = [p]$. We can't have both these things, for, if we did then

$$[p] = \zeta[L'] = 2[L][L'] = 2[p]$$

since ζ is rationally equivalent to the union of two lines through p. This contradicts the fact that the degree map is well defined on $A_0(X)$.

In some sense the problem here is that the intersection of the lines L and L', though reduced, takes place at a singular point of X. Note that in our requirement (*) of Section 5.1 we only insisted that $[A][B] = [A \cap B]$ when the generic points of the intersection were smooth points of X. And in fact we can define at leaset a numerical intersection product on $A(X)$ for any normal surface, satisfying (*), as long as we allow intersection numbers to be rational rather than integral. In the example above, for instance, we could have defined the intersection number of two lines through the p to be $1/2$ instead of 1, and everything would have worked. See (Example 8.3.11 of Fulton [1984]) for a general statement.

In higher dimensions, however, the situation becomes intractable, as the following example shows.

Example 5.23. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface and let $X = \overline{p, Q}$ be the cone in $\mathbb{P}^4$ with vertex $p \notin \mathbb{P}^3$. The quadric Q contains two families of lines $\{M_t\}$ and $\{N_t\}$, and the cone X is correspondingly swept out by the two families of 2-planes $\{\Lambda_t = \overline{p, M_t}\}$ and $\{\Gamma_t = \overline{p, N_t}\}$; see Figure 3.27.

Now, any line $L \subset X$ not passing through the vertex p maps, under projection from p, to a line of Q; that is, it must lie either in a plane Λ_t or in a plane Γ_t; lines on X that do pass through p lie on one plane of each type. Note that since lines M_t and $M_{t'} \subset Q$ of the same ruling are disjoint for $t \neq t'$, while lines M_t and $N_{t'}$ of opposite rulings meet in a

point, a general line $M \subset X$ lying in a plane Λ_t is disjoint from $\Lambda_{t'}$ for $t \neq t'$ and meets each plane Γ_s transversely in a point, so that if there were any intersection product on $A(X)$ satisfying the fundamental condition (*) of Section 5.1, we would have

$$[M][\Lambda_t] = 0 \quad \text{and} \quad [M][\Gamma_t] = [\{q\}]$$

for some point $q \in X$. Likewise, for a general line $N \subset X$ lying in a plane Γ_t, the opposite would be true; that is, we would have

$$[N][\Lambda_t] = [\{r\}] \quad \text{and} \quad [N][\Gamma_t] = 0.$$

But the lines M and N—indeed, any two lines on X—are rationally equivalent! Specifically, the line M can be rotated in the plane Λ_t until it passes through the point p, at which point it also lies in a plane Γ_s; it can then be rotated to an arbitrary line in Γ_s. Since by Corollary 4.15 a point cannot be rationally equivalent to zero on X, we have a contradiction.

How much can one recover from these problems? The current state of the art is to define an intersection product $[A][B]$ for subvarieties of a singular variety only when one of A, B is "regularly embedded," or in other words "locally a complete intersection" in X. This leads to a notion of *Chow cohomology groups* $A^*(X)$, which play a role relative to the Chow groups analogous to that of cohomology relative to homology in the topological context: we have intersection products

$$A^c(X) \otimes A^d(X) \to A^{c+d}(X)$$

and

$$A^c(X) \otimes A_k(X) \to A_{k-c}(X)$$

analogous to cup and cap products in topology. In the present volume we will avoid all this by sticking for the most part to the case of intersections on smooth varieties, where we can simply define $A^c(X) = A_{\dim X - c}(X)$; for the full treatment, see Fulton [1984] Chapters 6, 8 and 17, and, for a visionary account of what might be possible, Srinivas [to appear].

5.5 Intersection multiplicities

We have defined intersection products by moving the intersecting cycles until they are generically transverse. But this is not always necessary. If A and B are subvarieties of a smooth quasiprojective variety X that are dimensionally transverse in the sense of Definition 5.1 above then it is possible to assign to each irreducible component Z of $A \cap B$ an *intersection*

multiplicity $m_Z(A, B)$ in such a way that

$$[A][B] = \sum_Z m_Z(A, B)[Z].$$

For example, when A and B are generically transverse, Theorem 5.4 asserts that we can take all $m_Z(A, B) = 1$. More generally, if A, B are smooth, or locally complete intersections in X, or even just locally Cohen-Macaulay at a general point of each component Z, we can take $m_Z(A, B)$ to be the multiplicity of the subscheme $A \cap B$ along Z. This can be expressed in the following attractive form:

Theorem 5.24. *Let X be a smooth quasiprojective variety. If A, B are subvarieties on X that meet dimensionally transversely, and are locally Cohen-Macaulay at the generic point of every component of $A \cap B$, then*

$$[A][B] = [A \cap B].$$

For example, since plane curves are complete intersections, we have:

Corollary 5.25 (Strong Bézout)**.** *If C and $D \subset \mathbb{P}^2$ are any two plane curves of degrees d and e with no common components, then*

$$\sum_{p \in C \cap D} m_p(C, D) = de;$$

that is, the degree of the scheme $C \cap D$ is de. □

Proof of Theorem 5.24. By Theorem 5.15 there is an equality of cycles $\langle A \rangle = \langle D \rangle - \langle A' \rangle$ where D is an ambient cycle and A' is a scheme that is generically transverse to B and does not contain any component of $A \cap B$. Since A' is generically transverse to B, the intersection $A' \cap B$ is generically Cohen-Macaulay. Because of this we do not change the associated cycle $D \cap B$ by removing from X the closed subset of $D \cap B$ where it is not Cohen-Macaulay. Thus, we may assume from the outset that $D \cap B$ is Cohen-Macaulay.

Since D is ambient there is a subscheme $\mathcal{D} \subset \mathbb{P}^1 \times X$ whose fiber $\mathcal{D}$ over $0 \in \mathbb{P}^1$ is D and whose fiber over a general $t \in \mathbb{P}^1$ is generically transverse to B. Consider the subscheme $\mathcal{B} := \mathcal{D} \cap (\mathbb{P}^1 \times B)$. The fiber $\mathcal{B}_0$ of $\mathcal{B}$ over $0 \in \mathbb{P}^1$ is $D \cap B$. Since $\mathcal{B}_0$ is a Cartier divisor in $\mathcal{B}$, it follows that $\mathcal{B}$ is Cohen-Macaulay in a neighborhood of $\mathcal{B}_0$, and thus flat over a neighborhood of $0 \in \mathbb{P}^1$. Thus $\mathcal{B}$ is a rational equivalence between $\mathcal{B}_0$ and $\mathcal{B}_t$ for generic $t \in \mathbb{P}^1$.

It follows that

$$[A \cap B] + [A' \cap B] = [\mathcal{B}_0] = [\mathcal{B}_t] = [D_t \cap B].$$

By generic transversality,

$$[D_t \cap B] = [D_t][B] = [D_0][B] = [A][B] + [A'][B].$$

Again by generic transversality $[A' \cap B] = [A'][B]$, concluding the proof. □

If we do not assume that A and B are Cohen-Macaulay, the conclusion of Theorem 5.24 may not hold; that is the multiplicity $m_Z(A, B)$ may not be equal to the multiplicity of the scheme $A \cap B$ along Z (see Example 5.27). There are several ways of computing the multiplicity in general, but perhaps the most concise is that of Serre [2000], originally published in 1957:

Theorem 5.26 (Serre's Formula)**.** *Suppose that $A, B \subset X$ are dimensionally transverse subschemes of a smooth scheme X, and Z is an irreducible component of $A \cap B$. The intersection multiplicity of A and B along Z is*

$$m_Z(A, B) = \sum_{i=0}^{\dim X} (-1)^i \text{length}_{\mathcal{O}_{A\cap B,Z}}(\text{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{A,Z}, \mathcal{O}_{B,Z})) \quad \square$$

The first term of the alternating sum in Serre's Formula is

$$\text{length}_{\mathcal{O}_{A\cap B,Z}} \text{Tor}_0^{\mathcal{O}_{X,Z}}(\mathcal{O}_{A,Z}, \mathcal{O}_{B,Z}) = \text{length}_{\mathcal{O}_{A\cap B,Z}} \mathcal{O}_{X,Z}/(\mathcal{I}_A + \mathcal{I}_B)$$

which is precisely the multiplicity of Z in the subscheme $A \cap B$. The rest of the terms are zero in the Cohen-Macaulayu case, explaining the form of Theorem 5.24.

Example 5.27. Let $X = \mathbb{P}^4$, and let V_1, V_2 be general planes and let $A = V_1 \cup V_2$.

Consider the intersection of A with some 2-planes. Let $p = V_1 \cap V_2$ be the point where the two planes meet and let B_1 be a 2-plane that does not pass through p, and meets each of V_1, V_2 in a single (necessarily reduced) point. Let B_2 be a 2-plane that passes through p and does not meet A anywhere else. The cycles $[B_1]$ and $[B_2]$ are rationally equivalent in $\mathbb{P}^4$. The cycle $[B_1 \cap A]$ consists of two reduced points, the intersections of B_1 with V_1 and with V_2, so $\deg[B_1 \cap A] = 2$.

By contrast, it's not hard to see that $\deg[B_2 \cap A] \geq 3$: since the Zariski tangent space to the scheme $A = V_1 \cup V_2$ at the point p is all of $T_p(\mathbb{P}^4)$, the tangent space to the intersection $B_2 \cap A$ at p must be all of $T_p(B_2)$. In other words, $B_2 \cap A$ must contain the "fat point" at p in the plane B_2 (that is, the scheme defined by the square of the ideal of p in B_2), and so must have degree at least 3.

Since the degree function is well-defined on rational equivalence classes of zero-cycles, this implies that $[B_1 \cap A]$ is not rationally equivalent to $[B_2 \cap A]$.

In fact, we can see that the degree of $B_2 \cap A$ is exactly 3, by a local calculation. Since B_2 meets A only at the point p, we have to show that the length of the artinian ring $\mathcal{O}_{\mathbb{P}^4,p}/(\mathcal{I}(B_2) + \mathcal{I}(A))\mathcal{O}_{\mathbb{P}^4,p}$ is 3. Let $S =$

$k[x_0, \ldots, x_4]$ be the homogeneous coordinate ring of $\mathbb{P}^4$. We may choose V_1, V_2 and B_2 to have homogeneous ideals:

$$I(A) = (x_0, x_1) \cap (x_2, x_3) = (x_0x_2, x_0x_3, x_1x_2, x_1x_3),$$
$$I(B_2) = (x_0 - x_2, x_1 - x_3).$$

Modulo $I(B_2)$ we can eliminate the variables x_2, x_3 and the ideal $I(A)$ becomes (x_0^2, x_0x_1, x_1^2). Passing to the affine open subset where $x_4 \neq 0$, this is the square of the maximal ideal corresponding to the origin in the plane B_2. Thus $\mathcal{O}_{\mathbb{P}^4,p}/(\mathcal{I}(X_2)+\mathcal{I}(A))\mathcal{O}_{\mathbb{P}^4,p}$ has basis $\{1, x_0/x_4, x_1/x_4\}$, and thus its length is 3, as claimed. See Exercise 5.32 for a computation of the intersection multiplicity using Serre's formula.

5.6 Refinements and Other Approaches

5.6.1 Refined and Excess Intersection

We consider now the case where X is smooth, but the intersection of two subvarieties $A, B \subset X$ has dimension strictly larger than the expected $a + b - n$. (This is often called *excess intersection*.) The definition we have given leads to an intersection product represented by a cycle that may not be supported on $A \cap B$ (though if A is smooth then the pullback of $[B]$ via the inclusion map does give us a well defined cycle in $A(A)$), but there are other ways to define the intersection product so that it is supported on $A \cap B$, and indeed is well-defined as a class in $A \cap B$. This is described by Fulton as the *refined intersection class*.

An extreme example of this occurs when $A = B$. For example, suppose that X is the blowup of $\mathbb{P}^2$ at a point and E is the exceptional divisor. Taking $n = 2$ in Proposition **??**, we see that $[E][E]$ is equal to the negative of the class of a point, which can be taken to be a point of E. But which point? The automorphisms of X all fix E setwise, and act transitively on the points of E, so it would be impossible to choose a natural cycle—only the rational equivalence class on E can be well-defined.

There is another way that excess intersections appear that has great importance in the theory. It may be that when we intersect A and B we get some components of the proper dimension ($= \dim A + \dim B - \dim X$) and other components of too large dimension. This is, for example, what happens in the case of the famous classical problem of finding the smooth (plane) conics tangent to five given general conics. One easily constructs the hypersurface in the 5-dimensional projective space of (not necessarily smooth) plane conics consisting of those tangent to a given conic—it has degree 6. One might think that the answer would therefore be $6^5 = 7776$ smooth conics. But such a hypersurface contains the locus of double lines,

which form a surface in the $\mathbb{P}^5$ of all conics. Thus the intersection of five of these consists of the points that correspond to the answer to the problem—a dimensionally transverse intersection—but also the 2-dimensional family of double lines. We will see in Chapter 10 how to deal with this problem (there are 3264 smooth conics, whence the title of this book.)

5.6.2 Reduction to the Diagonal and Deformation to the Normal Cone

A classical approach to Theorem 5.4 uses the very powerful technique of *reduction to the diagonal*. Again, let X be a smooth variety, and suppose $A, B \subset X$ are subvarieties. The idea is to exploit the obvious formula

$$A \cap B \cong (A \times B) \cap \Delta$$

where $X \cong \Delta \subset X \times X$ is the diagonal. We can use this formula to replace the problem of defining an intersection class $[A][B]$ with an intersection class $[A \times B][\Delta]$. Of course this is apparently in the wrong variety: $[A \times B][\Delta] \in A(X \times X)$, not $A(X)$. But if the class can be represented by a cycle supported on $(A \times B) \cap \Delta$, then it can at least be interpreted as a cycle on $A \cap B$, and its class computed there (and then pushed forward into X if desired.)

It might seem that there is no advantage in this round-about procedure—after all, we still have to define an intersection class, and one supported on the intersection, to boot. But we have gained something important: if X is smooth, Δ is locally a complete intersection in $X \times X$. So we only have to understand intersections in the situation where one of the two varieties being intersected is nice in this way.

This is indeed how things were done classically for dimensionally transverse intersections. The multiplicity formula given above takes a simpler, non-homological form (due to Samuel) in this special case. In the modern period, Fulton and others understood how to use this idea more generally. The idea can be explained as follows:

If A is a Cartier divisor on a (not necessarily smooth or projective) variety X then A is the vanishing locus of a section of a line bundle $\mathcal{L}$ on X, and one can define the intersection of A with another subvariety B (inside *any* scheme X) to be the divisor associated to the restriction $\mathcal{L}|_B$. Using Chern classes one can extend this idea, and define an intersection product with the class of any subvariety A that is the zero section of a section of a vector bundle on X of rank equal to $\operatorname{codim} A$.

The technique of "deformation to the normal cone" (see Section 15.3.1, plus Fulton [1984] and the history there) allows one to extend this idea further to the case of an intersection with a subscheme A that is locally a

complete intersection in X: roughly speaking, one deforms X, along with the subvarieties A, B, to a new variety X', the *normal cone* of A, with a vector bundle of the right rank defined on it, and a section of that vector bundle vanishing only at A. Using the idea of reduction to the diagonal, this suffices to define an intersection theory on any smooth variety.

5.7 Proofs of the results on Pullbacks

We will prove Theorem 5.10 and Lemma 5.9 under the hypothesis that f is generically separable. Thus we let $f : Y \to X$ be a map of smooth projective varieties, and suppose that the field of rational functions on Y is separable over the field of rational functions on $f(Y)$, or equivalently, that the generic point of Y is a smooth point in its fiber. ****this hypothesis is slightly different than what we need in 5.9. Maybe change this intro para, or move it next to the proof of the Theorem****.

Proof of Lemma 5.9. Suppose that the fiber of f over a general point $x \in f(Y)$ has dimension k, so that $\dim Y = k + \dim f(Y)$. For each $l > k$, let Φ_l be the closure of

$$\{x \in f(Y) : \dim f^{-1}(x) \geq l\}.$$

We claim that Φ_l has codimension $\geq l - k + 1$. This is because, for $l > k$,

$$Y_l = \{y \in Y \mid \dim f^{-1}(x) \geq l\}$$

is a proper closed subset, so $\dim \Phi_l + l < \dim Y = \dim f(Y) + k$, (Eisenbud [1995] Corollary 14.5.)

Also, let Ψ be the closure of the set

$$\{x \in X : f^{-1}(x) \text{ has an irreducible component that is not reduced}\}.$$

To see that $\Psi \neq X$, let $U \subset X$ be the open dense set $X \setminus \Phi_{k+1}$, so that for each $x \in U$, the dimension of $f^{-1}(x)$ is minimal. Set $Y_U = f^{-1}(U)$. Because the dimension of $f^{-1}f(y)$ is semi-continuous, every component of every fiber of $f|_{Y_U} \to U$ has the same dimension k. The set

$$Z = \{y \in Y_U \mid y \text{ is a singular point of its fiber } f^{-1}f(y)\}$$

is closed in Y_U because it is the locus where the map of vector bundles $T_{Y_U} \to (f^*T_X)|_{Y_U}$ drops rank. By separability, Z does not contain the generic fiber, so $Z \neq Y_U$, and since Y_U is irreducible, $\dim Z < \dim Y_U$.

If, for some $x \in U$ the fiber $f^{-1}(x)$ has a non-reduced component, then $f^{-1}(x) \cap Z$ contains that component, so the fiber of Z over x has dimension

k; conversely if the fiber of Z over x has dimension as large as k, then it must contain a component, which must then be nonreduced; that is,

$$\Psi \cap U = \{x \in U \mid \dim f|_Z^{-1}(x) \geq k\}.$$

Since $\dim Z < \dim Y_U$, we see that $\Psi \cap U \neq U$.

We claim that if a subvariety $V \subset X$ is generically transverse to each of the subvarieties $\Psi, \Phi_l, f(Y)$, then V is generically transverse to f. First, to show that the preimage of V has the correct dimension, we use the estimate on the codimension of Φ_l. For $l > k$ this yields

$$\dim(V \cap \Phi_l) \leq \dim V - \operatorname{codim} f(Y) - l + k - 1,$$

from which it follows that

$$\dim\big(f^{-1}(V \cap \Phi_l)\big) \leq \dim V - \operatorname{codim} f(Y) + k - 1.$$

Since every component of $f^{-1}(V)$ will have dimension at least $\dim V - \operatorname{codim} f(Y) + k$, it follows that no component of $f^{-1}(V)$ is contained in $f^{-1}(\Phi_l)$ for $l > k$, and hence

$$\dim f^{-1}(V) = \dim V - \operatorname{codim} f(Y) + k,$$

as required.

Let $W \subset f^{-1}(V)$ be an irreducible component, and let f_W be the restriction of f to W. We must show that W is generically reduced. Since the general fiber of W has dimension k, the image $f(W)$ has the same dimension as $V \cap f(Y)$, and thus the image $f(W)$ is a component of $V \cap f(Y)$.

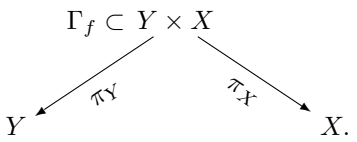
Let $w \in W$ be a general point; we will show that W is smooth at w. We see from the previous paragraph that $f(w)$ is a general point of a component of $f(Y) \cap V$. By generic transversality $f(w)$ is a smooth point of $f(Y) \cap V$. Further, by generic transversality, $f(w)$ is not contained in Ψ. Thus w is a smooth point of $f^{-1}f(w)$, and the kernel of the differential

$$T_{W,w} \to T_{V \cap f(Y), f(w)}$$

has dimension k. Since $\dim T_{V \cap f(Y), f(w)} = \dim(V \cap f(Y))$ we see that $\dim T_{W,w} \leq k + \dim(V \cap f(Y)) = \dim W$, so $w \in W$ is a smooth point. □

Proof of Theorem 5.10 Part (a). By the Moving Lemma and Lemma 5.9, the classes of subvarieties A that are generically transverse to f generate $A(X)$ as an abelian group, so there is at most one group homomorphism $f^* : A(X) \to A(Y)$ such that $f^*[A] = [f^{-1}(A)]$ for such subvarieties.

Let $\Gamma_f \subset Y \times X$ be the graph of f and consider the projections



Since π_X is flat and π_Y is proper, there are group homomorphisms $(\pi_Y)_*$ is as in Theorem 4.14 and $(\pi_X)^*$ is as in Theorem 4.20. We define f^* to be the group homomorphism

$$f^* = (\pi_Y)_* \circ (\pi_X)^*.$$

If A is a subvariety of X then $(\pi_X) * ([A]) = [f^{-1}(A)]$ by definition. If A is generically transverse to f, then $f^{-1}(A)$ is generically transverse to Γ_f, so

$$(\pi_Y)_*([f^{-1}(A)]) = [\pi_Y(\pi_X^{-1}(A))] = [f^{-1}(A)],$$

so the group homomorphism f^* satisfies the condition of Theorem 5.10.

To show that f^* is a ring homomorphism, let $\alpha, \beta \in A(X)$. Let $\{B_i\}$ be, as in Lemma 5.9, a finite collection of cycles such that a cycle is generically transverse to f if it is generically transverse to each B_i. By the Moving Lemma we can choose cycles $A, B \in Z^*(X)$ representing α and β such that A is generically transverse to each B_i and such that B is generically transverse to A, B_i and $A \cap B_i$ for each i. It follows that $A \cap B$ is generically transverse to each B_i, and thus A, B and $A \cap B$ are generically transverse to f. Under these circumstances, $f^{-1}(A)$ and $f^{-1}(B)$ intersect generically transversely and hence

$$\begin{aligned}(f^*\alpha)(f^*\beta) &= [f^{-1}(A)][f^{-1}(B)] \\ &= [f^{-1}(A) \cap f^{-1}(B)] \\ &= [f^{-1}(A \cap B)] \\ &= f^*(\alpha\beta).\end{aligned}$$

This completes the proof of Part (a) of Theorem 5.10

Part (b) The definition of the pullback map allows us to reduce statements about intersection products to statements about the intersections of subsets. For example, the push-pull formula

$$f_*\big(f^*(\alpha)\beta\big) = \alpha f_*(\beta)$$

corresponds to the set-theoretic equality

$$f(f^{-1}(A) \cap B) = A \cap f(B),$$

for subsets $A \subset X$ and $B \subset Y$. Of course to prove the equality of cycles that corresponds to the push-pull formula, we need to have enough transversality (which is possible through the Moving Lemma) and to take care of degrees.

To carry this out, note first that the formula is linear in β, so we can immediately reduce to the case where β is the class of an irreducible subvariety B. We may assume by the Moving Lemma that B is generically transverse to the generic fiber of f and thus is generically separable over its image in X (of course this is automatic in characteristic zero). Using the Moving Lemma and linearity, together with Lemma 5.9, we may assume

further that $\alpha = [A]$ where $A \subset X$ is a variety generically transverse to f to $f|_B$, and to $f(B)$. Because A is generically transverse to $f|_B$ we see that $f^{-1}(A)$ is generically transverse to B.

Suppose now that B is generically finite of degree d over $f(B)$. Because A is transverse to $f|_B$ the cycle $f^{-1}(A) \cap B$ is also generically finite, and of the same degree d, over its image $A \cap f(B)$. In this case, the proof is concluded by the series of equalities below, whose justifications are given on the right.

$$\begin{aligned} f_*(f^*(\alpha)\beta) &= f_*([f^{-1}(A)]\beta) && A \text{ generically transverse to } f \\ &= f_*([f^{-1}(A) \cap B]) && f^{-1}(A) \text{ generically transverse to } B \\ &= d[f(f^{-1}(A) \cap B)] && \text{definition of } f_* \\ &= d[A \cap f(B)] && \text{set-theoretic equality} \\ &= [A \cap f_*(B)] && \text{definition of } f_* \\ &= \alpha f_*(\beta) && A \text{ generically transverse to } f(B). \end{aligned}$$

If on the other hand B is not generically finite over $f(B)$, then by the semi-continuity of fiber dimension, $f^{-1}(A) \cap B$ is not generically finite over $A \cap f(B)$. Thus the chain of equalities above is valid with $d = 0$. □

Proof of Proposition 5.11. As in the last part of the proof of part a) of Theorem 5.10, we see from Lemma 5.9 that any classes $\alpha, \beta \in A(X)$ may be represented by cycles A, B such that A, B are generically transverse and A, B and $A \cap B$ are generically transverse to f. It follows that $f^{-1}(A)$ is generically transverse to $f^{-1}(B)$, so while $\alpha\beta$ is represented by $A \cap B$ $f^*(\alpha)f^*(\beta)$ is represented by $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$. □

5.8 Exercises

Exercise 5.28. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d, and $L \subset S$ a line. Find the degree of the self-intersection $[L]^2 \in A^2(S)$ by linkage, considering the intersection of S with a general plane containing L.

Exercise 5.29. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface, and C a twisted cubic curve lying on Q; let $c \in A^1(Q)$ be its class. Find the self-intersection c^2 in the Chow ring of Q.

Exercise 5.30. Now let $S \subset \mathbb{P}^3$ be a smooth cubic surface, and C a twisted cubic curve lying on S; again, let $c \in A^1(S)$ be its class. Find the self-intersection c^2 in the Chow ring of S.

Exercise 5.31. Show that if $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ is the d-th veronese mapping and $L \subset \mathbb{P}^N$ is a linear subspace of dimension $e < d$, then the length of the scheme $L \cap X$ is at most $e + 1$.

Exercise 5.32. In Example 5.27, calculate the length of the Tor modules in Serre's formula, and verify that the correct multiplicity $m_p(X_2, V)$ is 2. (Hint: this can be calculated directly, using the fact that all the ideals involved are monomial; alternatively, we can take the Koszul resolution of X_2 and tensor with $\mathcal{O}_{V,p}$.)

Exercise 5.33. Let $f : Y \to X$ be a generically finite surjective morphism of smooth projective varieties. Show that the composition

$$f_* f^* : A_k(X) \to A_k(X)$$

is multiplication by the degree of f; deduce in particular that the map $f^* : \mathbb{Q} \otimes A_k(X) \to \mathbb{Q} \otimes A_k(Y)$ is injective. Show by example that $f^* f_* \alpha$ may not be a multiple of α, so no analogous statement holds.

6

Interlude: Vector Bundles and Direct Images

We will now introduce some ideas that are of the utmost usefulness in intersection theory, and in algebraic geometry generally: vector bundles and direct images. We will use these ideas to solve enumerative problems by *linearizing* them: that is, replacing a polynomial equation with a family of systems of linear equations, that is, a family of linear maps between families of vector spaces parameterized by the points of a variety B. It is not an exaggeration to say that in the rest of this book the different chapters are organized around the treatment of problems using different families of vector bundles.

After introducing the main idea with an example, we review the ideas of vector bundles and pullbacks. Then we present the notion of the direct image in a more systematic and leisurely way. Chapter 17 provides a more technical version with a still greater range of applications.

6.1 Vector Bundles and Locally Free Sheaves

This section and the next are intended to review some basic notions and set terminology that we will use. The reader who has not seen these things before might consult ???? for a more leisurely treatment and additional information. The last section of the Chapter, on direct images, is a systematic introduction.

- A *vector bundle* of rank n on a scheme X is a scheme E and a morphism $\pi : E \to X$ with a section $\sigma : X \to E$ called the zero section, such that, for some open affine cover $\{U_i\}$ of X there are isomorphisms $\alpha_i : E_{U_i} := \pi^{-1}(U_i) \to \mathbb{A}^n_{U_i} = V \times U_i$, where V_i is a vector space, and such that for each $x \in U_i \cap U_j$ the induced map $\alpha_i(x)\alpha_j^{-1}(x) : \mathbb{A}^n \to \mathbb{A}^n$ is a linear isomorphism. A *section* of E (really of the pair (E, π)) over $U \subset X$ is a map $\sigma : U \to \pi^{-1}(U)$ such that $\pi\sigma = 1_U$.
  This looks a little simpler in the case of a scheme X of finite type over an algebraically closed field K, the case treated in this book, since in this case we may identify the closed points of the affine space $\mathbb{A}^n_{U_i}$ with the points of $V \times U_i$, where V is an n-dimensional vector space over K. Sometimes—for example in the definition of the tangent bundle of a smooth variety—we don't give E and π globally, but start from only give the $E_{U_i} = V_i \times U_i$, with $\pi_i : E_{U_i} \to U_i$ defined as the projection. In this case we need to give isomorphisms
  $$V_j \times U_j \xrightarrow{\beta_{i,j}} V_i \times U_i$$
  (which, in our previous construction were $\beta_{i,j} = \alpha_i\alpha_j^{-1}$) such that for each $x \in U_i \cap U_j$ the induced map $\beta_{i,j}(x) : V_j \to V_i$ is linear; put informally, the $\beta_{i,j}(x)$ are regular functions from $U_i \cap U_j$ to the vector space $Hom(V_j, V_i)$ with values in the set of isomorphisms. These maps must satisfy a compatibility condition on triple overlaps:
  $$\beta_{i,j}\beta_{j,k} = \beta_{i,k} \quad \text{on } U_i \cap U_j \cap U_k.$$
  We can then construct the scheme E as the quotient of the disjoint union of the $V_i \times U_i$ by the equivalence relation generated by the $\beta_{i,j}$.
- A *locally free sheaf* of rank n on a scheme X is a coherent sheaf $\mathcal{E}$ on X such that for some open affine cover $\{U_i\}$ of X there are isomorphisms $\mathcal{E}|_{U_i} \cong \mathcal{O}_X(U_i)^n$. A section of $\mathcal{E}$ over U is just an element of $\mathcal{E}(U)$.

Note that we are only dealing with vector bundles of finite rank, and locally free sheaves that are coherent. Nothing prevents the extension of these notions to the infinite rank/quasicoherent case, but we will never need this, and we will assume finite rank or coherence throughout.

A locally free sheaf $\mathcal{E}$ can be made from a vector bundle E by taking the sheaf of sections: $\mathcal{E}(U) := H^0(E|_U)$. A vector bundle E can be made from a locally free sheaf $\mathcal{E}$ by setting $E = \operatorname{Spec} Sym(\mathcal{E}^*)$, the global spectrum of the symmetric algebra of the dual of $\mathcal{E}$.

This construction becomes clearer if we work locally: on an affine open set U of X where $\mathcal{E}$ is free we have $\mathcal{E}|_U = \mathcal{O}_X(U)^n$. Thus $(\mathcal{E}|U)^* = \mathcal{E}^*|_U$ may be thought of as the linear functions on the fibers of the trivial vector bundle over U, $U \times K^n$, with coefficients that are regular functions on U. Thus $\operatorname{Sym}(\mathcal{E}^*)|_U = \operatorname{Sym}((\mathcal{E}|_U)^*)$ is the ring of polynomial functions on the fibers with coefficients that are regular functions on U, as required.

These identifications, which are inverse to one another, show that the notions of "vector bundle" and "locally free coherent sheaf" are equivalent. *From now on we will use the terms and the notation interchangeably, though we sometimes write $\mathcal{E}$ to emphasize that we are taking the sheaf point of view or E to emphasize that we are taking the vector bundle point of view.*

The category of vector bundles on a scheme X is closed under the functors Hom and $\otimes_X$, interpreted fiberwise (in the first description) or, equivalently, as operations on coherent sheaves. However, the category of vector bundles is not generally an abelian category: the quotient $\mathcal{E}/\mathcal{F}$ of locally free sheaves may not be locally free. From the vector bundle point of view, a map $\varphi : E \to F$ of vector bundles over X (that is, a morphism of schemes that commutes with the projection to X and is linear on each fiber), may have the same rank (as a linear transformation) on all the fibers; in this case we say that φ *has constant rank*, and its fiberwise kernel and cokernel are again vector bundles. But it may also have ranks that vary with the fibers; then its kernel and cokernel can be regarded as coherent sheaves, but not as vector bundles. We say that E is a sub-bundle of F if $E \to F$ has constant rank equal to the rank of E, and similarly for a quotient bundle. We regard two sub-bundles as the same if their images in F are the same.

Vector bundles very commonly arise through what one might informally call the "Principal of Naturality". One starts with a space X and a "natural" way of describing a vector space of dimension n for each point of the space. If the description is natural enough, these spaces turn out to be identified with the fibers of a vector bundle over X; one feels that it simply could not be otherwise. We don't know a sharp theorem to this effect that is worth stating, but the reader will see many examples below.

6.2 Pullbacks

For a detailed treatment of the ideas in this section see Hartshorne [1977] Section II.5.

Given a morphism of varieties $\pi : X \to B$ and a vector bundle or coherent sheaf $\mathcal{G}$ on B, we wish to define a natural vector bundle $\mathcal{F} = \pi^*\mathcal{G}$ on X, called the pullback of $\mathcal{G}$ under the map π, whose fiber at a closed point $p \in X$ is the same as the fiber of $\mathcal{G}$ at $\pi(p)$.

In fact, given a (quasi)coherent sheaf $\mathcal{G}$ on B, there is a (quasi)coherent sheaf $\pi^*\mathcal{G}$ on X characterized by the property that for each pair of open affine sets $U \subset X$ and $V \subset B$ such that $\pi(U) \subset V$ we have $(\pi^*\mathcal{G})(U) = \mathcal{O}_X(U) \otimes_{\mathcal{O}_B(V)} \mathcal{G}(V)$ as $\mathcal{O}_X(U)$ modules. Existence is proven by observing that the module $(\pi^*\mathcal{G})(U)$ defined above is independent of the choice of V, and glues appropriately along open coverings.

To check that this construction implies the desired property for vector bundles, note that the fiber of $\pi^*\mathcal{G}$ at a closed point $x \in U$ is by definition $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \otimes_{\mathcal{O}_{X,x}} \pi^*\mathcal{G}(U)$, and

$$\begin{aligned}\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} &\otimes_{\mathcal{O}_{X,x}} \pi^*\mathcal{G}(U) \\ &= \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_X(U) \otimes_{\mathcal{O}_B(V)} \mathcal{G}(V) \\ &= \mathcal{O}_{B,\pi(x)}/\mathfrak{m}_{B,\pi(x)} \otimes_{\mathcal{O}_B(V)} \mathcal{G}(V)\end{aligned}$$

since (by our standing assumption that the ground field is algebraically closed) the map

$$\mathcal{O}_{B,\pi(x)}/\mathfrak{m}_{B,\pi(x)} \to \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$$

induced by π is an isomorphism.

For example, it follows at once from the definitions that if $\pi : X \to B$ is a morphism of schemes, then $\pi^*\mathcal{O}_B = \mathcal{O}_X$.

6.3 Flat Families of Sheaves

A family of schemes is simply a morphism $\pi : X \to B$. But for the fibers to have much "family resemblance" some condition is necessary. Of course one could assume that π was locally a product over B in some sense[1] but for many purposes in algebraic geometry this notion is too restrictive; for example it eliminates families in which smooth varieties degenerate to singular varieties, such as the family of plane curves of a given degree. Serre introduced a very algebraic but much more flexible notion, which has proven itself in countless applications: *flatness*.

Recall that X is flat over B or, more precisely, π is a *flat* morphism, if for every point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a flat module over the local ring $\mathcal{O}_{B,\pi(x)}$ via the map $\mathcal{O}_{B,\pi(x)} \to \mathcal{O}_{X,x}$ via the map taking a germ h of a function on B to the germ $h \circ \pi$ on X. (For more intuition about this opaque definition, see for example Eisenbud and Harris [2000].)

Similarly, if $\pi : X \to B$ is any morphism of schemes and $\mathcal{F}$ is a (quasi)coherent sheaf on X, then we say that $\mathcal{F}$ is flat over B if for every point $x \in X$ the stalk $\mathcal{F}_x$ is a flat module over the local ring $\mathcal{O}_{B,\pi(x)}$ via π.

One of the nice consequences of flatness, which we will prove in Section6.5, is that the restrictions of a flat sheaf to different fibers have the same Euler characteristic:

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[1] Perhaps in the analytic topology, if we were working over $\mathbb{C}$, or in the étale topology generally.

Theorem 6.1. *Let $\pi : X \to B$ be a projective morphism of schemes, with B connected, and let $\mathcal{F}$ be a sheaf on X that is flat over B. The Euler characteristic*

$$\chi(\mathcal{F}|_{\pi^{-1}(b)}) = \sum_i (-1)^i \dim H^i(\mathcal{F}|_{\pi^{-1}(b)})$$

is independent of the closed point $b \in B$.

If B is reduced and π is a projective morphism, that is, the projection of $X \subset \mathbb{P}^n_B$ to B, then the constancy of the Euler characteristic for all $\big(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n_B}(d)\big)|_{\pi^{-1}(b)}$ is necessary and suffient for flatness (see for example Eisenbud and Harris [2000] III-56, where the result is proved in the special case $\mathcal{F} = \mathcal{O}_X$; the proof in general follows exactly the same lines), but we will not need this fact.

6.4 Direct Images

Suppose that we are given a family of varieties and a family of sheaves on them—in other words, a morphism $X \to B$ and a sheaf $\mathcal{F}$ on X. For many purposes in this book we would *like* to define a sheaf $\mathcal{G}$ on B whose fiber at each point $b \in B$ is the space $H^0(\mathcal{F}|_{X_b})$ of global section of $\mathcal{F}$ on the fiber X_b of X over b, and such that algebraic families of elements of $H^0(\mathcal{F}|X_v)$ give rise to sections of $\mathcal{G}$ in a way that is compatible with the identification of $\mathcal{G}_b$ with $H^0(\mathcal{F}|_{X_b})$.

Does such a sheaf exist?

Our applications will be to projective morphisms $X \to B$ and coherent sheaves $\mathcal{F}$, so we'll focus on this case. With a little more effort, the results can all be extended to proper morphisms; see Grothendieck [1963] Theorem 3.2.1.

It is reasonable to interpret the phrase "algebraic family of elements" in the description above to mean "section defined over an open set of B", and this idea leads naturally to the construction below as giving the best hope at finding a sheaf with the desired properties:

Definition 6.2. Given a morphism of schemes $\pi : X \to B$ and a sheaf $\mathcal{F}$ on X, we define the *direct image* $\pi_*\mathcal{F}$ of $\mathcal{F}$ to be the presheaf on B that assigns to each open subset $U \subset B$ the space of sections of $\mathcal{F}$ on the open set $X_U = \pi^{-1}(U)$.

It follows easily from the definition that $\pi_*\mathcal{F}$ is actually a sheaf, and also that $H^0(\pi_*(\mathcal{F})) = H^0(\mathcal{F})$. If B is an affine space and $\mathcal{F}$ is quasicoherent,

then we can represent the quasicoherent sheaf $\pi_*(\mathcal{F})$ by its module of global sections, and we see at once that this is

$$\pi_*(\mathcal{F})(B) = \mathcal{F}(\pi^{-1}(B) = H^0(\mathcal{F}).$$

We also see that there is a natural map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ which, restricted to the preimage of an open set $U \subset B$, is the tautogical map $\mathcal{O}_{X_U} \otimes H^0(\mathcal{F}|_{X_U}) \to \mathcal{F}$.

By construction, an algebraic family of sections of $\mathcal{F}$ over an open set $U \subset B$ gives rise to a section of $\pi_*\mathcal{F}$ over U, fulfilling one of our desiderata, and doing so in the most efficient way. In general the fiber $\pi_*\mathcal{F}$ at a point $b \in B$ will *not* be $H^0(\mathcal{F}|_{X_b})$. ****we either need to give an example here or refer to one in Ch 17.**** But at least we get a natural map: for any point $b \in B$ and any neighborhood U of b in B there is a restriction map

$$\pi_*\mathcal{F}(U) = \mathcal{F}(X_U) \ \to \ H^0(\mathcal{F}|_{X_b}).$$

The map $\pi_*\mathcal{F}(U) = \mathcal{F}(X_U) \ \to \ H^0(\mathcal{F}|_{X_b})$ induces a map from the fiber of $\pi_*\mathcal{F}$ at b to $H^0(\mathcal{F}|_{X_b})$, and thus a map

$$\varphi_b : (\pi_*\mathcal{F})|_b \ \to \ H^0(\mathcal{F}|_{X_b}).$$

It follows from the definition that the image of φ_b consists of those sections of $\mathcal{F}|_{X_b}$ that extend to sections of $\mathcal{F}|_{X_U}$ for some neighborhood U of $b \in B$, while the kernel consists of sections of defined in some small neighborhod U of $\pi^{-1}(b)$ that vanish on $\pi^{-1}(b)$, but cannot be expressed in terms of functions pulled back from any small neighborhood of b and vanishing at b.

We will be particularly interested in cases where we can show that $\pi_*(\mathcal{F})$ is a vector bundle. To get an idea what to expect, consider the simplest case, where both X and B are affine varieties and the map π is finite (and thus projective as well as affine). In this simple situation it will always be the case that φ_b is an isomorphism for every b, and we can say just when $\pi_*(\mathcal{F})$ is a vector bundle—that is, M is a locally free S-module:

Proposition 6.3. *Suppose that $\pi : X \to B$ is a finite morphism of affine varieties, and $\mathcal{F}$ is a coherent sheaf on X, the maps $\varphi_b : (\pi_*\mathcal{F})|_b \ \to H^0(\mathcal{F}|_{X_b})$ are isomorphisms for all closed points $b \in B$. Moreover, the following are equivalent:*

(a) *$\pi_*\mathcal{F}$ is a vector bundle on B.*

(b) *$\mathcal{F}$ is flat over B.*

(c) *The dimension of $H^0(\mathcal{F}|_{X_b})$ as a vector space over the residue class field $\kappa(b)$ is independent of the closed point $b \in B$.*

Proof. Let $X = \operatorname{Spec} R$ and $B = \operatorname{Spec} S$, and suppose that the coherent sheaf $\mathcal{F}$ is represented by a finitely generated R-module M. In this case $\pi_*\mathcal{F}$

is represented by M regarded as a module over S via the map $\pi^* : S \to R$. Note that M is finitely generated over S because R is, by hypothesis, a finitely generated S-module. The maps φ_b are isomorphisms because, writing $\mathfrak{m}_b \subset S$ for the maximal ideal corresponding to b, both $(\pi_*\mathcal{F})|_b$ and $H^0(\mathcal{F}|_{X_b})$ may be identified canonically with $M/\mathfrak{m}_b M$. The equivalence of parts a) and c) is proven in Proposition 6.8 below. The equivalence of a) and b) is proven in Eisenbud [1995] Theorem ***. □

In case we suppose only that π is a projective morphism, not necessarily finite, neither condition b) nor c) of Proposition6.3 alone will imply either that $\pi_*\mathcal{F}$ is a vector bundle nor that the maps φ_b are isomorphisms. But conditions b) and c) together do imply both of these conclusions. This result is the most useful special case of the "Theorem on Cohomology and Base Change", Theorem 6.6.

Theorem 6.4 (Cohomology and Base Change—version 1)**.** *Let $\pi : X \to B$ be a projective morphism of varieties, and let $\mathcal{F}$ be a sheaf on X that is flat over B. If the dimension of $H^0(\mathcal{F}|_{\pi^{-1}(b)})$ is independent of the closed point $b \in B$, then $\pi_*(\mathcal{F})$ is a vector bundle and for every closed point $b \in B$ the comparison map*

$$\varphi_b : \pi_*(\mathcal{F})|_b \to H^0(\mathcal{F}|_{\pi^{-1}(b)})$$

is an isomorphism. If $\mathcal{F}$ is finite over B—that is, if $\mathcal{F}|_{\pi^{-1}(b)}$ is a finite scheme for each $b \in B$—then the flatness hypothesis is unnecessary. □

Proof. This is all covered in the more general Theorem 6.6 except the very last statement, so it suffices to show that if $\mathcal{F}$ is finite over B and the dimension of $H^0(\mathcal{F}|_{\pi^{-1}(b)})$ is independent of the closed point $b \in B$, then $\mathcal{F}$ is flat over B.

Without loss of generality we may assume that X is equal to the support of $\mathcal{F}$. Under our hypothesis this support is itself quasi-finite and projective over B, and thus is affine and finite over B. Replacing B with an open affine subset $\operatorname{Spec} R$, we may think of $\mathcal{F}$ as a finitely generated R-module M. Our hypothesis then says that $M/\mathfrak{m}_b M$ has dimension over $R/\mathfrak{m}_b$ that is independent of b. Such a module is free over R (Eisenbud [1995] Theorem***) and thus flat. □

****I moved the following material from Chapter 17.11, Before just part a of the corollary was here, but the rest is used in the Projective Bundle chapter. Should it move someplace else?****

Using Theorem 6.6 we can easily derive some results about line bundles on families of varieties that are particularly useful in the case of projective bundles. ****Maybe this next bit should move earlier**** We will make use of the following natural "adjunction" maps, defined for any morphism $\pi : X \to B$ of schemes:

(a) If $\mathcal{F}$ is a quasi-coherent sheaf on X there is a natural map

$$\epsilon : \pi^*\pi_*\mathcal{F} \to \mathcal{F},$$

which, on the preimage of an affine set $U \subset B$ agrees with the map

$$\mathcal{O}_X(\pi^{-1}(U)) \otimes_{\mathcal{O}_Y(U)} H^0(\mathcal{F}|_U) \longrightarrow \mathcal{F}(\pi^{-1}(U))$$

sending each section to itself. This is well-defined because $\mathcal{F}(\pi^{-1}(U))$ is an $\mathcal{O}_X(\pi^{-1}(U))$-module.

(b) Given a ****sheaf $\mathcal{G}$ on B, there is a natural map

$$\eta : \mathcal{G} \to \pi_*\pi^*\mathcal{G},$$

which, on any open set $U \subset B$, agrees with the map

$$\begin{aligned}\mathcal{G}(U) = \mathcal{O}_B(U)\otimes_{\mathcal{G}(U)} &\xrightarrow{\pi^*\otimes 1} \mathcal{O}_X(\pi^{-1}(U) \otimes_{\mathcal{O}_B(U)} \mathcal{G}(U) \\ &= \pi^*(\mathcal{G})(\pi^{-1}(U) = (\pi_*\pi^*\mathcal{G})(U)\end{aligned}$$

Corollary 6.5. *Suppose $\pi : X \to B$ is a flat, projective morphism, and that all the fibers of π are reduced and connected.*

(a) $\pi_*\mathcal{O}_X = \mathcal{O}_B$.

(b) *If $\mathcal{L}, \mathcal{L}'$ are line bundles on X, then $\mathcal{L}|_{\pi^{-1}(b)} \cong \mathcal{L}'|_{\pi^{-1}(b)}$ for all $b \in B$ if and only if $\mathcal{L}$ and $\mathcal{L}'$ differ by tensoring with a line bundle pulled back from B.*

Remark. The result fails without flatness, for example in the case when π is the embedding of a proper closed subscheme of B.

Proof. To say that X is flat over B means that $\mathcal{O}_X$ is flat over B. Since flatness is a local property, this implies that any line bundle on X is flat over B, so we may apply Theorem 6.6 to line bundles on X.

(a): We will show that the natural map $\pi^* : \mathcal{O}_B \to \pi_*\mathcal{O}_X = \mathcal{O}_B$ is an isomorphism. Because X is flat, $1 \in \pi_*\mathcal{O}_X$ is not annihilated by any (local) section of $\mathcal{O}_B$. The map π^* takes $1 \in H^0(\mathcal{O}_B)$ to $1 \in H^0(\pi_*\mathcal{O}_X)$ so the map $\pi^* : \mathcal{O}_B \to \pi_*\mathcal{O}_X$ is injective.

To show surjectivity we use Theorem 6.6. Since $H^0\big((\mathcal{O}_X)|_{\pi^{-1}(b)}\big) = H^0(\mathcal{O}_{\pi^{-1}(b)})$ is 1-dimensional for every $b \in B$, the sheaf $\pi_*\mathcal{O}_X$ is a line bundle with fiber $H^0(\mathcal{O}_{\pi^{-1}(b)})$. Thus π^* is onto fiber by fiber. By Nakayama's Lemma, $\pi^* : \mathcal{O}_B \to \pi_*\mathcal{O}_X$ is surjective as desired.

(b): If $\mathcal{M}$ is a line bundle on B then $\big(\pi^*(\mathcal{M}) \otimes \mathcal{L}\big)|_{\pi^{-1}(b)} = \mathcal{M}_b \otimes \big((\mathcal{O}_X)|_{\pi^{-1}(b)}\big) \cong \big((\mathcal{O}_X)|_{\pi^{-1}(b)}\big)$.

For the converse we may replace $\mathcal{L}$ by $\mathcal{L}^{-1}\otimes\mathcal{L}'$, and it suffices to consider the case where $\mathcal{L}|_{\pi^{-1}(b)}$ is trivial for each $b \in B$. Our hypothesis implies that $H^0(\mathcal{L}|_{\pi^{-1}(b)})$ is 1-dimensional for every $b \in B$, so by Theorem 6.6, $\pi_*\mathcal{L}$ is a line bundle.

We will complete the proof by showing that the natural map $\epsilon : \pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is an isomorphism. Since both sheaves are line bundles, it suffices to show that ϵ is surjective, and for this we may, by Nakayama's Lemma, restrict to a fiber. By Theorem 6.6, the fiber of $\pi_*\mathcal{L}$ at a point b is $H^0(\mathcal{L}|_{\pi^{-1}(b)})$, so $(\pi^*\pi_*\mathcal{L})|_{\pi^{-1}(b)}$ is the trivial line bundle of rank 1, generated by any nonzero global section of $\mathcal{L}|_{\pi^{-1}(b)}$. Since the latter is also a trivial line bundle of rank 1, we are done. □

We will see some examples of direct images and higher direct images in situations where the dimension of $H^0(\mathcal{F}|_{\pi^{-1}(b)})$ does jump in Exercises 6.13 and 6.14

6.5 Cohomology and Base Change

In this section we will study how the i-th cohomology of a sheaf can vary as the sheaf varies in a family. In particular, we will give a basic criterion for the family of i-th cohomology spaces to "fit together" to form a vector bundle, or more generally to be the fibers of a coherent sheaf.

For example, suppose that $\pi : X \to B$ is a projective morphism, and that $\mathcal{F}$ is a sheaf on X, flat over B, which we view as a family of sheaves $\mathcal{F}|_b$ on the fibers $X_b := \pi^{-1}(b)$. Suppose for simplicity that B is connected. An obvious necessary condition for the cohomology groups $H^i(\mathcal{F}|_b)$ to be the fibers of a vector bundle $\mathcal{E}$ on B is that they all have the same dimension. We will show, under some commonly satisfied hypotheses, that this is enough, and that $\mathcal{E} = R^i\pi_*(\mathcal{F})$.

Theorem 6.6 (Cohomology and Base Change)**.** *Let $\pi : X \to B$ be a projective morphism of schemes, with B connected, and let $\mathcal{F}$ be a sheaf on X that is flat over B.*

(a) *For each i, the dimension $\dim_{\kappa(b)} H^i(\mathcal{F}|_{\pi^{-1}(b)})$ is an upper semicontinuous function of b (that is the set where it is $\geq m$ is closed for any m.) The alternating sum of these dimensions, which is the Euler characteristic $\chi(\mathcal{F}|_{\pi^{-1}(b)})$, is independent of $b \in B$.*

(b) *Suppose that B is reduced. If, for some i, $\dim_{\kappa(b)} H^i(\mathcal{F}|_{\pi^{-1}(b)})$ is a constant function of b then $R^i\pi_*(\mathcal{F})$ is a vector bundle and for every point $b \in B$ the comparison maps*

$$\varphi_b^i : R^i\pi_*(\mathcal{F})|_b \to H^i(\mathcal{F}|_{\pi^{-1}(b)})$$
$$\varphi_b^{i-1} : R^{i-1}\pi_*(\mathcal{F})|_b \to H^{i-1}(\mathcal{F}|_{\pi^{-1}(b)})$$

are isomorphisms. If B is quasi-projective, then it is enough to check the constancy at the closed points $b \in B$.

****For an exercise: On the blowup of $\mathbb{P}^2$ at a point, $\mathrm{H}^0\mathcal{O}(E)_b$ fails semi-cont and const of χ. I guess $\mathcal{O}(E) \oplus \mathcal{O}_p$, with p a point on E, is a case of constancy where part (b) fails. Can we do it with just a line bundle on the blowup?****

Corollary 6.7. *Suppose that $\pi : X \to B$ is a projective morphism to an integral scheme, and that $\mathcal{F}$ is a coherent sheaf on X (not necessarily flat over B). There is a dense open set U of points $b \in B$ such that $R^i\pi_*\mathcal{F}|_U$ is a vector bundle, and such that for $b \in U$ the fiber $(R^i\pi_*\mathcal{F}|_U)_b$ is equal to $H^i(\mathcal{F}|_b)$.*

Proof. There is an open set $U_1 \subset B$ over which $\mathcal{F}$ is flat, and within that set the set U of points $b \in U_1$ where $\dim_{\kappa(b)} H^i(\mathcal{F}|_b)$ takes on its smallest value is open by part (a) of the Theorem. The rest follows from part (b) of the Theorem. □

All these useful statements follow easily from the abstract-looking Proposition 6.8 and Corollary 6.10 below, as we shall soon see.

We digress to explain what Theorem6.6 has to do with "base change". Suppose we have a family of schemes parametrized by B—that is, a map $\pi : X \to B$—and a sheaf $\mathcal{F}$ on X. For any map $\alpha : B' \to B$, we can consider the pullback of the family $X \to B$ to B'—that is, the morphism

$$\pi' : X' = X \times_B B' \ \to \ B'$$

and the pullback sheaf

$$\mathcal{F}' = \alpha^*\mathcal{F}$$

on X'. In these circumstances we say that the new map $\pi' : X' \to B'$ and sheaf $\mathcal{F}'$ are obtained from the map $\pi : X \to B$ and sheaf $\mathcal{F}$ by *base change*, and ask: if we take the direct image $\pi'_*\mathcal{F}'$ of $\mathcal{F}'$ to B', do we get the pullback $\alpha^*\pi_*\mathcal{F}$ of $\pi_*\mathcal{F}$ to B'? In other words, does the formation of the direct image of $\mathcal{F}$ commute with base change? Of course, we can ask the analogous questions with respect to the formation of the higher direct images $R^i\pi_*\mathcal{F}$.

The question of when the maps φ_b^i are isomorphisms is the special case in which the map $\alpha : B' \to B$ is the inclusion of the point $B' = \{b\}$ in B. Thus, we sometimes refer to the property that the map φ_b^i are isomorphisms for all $b \in B$ as the "base change property" for $R^i\mathcal{F}$. The ideas we will develop can also be used to answer the more general question; see for example Mumford [2008], from which much of our treatment of Theorem6.6 is derived.

6.5.1 Tools From Commutative Algebra

The proof of Theorem6.6 requires two tools. First, a fundamental method for proving that a sheaf is a vector bundle (that is, is locally free):

Proposition 6.8. *A coherent sheaf $\mathcal{G}$ on a reduced scheme B is a vector bundle if and only the dimension of the $\kappa(b)$-vector space $\mathcal{G}|_b$ is the same for all points $b \in B$; if B is quasi-projective, then it even suffices to check this for closed points. These conditions are satisfied, in particular, if $\mathcal{G}$ has a resolution*

$$\mathcal{P}^\bullet : \quad \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow \mathcal{G} \longrightarrow 0$$

by vector bundles of finite rank that remains a resolution when tensored with $\kappa(b)$ for every $b \in B$.

Proof. If $\mathcal{G}$ is a vector bundle, then it has constant rank because B is connected, and this rank is the common dimension of $\mathcal{G}|_b$ over $\kappa(b)$.

To prove the converse, and also the last statement of the Proposition, we note that the problem is local, so we may assume that $B = \operatorname{Spec} A$, where A is a local ring with maximal ideal $\mathfrak{m}$ corresponding to the closed point $b_0 \in B$, and that $\mathcal{G} = \tilde{G}$, where G is a finitely generated A-module. By Nakayama's Lemma, a minimal set of generators of G corresponds to a map $f : F \to G$ from a free A-module F such that the induced map $(A/\mathfrak{m}) \otimes F \to (A/\mathfrak{m}) \otimes G$ is an isomorphism. In particular, the rank of the free module F is $\dim_{A/\mathfrak{m}}(A/(\mathfrak{m}) \otimes G = \dim_{\kappa(b)} \mathcal{G}|_b$.

Let $K = \operatorname{Ker} \varphi$, and let P be a minimal prime of A. Since A is reduced, A_P is a field. Localizing at P, we get an exact sequence of finite dimensional vector spaces

$$0 \to K_P \to F_P \to G_P \to 0,$$

whence $\operatorname{rank} F = \dim_{A_P} G_P + \dim_{A_P} K_P$. By hypothesis, $\dim_{A_P} G_P = \dim_{A/\mathfrak{m}} P/\mathfrak{m}P = \operatorname{rank} F$, so $K_P = 0$. Since A is reduced, the only associated primes of F are the minimal primes of A, and thus $K \subset F$ itself must be zero.

For the last statement, suppose that $\mathcal{G}$ has a resolution $\mathcal{P}^\bullet$ with the given property. Since B is the spectrum of the local ring A, we may identify $\mathcal{P}^\bullet$ with a free resolution of $\mathcal{G}$. Since a minimal free resolution is a summand of any resolution (Eisenbud [1995] Theorem 20.2) the minimal free resolution $\mathcal{P}'^\bullet$ of $\mathcal{G}$ has the same property. But after tensoring with the residue class field $\kappa(b_0)$ the differentials in $\mathcal{P}'^\bullet$ become zero. Since by hypothesis the resolution remains acyclic, we must have $P'^0 = \mathcal{G}$, so $\mathcal{G}$ is free. □

The second tool we need is a way of approximating a complex by a complex of free modules with good properties.

Proposition 6.9. *Let A be a noetherian ring, and let*

$$\mathcal{K}^\bullet : \cdots \xrightarrow{d} K^i \xrightarrow{d} K^{i+1} \xrightarrow{d} \cdots$$

be a complex of (not necessarily finitely generated) flat A-modules whose homology modules are finitely generated and such that $K^m = 0$ for $m \gg 0$. There is a complex of finitely generated free A-modules $\mathcal{P}^\bullet$ with $P^m = 0$ for $m \gg 0$, and a map of complexes $r : \mathcal{P}^\bullet \to \mathcal{K}^\bullet$ such that, for every A-module M, the map

$$r \otimes_A M : \mathcal{P}^\bullet \otimes_A M \longrightarrow \mathcal{K}^\bullet \otimes_A M$$

induces an isomorphism on homology.

Proof. We will construct a complex of finitely generated free modules $\mathcal{P}^\bullet$ with a map r to $\mathcal{K}^\bullet$ inducing an isomorphism on homology without the assumption of flatness; and then we will use the flatness hypothesis to show that any such $r : \mathcal{P}^\bullet \to \mathcal{K}^\bullet$ induces an isomorphism on homology after tensoring with the arbitrary module M.

We will construct the complex $\mathcal{P}^\bullet$ by a downward induction. Suppose that for some i a map of complexes

$$\begin{array}{ccccccc} K^{i+1} & \xrightarrow{d^{i+1}} & K^{i+2} & \xrightarrow{d^{i+2}} & \cdots \\ \uparrow{\scriptstyle r_{i+1}} & & \uparrow{\scriptstyle r_{i+2}} & & \\ P^{i+1} & \xrightarrow{e^{i+1}} & P^{i+2} & \xrightarrow{e^{i+2}} & \cdots \end{array}$$

inducing isomorphisms $\mathrm{H}^j(\mathcal{P}^\bullet) \to \mathrm{H}^j(\mathcal{K}^\bullet)$ for all $j \geq i+2$ has been constructed, with the additional property that the composite map $\operatorname{Ker} e^{i+1} \to \operatorname{Ker} d^{i+1} \to \mathrm{H}^{i+1}(\mathcal{K}^\bullet)$ is surjective.

If i is sufficiently large that that $K^m = 0$ for all $m \geq i+1$, then the choice $P^m = 0$ and $r_m = 0$ for $m \geq i+1$ satisfies these conditions, giving a base for the induction.

To make the inductive step from $i+1$ to i, we choose P^i to be the direct sum of two projective modules, $P^i = P_1^i \oplus P_2^i$ where P_1^i is chosen to map onto the kernel of the composite $\operatorname{Ker} e^{i+1} \to \operatorname{Ker} d^{i+1} \to \mathrm{H}^{i+1}(\mathcal{K}^\bullet)$, and P_2^i is chosen to map onto $\mathrm{H}^i(\mathcal{K}^\bullet)$. We define the differential e^i to be the given map on P_1^i and zero on P_2^i. Also, we define r_i on P_2^i by lifting the chosen map $P_2^i \to \mathrm{H}^i(\mathcal{K}^\bullet)$ to a map $P_2^i \to \operatorname{Ker} d^i$ and composing with the inclusion $\operatorname{Ker} d^i \subset K^i$. On the other hand, since r_{i+1} carries the image of P_1^i to the kernel of the map $\operatorname{Ker} d^{i+1} \to \mathrm{H}^{i+1}(\mathcal{K}^\bullet)$, which is by definition $\operatorname{Im} d^i$, we may define r_i on P_1^i to be the lifting of this map $P_1^i \to \operatorname{Im} d^i$ to a map $P_1^i \to K^i$.

This gives a map of complexes

$$\begin{array}{ccccccccc}
K^i & & \xrightarrow{d^i} & & K^{i+1} & \xrightarrow{d^{i+1}} & K^{i+2} & \xrightarrow{d^{i+2}} & \cdots \\
\uparrow{\scriptstyle r_i} & & & & \uparrow{\scriptstyle r_{i+1}} & & \uparrow{\scriptstyle r_{i+2}} & & \\
P^i = P_1^i \oplus P_2^i & & \xrightarrow{e^i} & & P^{i+1} & \xrightarrow{e^{i+1}} & P^{i+2} & \xrightarrow{e^{i+2}} & \cdots
\end{array}$$

It is clear from the construction that the r_i induce isomorphisms $\mathrm{H}^j(\mathcal{P}^\bullet) \to \mathrm{H}^j(\mathcal{K}^\bullet)$ for all $j \geq i+1$ and that the composite map $\operatorname{Ker} e^{i+1} \to \operatorname{Ker} d^{i+1} \to \mathrm{H}^{i+1}(\mathcal{K}^\bullet)$ is surjective, so the induction is complete.

We now use the hypothesis that the K^i are flat, and suppose that we have proven that r_j induces an isomorphism $\mathrm{H}^j(\mathcal{P}^\bullet \otimes M) \to \mathrm{H}^j(\mathcal{K}^\bullet \otimes M)$ for every $j > i$ and for every module M. This is trivial in the range where K^j and P^j are both zero, so again we can do a downward induction.

Choose a surjection $F \to M$ from a free A-module, and let L be the kernel, so that

$$0 \to L \to F \to M \to 0$$

is a short exact sequence. Since all the K^i and the P^i are flat, we get short exact sequences of complexes by tensoring with $\mathcal{P}^\bullet$ and $\mathcal{K}^\bullet$, from which we get two long exact sequences by applying the higher direct image functors. and the comparison map $r : \mathcal{P}^\bullet \to \mathcal{K}^\bullet$ induces a commutative diagram

$$\begin{array}{ccccccccc}
H^i(\mathcal{K}^\bullet \otimes L) & \to & H^i(\mathcal{K}^\bullet \otimes F) & \to & H^i(\mathcal{K}^\bullet \otimes M) & \to & H^{i+1}(\mathcal{K}^\bullet \otimes L) & \to & H^{i+1}(\mathcal{K}^\bullet \otimes F) \\
\uparrow{\scriptstyle r_i \otimes L} & & \uparrow{\scriptstyle r_i \otimes F} \cong & & \uparrow{\scriptstyle r_i \otimes M} & & \uparrow{\scriptstyle r_{i+1} \otimes L} \cong & & \uparrow{\scriptstyle r_{i+1} \otimes F} \cong \\
H^i(\mathcal{P}^\bullet \otimes L) & \to & H^i(\mathcal{P}^\bullet \otimes F) & \to & H^i(\mathcal{P}^\bullet \otimes M) & \to & H^{i+1}(\mathcal{P}^\bullet \otimes L) & \to & H^{i+1}(\mathcal{P}^\bullet \otimes F)
\end{array}$$

where, for any module N, we have written $r_i \otimes N$ to denote the map $H^i(\mathcal{P}^\bullet \otimes N) \to H^i(\mathcal{K}^\bullet \otimes N)$ induced by r_i. The maps marked "$\cong$" are isomorphisms: $r_i \otimes F$ and $r_{i+1} \otimes F$ are isomorphisms because F is free, while $r_{i+1} \otimes L$ is an isomorphism by induction. A diagram chase (sometimes called the "five-lemma") immediately shows that the map $r_i \otimes M$ is a surjection. Since the module M was arbitrary, $r_i \otimes L$ is a surjection as well. Using this information, a second diagram chase shows that $r_i \otimes M$ is injective, completing the induction. □

We will use Proposition6.9 via the following Corollary. To simplify the notation, we will identify quasi-coherent sheaves over an affine scheme $B = \operatorname{Spec} A$ with their modules of global sections.

Corollary 6.10. *Let $\pi : X \to B$ be a projective morphism to an affine scheme $B = \operatorname{Spec} A$, and let $\mathcal{F}$ be a sheaf on X that is flat over B. There is a complex*

$$\mathcal{P}^\bullet : \quad \cdots \longrightarrow P^0 \longrightarrow \cdots \longrightarrow P^n \longrightarrow 0$$

of finitely generated projective A-modules such that

(a) $R^i\pi_*(\mathcal{F}) \cong \mathrm{H}^i(\mathcal{P}^\bullet)$ *for all* i.

(b) *For every point* $b \in B$ *there is an isomorphism*

$$H^i(\mathcal{F}|_{\pi^{-1}(b)}) \cong \mathrm{H}^i(\kappa(b) \otimes_A \mathcal{P}^\bullet).$$

Moreover, if B *is reduced, then we may choose* $\mathcal{P}^\bullet$ *so that* $P^i = 0$ *for* $i < 0$.

Proof. Since π is projective we may write $X \subset \mathbb{P} := \mathbb{P}^n_A$ for some n, and we let $U_i,\ i = 0, \ldots, n$ be the standard open covering of $\mathbb{P}$ as in Section17.4. Let $\mathcal{C}^\bullet$ be the Čech complex defined there. Since $\mathcal{F}$ is flat and $(\mathcal{F} \otimes \mathcal{O}_{U_i \cap U_j \cap \cdots})_0$ is the module corresponding to the restriction of $\mathcal{F}$ to the affine open set $U_i \cap U_j \cap \cdots$, the modules of the complex $(\mathcal{F} \otimes \mathcal{C}^\bullet)_0$ are flat. By Theorem17.10 the homology of this complex is finitely generated, so we may apply Proposition6.9 and obtain a complex $\mathcal{P}^\bullet$ whose i-th homology is $R^i\pi_*\mathcal{F}$. Taking $M = \kappa(b)$ in the Proposition, we see that $\mathcal{P}^\bullet$ has the second required property as well.

Finally, to show that we may choose $\mathcal{P}^\bullet$ with $P^i = 0$ for $i < 0$, note that for any choice of $\mathcal{P}^\bullet$ satisfying the Proposition the homology $\mathrm{H}^i(\mathcal{P}^\bullet) = 0$ for $i < 0$. The last statement of Proposition **??** shows that $P'^0 := \mathrm{coker}(P^{-1} \to P^0$ is projective. The map r_0 induces a map $P'^0 \to \mathrm{coker}(K^{-1} \to K^0)$, and since P'^0 is projective we may lift this to a new map $r'_0 : P'^0 \to K^0$. It follows from the construction that

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & K^{-1} & \longrightarrow & K^{-1} & \longrightarrow & K^0 & \longrightarrow & K^1 & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle r'_0} & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P'^0 & \longrightarrow & P^1 & \longrightarrow & \cdots
\end{array}$$

again induces an isomorphism on homology. □

6.5.2 Proof Of the Theorem On Cohomology and Base Change

Proof of Theorem6.6. It suffices to check each of the parts in the case where B is affine, say $B = \mathrm{Spec}\, A$. Let $\mathcal{P}^\bullet$ be a complex of projective A-modules with the properties given in Corollary 6.10. For each $b \in B$ we get a complex of vector spaces by taking the fiber of $\mathcal{P}^\bullet$ at b. Since B is connected, the rank of each vector space $P^i|_b$ is independent of b.

(a): we may begin by restricting $\mathcal{F}$ to the preimage of B_{red}, and thus assume that B is reduced. In this case Corollary 6.10 gives us a finite complex $\mathcal{P}^\bullet$. The ranks of the differentials in the sequence

$$P^{i-1}_b \xrightarrow{d^{i-1}_b} P^i_b \xrightarrow{d^i_b} P^{i+1}_b$$

are lower semicontinuous functions of b, and the ranks of the homology vector space

$$\dim_{\kappa(b)} \mathrm{H}^i(\mathcal{P}^\bullet|_b) = \operatorname{rank} P_b^i - \operatorname{rank} d_b^{i-1} - \operatorname{rank} d_b^i$$

are thus upper semicontinuous.

The Euler characteristic $\chi(\mathcal{F}_{\pi^{-1}(b)})$ is equal to the Euler characteristic $\sum_i (-1)^i \dim_{\kappa(b)} \mathrm{H}^i(\mathcal{P}^\bullet|_b)$, and this is constant because it is equal to the alternating sum of the ranks of the modules $P^i|_b$.

(b): Suppose that the rank of the differential d_b^i is constant. Since taking fibers commutes with taking images, the fibers of the sheaf $\operatorname{Im} d^i$ have constant rank. By Proposition 6.8, the module $\operatorname{Im} d^i$ is projective, and the short exact sequence

$$0 \to \operatorname{Ker} d^i \to P^i \to \operatorname{Im} d^i \to 0$$

splits. We deduce that $(\operatorname{Ker} d^i)_b = \operatorname{Ker}(d_b^i)$, and it follows that the map

$$R^i\pi_*(\mathcal{F})_b = \mathrm{H}^i(\mathcal{P}^\bullet)_b \xrightarrow{\varphi_b^i} \mathrm{H}^i(\mathcal{P}^\bullet{}_b) = H^i(\mathcal{F}|_{\pi^{-1}(b)})$$

is an isomorphism.

Now suppose that $\dim_{\kappa(b)} H^i(\mathcal{F}_{\pi^{-1}(b)})$ is constant. Since the ranks of the differentials d_b^{i-1} and d_b^i are lower semicontinuous, they must both constant as well, and we see that both displayed maps in the statement of Part (b) of the Theorem are isomorphisms. Of course then $\dim_{\kappa(b)} R^i\pi_*(\mathcal{F})_b$ is constant, and it follows from Proposition 6.8 that $R^i\pi_*(\mathcal{F})$ is a vector bundle. The Proposition also assures us that it suffices to check the constancy of the rank at closed points in the quasi-projective case. □

Remark. Suppose that $\pi : X \to B$ is a projective morphism and $\mathcal{F}$ is a coherent sheaf on X, flat over B, as in Theorem 6.6, and suppose that $b \in B$ is a point at which $\dim_{\kappa(b)} H^i(\mathcal{F}_b)$ "jumps"—that is, is larger than it is for some points in any open neighborhood of b. From the constancy of the Euler characteristic $\chi(\mathcal{F}_{\pi^{-1}(b)})$ it follows that some $\dim_{\kappa(b)} H^j(\mathcal{F}_b)$ with $j \neq i \bmod 2$ must jump too. But the proof above gives a tiny bit more: since the rank of d_b^i or of d_b^{i-1} must have gone down, either $\dim_{\kappa(b)} H^{i+1}(\mathcal{F}_b)$ or $\dim_{\kappa(b)} H^{i-1}(\mathcal{F}_b)$ must jump at b.

6.6 Examples

6.6.1 Applications to Line Bundles

We can now complete some of the results from Corollary 6.5.

Corollary 6.11. *Let $\mathcal{E}$ be a vector bundle of rank $r+1$ on a scheme B, and let $\mathbb{P} = \mathbb{P}(\mathcal{E})$ is the corresponding projective bundle.*

(a) $\pi^* : \operatorname{Pic} B \to \operatorname{Pic} \mathbb{P}$ *is a monomorphism of groups, and*

$$\operatorname{Pic} \mathbb{P} = \pi^*(\operatorname{Pic} B)) \oplus \mathbb{Z} \cdot [\mathcal{O}_{\mathbb{P}}(1)],$$

where $[\mathcal{O}_{\mathbb{P}}(1)]$ denotes the isomorphism class of $\mathcal{O}_{\mathbb{P}}(1)$.

(b) $\pi_*(\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}}(d)) = \mathcal{L} \otimes \operatorname{Sym}^d(\mathcal{E}^*)$, *and* $R^i\pi_*(\mathcal{L} \otimes pi_*\mathcal{O}_{\mathbb{P}}(d)) = 0$ *for* $0 < i < r-1$.

Remark. Part (b) of the Corollary is often used with *relative duality*: setting

$$\omega_{\mathbb{P}/B} = \wedge^r \mathcal{E}(-r-1)$$

we have $R^r\pi_*(\omega_{\mathbb{P}/B}) = \mathcal{O}_B$, and more generally

$$R^r\pi_*(\mathcal{M}) = \operatorname{Hom}(\pi_*(\omega \otimes \mathcal{M}^{-1}), \mathcal{O}_B)$$

for any line bundle $\mathcal{M}$ on $\mathbb{P}$. We will not need this fact; but see **** for a proof. ****The first statement is easy from the Koszul complex; the second is a special case. Should we state this more generally? Are we really not going to use it, even for the jumping lines? Should we prove this special case?****

Proof. (a): It is enough to show that $\mathcal{L}$ and d can be recovered from the line bundle $\mathcal{L}(\delta) = \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}}(d)$. But d is the degree of the restriction of $\mathcal{L}(d)$ to any fiber. Moreover, we have $\pi_*(\mathcal{O}_{\mathbb{P}}) = \mathcal{O}_B$ by Corollary 6.5, so $\pi_*(\mathcal{L}(d) \otimes \mathcal{O}_{\mathbb{P}}(-d) = \mathcal{L} \otimes \pi_*(\mathcal{O}_{\mathbb{P}}) = \mathcal{L}$.

(b): By part (c) of 17.9 it suffices to treat the case where $\mathcal{L}$ is trivial. The map $\pi^*\mathcal{E}^* \to \mathcal{O}_{\mathbb{P}}(1)$ induces a map $\pi^* \operatorname{Sym}^d(\mathcal{E}^*) = \operatorname{Sym}^d(\pi^*(\mathcal{E}^*)) \to \mathcal{O}_{\mathbb{P}}(d)$ that becomes an isomorphism when restricted to any fiber (this is the statement that on $\mathbb{P}^r_K$ we have $H^0(\mathcal{O}_{\mathbb{P}^r_K}(d)) = Sym^d(K^{r+1})$.) By Theorem6.6, the direct images of these two sheaves are vector bundles of the same rank on B, and the induced map between them is an isomorphism on fibers, so it is an isomorphism. □

6.6.2 Double covers

By a *double cover* of a variety B we mean a finite flat map $\pi : X \to B$ from a reduced scheme X. We can make this description more concrete as follows: the hypotheses imply that if U is an affine open subset of X then $\mathcal{O}_X(\pi^{-1}(U)) = \pi_*(\mathcal{O}_X)(U)$ is a free module of rank 2 over $\mathcal{O}_B(U)$. Since the section 1 is nonzero locally everywhere we see that $\pi_*(\mathcal{O}_X)(U)$ is generated as a module, and thus as a ring, by 1 together with one more element z. It follows that $X|_U \subset U \times \mathbb{A}^1$, and is defined there by a monic,

quadratic equation $z^2 + py + q = 0$ with $p, q \in \mathcal{O}_X(U)$. If the characteristic is not 2, then changing variable to $y = z + p/2$ we may rewrite the equation in the form $y^2 - f = 0$ with $f \in \mathcal{O}_X(U)$. Since we have assumed that X is reduced, $f \neq 0$. Conversely, any scheme that given, locally on each affine subset of B by an equation of the form $y^2 - f = 0$ with $0 \neq f \in \mathcal{O}_B(U)$

For simplicity, we assume from now on that the characteristic is not 2.

If $p \in X$, and the cover restricts on an affine neighborhood U of p to one given by an equation $y^2 - f = 0$, then the fiber $\pi^{-1}(p)$ consists of two distinct points when $f(p) \neq 0$ and one double point when $f(p) = 0$. Thus the set of points p where f vanishes is determined by the map π. But in fact, even the divisor of zeros of f is determined. To see this, we compute the pushforward to U of the relative cotangent sheaf (the sheaf of differentials) $\Omega_{\pi^{-1}(U)/U}$. To simplify the notation we set $V = \pi^{-1}(U)$.

From embedding $V \subset B \times \mathbb{A}^1$ we see that $\Omega_{V/U} = \mathcal{O}_V dy/2y$. Pushing this forward, we get

$$\pi_*\Omega_{V/U} = \pi_*\mathcal{O}_V dy/2y = \operatorname{coker}\left(\pi_*\mathcal{O}_V \xrightarrow{\pi_*(2y)} \pi_*\mathcal{O}_V\right),$$

where $\pi_*(2y)$ denotes the pushforward of the endomorphism of $\mathcal{O}_V$ that is multiplication by $2y$. Now $\mathcal{O}_V$ is a free $\mathcal{O}_U$-module on the generators $1, y$, and writing the endomorphism in this basis we get

$$\pi_*\Omega_{V/U} \cong \operatorname{coker}\left(\mathcal{O}_U \oplus \mathcal{O}_U y \xrightarrow{\begin{pmatrix} 0 & 2f \\ 2 & 0 \end{pmatrix}} \mathcal{O}_U \oplus \mathcal{O}_U y\right) \cong \mathcal{O}_U/(2f).$$

Taking into account the fact that 2 is a unit, we see that the ideal (f) is determined by the map π, as claimed. Since the construction we have just made commutes with restriction to smaller affine open sets, we get a well defined divisor on all of B, called the *branch divisor* of π. We will denote it Δ. (In general, for a finite flat map $X \to B$ of varieties one defines the branch divisor to be the 0-th Fitting ideal of $\pi_*(\Omega_{X/B}$; see Eisenbud [1995] Section ****.)

Because the element $1 \in \pi_*\mathcal{O}_X$ is nonzero locally everywhere, and $\pi_*\mathcal{O}_X$ is locally free of rank 2, the quotient

$$\mathcal{L} := \frac{\pi_*\mathcal{O}_X}{\mathcal{O}_X \cdot 1}$$

is a line bundle locally generated by the element we called y above. But in fact, $\pi_*\mathcal{O}_X$ decomposes as a direct sum. Indeed any degree 2 field extension in characteristic not 2 is Galois, and restricting to an open set V as above, we see that the Galois group is generated by the map $\sigma : y \mapsto -y$, which fixes $\mathcal{O}_X$ inside its quotient field, and has ring of invariants equal to $\mathcal{O}_U$.

Thus

$$\frac{1}{2}(1+\sigma) : \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_X$$

is an $\mathcal{O}_B$-linear retraction of $\pi_*\mathcal{O}_X$ onto $\mathcal{O}_B \cdot 1$, proving that

$$\pi_*\mathcal{O}_X = \mathcal{O}_B \oplus \mathcal{L}.$$

Since $\mathcal{L}$ is locally generated by a section y whose square is a defining equation of the divisor Δ, we get

$$L^2 \;\cong\; \mathcal{O}_B(-\Delta),$$

It turns out that this data is enough to recover the double cover:

Proposition 6.12. *Let B be any variety over a field of characteristic $\neq 2$. There is a one-to-one correspondence between double covers $\pi : X \to B$ of B, and pairs $(\mathcal{L}, \Delta)$ consisting of a line bundle $\mathcal{L}$ on B and a divisor $\Delta \in |L^{-2}|$.*

Proof. We have seen the correspondence in one direction: the double cover $\pi : X \to B$ determines both the branch divisor Δ, and $\mathcal{L} = \pi_*\mathcal{O}_X/\mathcal{O}_B \cdot 1$.

Conversely, given $(\mathcal{L}, \Delta)$ as above, the divisor Δ is the vanishing locus of a section $\alpha \in \mathcal{L}^{-2}$. The section α defines a nonzero homomorphism $\tilde{\sigma} : \mathcal{L}^2 \to \mathcal{O}_B$. Using α we make the locally free sheaf

$$\mathcal{E} \;=\; \mathcal{O}_B \oplus \mathcal{L}$$

into a sheaf of $\mathcal{O}_B$-algebras: multiplication of the summand $\mathcal{O}_B$ by itself and by $\mathcal{L}$ is the obvious one, while products of sections of the summand $\mathcal{L}$ map to $\mathcal{O}_B$ via the map $\tilde{\sigma}$. We define the corresponding double cover to be $X = \operatorname{Spec}\mathcal{E}$. To see that these descriptions are inverse, one computes what they mean in terms of the local description of the cover over an affine open set $U \subset B$. □

In this situation the base change maps $\varphi_b : (\pi_*\mathcal{O}_X)|_b \to H^0(\mathcal{O}_X|_{X_b})$ are isomorphisms for every b. This is clear in case $b \notin \Delta$ is not in the branch divisor, so that the fiber X_b consists of two reduced points; and only slightly less clear when $b \in \Delta$, in which case X_b consists of a single point of degree 2.

****revised to here Jan 31, 2011****.

6.6.3 Projective bundles

****haven't we done this example in the projective bundle chapter? How could we not have??**** Our next example will be a fairly common one: we will take $\pi : X \to B$ the projectivization of a vector bundle E of rank

n on B, and $\mathcal{F}$ a line bundle on X. Since every line bundle on $\mathbb{P}E$ is of the form $\pi^*L \otimes \mathcal{O}_{\mathbb{P}E}(k)$ for some $L \in \mathrm{Pic}(B)$ and some integer k, in view of the push-pull formula c above we might as well stick to the case $\mathcal{F} = \mathcal{O}_{\mathbb{P}E}(k)$ for some k.

The first example, $k = 0$, is relatively uninteresting: since the fibers of $X \to B$ are projective spaces, any regular function on the inverse image X_U of an open set $U \subset B$ is constant along the fibers and hence the pullback of a function in U; thus $\pi_*\mathcal{O}_X = \mathcal{O}_B$. Similarly, since the preimages X_U of small enough open sets $U \subset B$ are products $U \times \mathbb{P}^{n-1}$, their structures sheaves will have no higher cohomology; thus $R^i\pi_*\mathcal{O}_X = 0$ for all $i > 0$.

The case $n = 1$—that is, $\mathcal{F} = \mathcal{O}_{\mathbb{P}E}(1)$—has much more going on. To begin with, recall the definition of the bundle $\mathcal{O}_{\mathbb{P}E}(1)$. Briefly, the points of the variety $X = \mathbb{P}E$ are pairs (b, ξ), with $b \in B$ and $\xi \subset E_b$ a one-dimensional subspace of the fiber E_b of E at b. By definition, the fiber of the pullback bundle π^*E at a point $(b, \xi) \in \mathbb{P}E$ is simply E_b, and so we can define a rank 1 subbundle T of π^*E by specifying

$$T_{(b,\xi)} = \xi \subset E_b.$$

The bundle $\mathcal{O}_{\mathbb{P}E}(1)$ is defined to be the dual of the bundle T; that is, the bundle on $\mathbb{P}E$ whose fiber at (b, ξ) is simply the space of linear forms on the vector space ξ.

Now suppose that $U \subset B$ is any open set, and σ a section of the dual bundle E^* over U. The section σ gives us, for each $b \in U$, a linear form $\sigma(b)$ on the fiber E_b of E at b, and hence by restriction a linear form on each one-dimensional subspace of E_b. In other words, a section of E^* over U gives a section $\tilde{\sigma}$ of $\mathcal{O}_{\mathbb{P}E}(1)$ over X_U by setting

$$\tilde{\sigma}(b, \xi) = \sigma(b)|_\xi.$$

We thus have a map

$$E^* \to \pi_*\mathcal{O}_{\mathbb{P}E}(1)$$

and we claim that this map is in fact an isomorphism.

6.6.4 Ideals of points in projective space

We promised an example where the base change maps were not necessarily isomorphisms; here it is. To start with, let p and $q \in \mathbb{P}^2$ be any two points in the plane, and $L \subset \mathbb{P}^2$ a line not containing either. We let $X = L \times \mathbb{P}^2$ be the product, and let $Z \subset X$ be the union of the three curves $L \times \{p\}$, $L \times \{q\}$ and $L \times L$. We let $\pi : X \to L$ and $\alpha : X \to \mathbb{P}^2$ be the projections, and consider the sheaf

$$\mathcal{F} = \alpha^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{I}_Z.$$

****this is also in the discussion in Chapter 4****

6.7 Applications I: Bundles revisited

Virtually all of the bundles we've introduced and analyzed in the preceding chapters are defined as direct images, and consequently we can use the Grothendieck-Riemann-Roch formula to calculate their Chern classes. In this first section of applications we'll give some examples of this. One warning before we get started: while it's nice to have a systematic approach to these calculations, in practice the sort of ad-hoc methods we used in earlier chapters are typically easier and faster than hauling out the GRR. This is in fact a fairly general phenomenon: 95% of the time that we can actually use GRR to determine the Chern classes of a bundle, there are simpler alternatives. We will see in the following two sections, however, examples of applications of GRR where there really is no simpler alternative.

6.7.1 Symmetric powers of tautological bundles

Let's start with one of the earliest example of a vector bundle we introduced. Recall that in Chapter **, where we wanted to find the number of lines on a smooth cubic surface $S \subset \mathbb{P}^3$, we introduced a vector bundle E on the Grassmannian $G = \mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$; informally, we described E by saying that for each line $L \subset \mathbb{P}^3$, the fiber of E at the point $[L] \in G$ was the vector space of homogeneous cubic polynomials on L: that is,

$$E_{[L]} = H^0(\mathcal{O}_L(3)).$$

Back then, we calculated the Chern classes of E by realizing it as the third symmetric power of the dual of the universal subbundle on G. But—as with virtually every vector bundle we introduced, and whose Chern classes we calculated, E may be realized as a direct image, and its Chern classes calculated from the Grothendieck-Riemann-Roch formula.

To do this, we introduce the universal family of lines over G, that is, the incidence correspondence

$$\Phi = \{(L, p) : p \in L\} \subset G \times \mathbb{P}^3;$$

we'll let $\alpha : \Phi \to G$ and $\beta : \Phi \to \mathbb{P}^3$ be the projection maps. Now, we want a vector bundle on G whose fiber $E_{[L]}$ is the space of global sections of $\mathcal{O}_L(3)$; this suggests that we find a line bundle on Φ whose restriction to each fiber L of α is $\mathcal{O}_L(3)$, and take its direct image. The first is easy: we simply take the line bundle $\mathcal{O}_{\mathbb{P}^3}(3)$ on $\mathbb{P}^3$, pull it back to the product $G \times \mathbb{P}^3$, and restrict to Φ—in other words, the line bundle $\mathcal{L} = \beta^* \mathcal{O}_{\mathbb{P}^3}(3)$.

What happens when we take the direct image of $\mathcal{L}$? Well, things are about as nice as they're ever going to get. First of all, $\mathcal{L}$ is flat over G: the projection $\alpha : \Phi \to G$ is flat, and $\mathcal{L}$ is a line bundle of Φ. Moreover, all the higher cohomology of the restriction $\mathcal{O}_L(3)$ of $\mathcal{L}$ to the fiber $\Phi_{[L]} = \alpha^{-1}([L]) = L$ vanishes, and the dimension of $H^0(\mathcal{L}|_{\Phi_{[L]}})$ is constant. Thus the direct image $\alpha_*\mathcal{L}$ is locally free, with fiber $H^0(\mathcal{O}_L(3))$ at $[L]$. In short, we can take

$$E = \alpha_*\mathcal{L} = \alpha_*\big(\beta^*\mathcal{O}_{\mathbb{P}^3}(3)\big).$$

Moreover, because of the vanishing of the higher cohomology of $\mathcal{L}$ on the fiber of α, Grothendieck-Riemann-Roch gives us a complete description of the Chern class of E: we have

$$\mathrm{Ch}(E) = \alpha_*\big(\mathrm{Ch}(\mathcal{L}) \cdot \mathrm{Td}(T^v_{\Phi/G})\big).$$

To evaluate this explicitly, we have to

(a) describe the Chow rings $A^*(G)$ and $A^*(\Phi)$, and the Gysin map $\alpha_* : A^*(\Phi) \to A^*(G)$;

(b) calculate the Chern classes of $\mathcal{L}$ and the relative tangent bundle $T^v_{\Phi/G}$;

(c) convert these to the Chern character and Todd class, respectively;

(d) take the Gysin image of their product, to arrive at $\mathrm{Ch}(E)$; and finally

(e) convert this back into the Chern classes of E.

For the first, we've seen how to do this in Chapters ** and **: $\Phi = \mathbb{P}S$ is the projectivization of the universal subbundle on G, so that we have

$$A^*(\Phi) = A^*(G)[\zeta]/(\zeta^2 - \sigma_1\zeta + \sigma_{1,1})$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}S}(1))$. As for the Gysin map $\alpha_* : A^*(\Phi) \to A^*(G)$, we have $\alpha_*(\zeta) = [G]$; the Gysin image of higher powers of ζ may be evaluated via applications of the relation $\zeta^2 = \sigma_1\zeta - \sigma_{1,1}$ satisfied by ζ and the image of any product of a power of ζ with the pullback of a class from G by the push-pull formula. Explicitly, we have

$$\begin{aligned}\alpha_*(\zeta^2) &= \alpha_*(\sigma_1\zeta - \sigma_{1,1})\\ &= \sigma_1,\end{aligned}$$

$$\begin{aligned}\alpha_*(\zeta^3) &= \alpha_*(\sigma_1\zeta^2 - \sigma_{1,1}\zeta)\\ &= \sigma_1^2 - \sigma_{1,1}\\ &= \sigma_2,\end{aligned}$$

and

$$\begin{aligned}\alpha_*(\zeta^4) &= \alpha_*(\sigma_1\zeta^3 - \sigma_{1,1}\zeta^2)\\ &= \sigma_1\sigma_2 - \sigma_{1.1}\sigma_1\\ &= 0.\end{aligned}$$

(The last one is just a check: we know that $\zeta^4 = 0$ in $A^*(\Phi)$, since ζ is the pullback of the hyperplane class on $\mathbb{P}^3$.)

One other remark we should make is that, since the fiber of the line bundle $\mathcal{O}_{\mathbb{P}S}(1)$ at a point $(L, p) \in \Phi$ is just the dual of the one-dimensional vector subspace of $\mathbb{C}^4$ corresponding to p, we also have

$$\zeta = c_1(\mathcal{O}_{\mathbb{P}S}(1)) = \beta^* c_1(\mathcal{O}_{\mathbb{P}^3}(1).$$

In particular, it follows that

$$c_1(\mathcal{L}) = 3\zeta$$

and so

$$\mathrm{Ch}(\mathcal{L}) = 1 + 3\zeta + \frac{9}{2}\zeta^2 + \frac{27}{6}\zeta^3$$

(again, higher powers of ζ must vanish.

As for the relative tangent bundle of Φ over G, we will see how to find its Chern classes in Chapter 7: if we denote by T the tautological subbundle on $\Phi = \mathbb{P}S$ (the letter "S" is taken), and by Q the tautological quotiuent bundle, we have

$$T^v_{\Phi/G} = T^* \otimes Q.$$

From the exact sequence

$$0 \to T \to \alpha^* S \to Q \to 0,$$

moreover, we see that

$$c_1(Q) = c_1(\alpha^* S) - c_1(T) = -\sigma_1 + \zeta$$

and hence

$$c_1(T^v_{\Phi/G}) = c_1(T^* \otimes Q) = \zeta + c_1(Q) = -\sigma_1 + 2\zeta.$$

Plugging this into the formula for the Todd class, we have

$$\mathrm{Td}(T^v_{\Phi/G}) = 1 + \frac{2\zeta - \sigma_1}{2} + \frac{(2\zeta - \sigma_1)^2}{12} - \frac{(2\zeta - \sigma_1)^4}{720}.$$

Taking the product, we calculate

$$\begin{aligned}\mathrm{Ch}(\mathcal{L})\,\mathrm{Td}(T^v_{\Phi/G}) = 1 &+ \frac{8\zeta - \sigma_1}{2} + \frac{1}{12}\big(94\zeta^2 - 22\sigma_1\zeta + \sigma_1^2\big)\\ &+ \frac{1}{12}\big(120\zeta^3 - 39\sigma_1\zeta^2 + 3\sigma_1^2\zeta\big)\\ &+ \frac{1}{720}\big(-2668\sigma_1\zeta^3 + 246\sigma_1^2\zeta^2 + 8\sigma_1^3\zeta - \sigma_1^4\big)\\ &+ \frac{1}{720}\big(198\sigma_1^2\zeta^3 + 24\sigma_1^3\zeta^2 - 3\sigma_1^4\zeta\big)\end{aligned}$$

and applying the Gysin map we find that, by Grothendieck-Riemann-Roch,

$$\mathrm{Ch}(E) = 4 + 6\sigma_1 + (7\sigma_2 - 3\sigma_{1,1}) - 3\sigma_{2,1} + \frac{1}{3}[pt].$$

It remains to convert this to the Chern class of E. Once more, this can be messy: we have

$$c_1(E) = \mathrm{Ch}_1(E) = 6\sigma_1$$

and

$$\begin{aligned} c_2(E) &= \frac{\mathrm{Ch}_1(E)^2}{2} - \mathrm{Ch}_2(E) \\ &= 18\sigma_1^2 - (7\sigma_2 - 3\sigma_{1,1}) \\ &= 11\sigma_2 + 21\sigma_{1,1}. \end{aligned}$$

Similarly,

$$\begin{aligned} c_3(E) &= \frac{\mathrm{Ch}_1(E)^3}{6} - \mathrm{Ch}_1(E)\,\mathrm{Ch}_2(E) + 2\,\mathrm{Ch}_3(E) \\ &= 36\sigma_1^2 - 6\sigma_1(7\sigma_2 - 3\sigma_{1,1}) - 6\sigma_{2,1} \\ &= 72\sigma_{2,1} - 24\sigma_{2,1} - 6\sigma_{2,1} \\ &= 42\sigma_{2,1} \end{aligned}$$

and finally—the payoff, at last!—

$$c_4(E) =$$

6.8 Exercises

Exercise 6.13. Let E be an elliptic curve, $p \in E$ any point, and $\pi : E \times E \to E$ projection on the first factor. Let $\Delta \subset E \times E$ be the diagonal, and $D = E \times \{p\}$; let $\mathcal{F}$ be the invertible sheaf

$$\mathcal{F} = \mathcal{O}_{E\times E}(\Delta - D).$$

Show that $\pi_*\mathcal{F} \otimes \kappa_p = 0$, even though $H^0(\mathcal{F}|_{\pi^{-1}(p)}) \neq 0$.

Exercise 6.14. Let $L \in \mathbb{P}^2$ be a line, and let $q, r \in \mathbb{P}^2$ be two points not lying on L. Let $X = L \times \mathbb{P}^2$, and $\pi : X \to L$ and $\alpha : X \to \mathbb{P}^2$ the two projections; let $\Gamma = \Gamma_1 \cup \Gamma_2 \subset X$ where

$$\Gamma_1 = L \times \{q, r\} \quad \text{and} \quad \Gamma_2 = \{(p, p) : p \in L\}$$

Let $\mathcal{F} = \alpha^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{I}_\Gamma$. For which $p \in L$ is it the case that $\pi_*\mathcal{F} \otimes \kappa_p \neq H^0(\mathcal{F}|_{\pi^{-1}(p)}) \neq 0$?

****pictures would be nice here****

7

Vector Bundles and Chern Classes

Keynote Questions

(a) Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. How many lines $L \subset \mathbb{P}^3$ are contained in S? Does a general quartic surface contain any lines?

(b) Let $\{S_t \subset \mathbb{P}^3\}_{t\in\mathbb{P}^1}$ be a general pencil of quartic surfaces—that is, let F and G be general homogeneous polynomials of degree 4 in four variables, and set $S_t = V(t_0F + t_1G) \subset \mathbb{P}^3$. How many members S_t of the pencil contain a line?

(c) Let $\{C_t \subset \mathbb{P}^2\}_{t\in\mathbb{P}^1}$ be a general pencil of plane curves of degree d—that is, let F and G be general homogeneous polynomials of degree d in three variables, and set $C_t = V(t_0F + t_1G) \subset \mathbb{P}^2$. How many of the curves C_t will be singular?
In this chapter we will introduce the machinery for answering these questions; the answers themselves will be found in Chapters *** and ****.

In Chapter **??** we saw a few examples of how Chern classes could be used to compute invariants of varieties, and, even more important for enumerative geometry, to linearize nonlinear problems. We shall see many more of these in the chapters to come, where Chern classes will play a central role.

In this chapter we will develop the necessary foundational material on vector bundles and Chern classes, and build up a vocabulary of bundles whose classes we know. The keynote questions above are among those that

will be answered in the following chapters, using the theory we've developed.

7.1 Chern Classes

****Why not go right away to allowing schemes for degen loci? And dealing with bundles generated by global sections? In any case, we need to add a section on basic elements for VB.****

We will begin by defining Chern classes of vector bundles with "enough sections" (in a sense made precise by Definition 7.1) because in this case the definition has a simple geometrical interpretation that provides an insight into what should be true. Since any bundle will have enough sections after twisting with a sufficiently ample line bundle, the case of bundles with enough sections determines the whole theory in the case of projective varieties, the generality in which we will work. The extension to arebitrary bundles on projective varieties is carried out explictly in Proposition-Definition 7.7.) Using the Whitney sum formula the definition extends to coherent sheaves on smooth varieties; see Chapter **??**. A more general approach to the theory, introduced by Grothendieck, is be to use Theorem **??** as the definition.

7.1.1 Chern Classes as Dependency Loci

****sometimes the bundle has rank n, sometimes r. Make r, so that X has dim n.****

Let X be a smooth variety, E a vector bundle of rank n on X and σ a section of E. We'll say that σ is dimensionally transverse to the zero section of E, or simply dimensionally transverse , if the zero locus of σ is generically reduced of codimension equal to the rank of E, and generically transverse if, in addition, it is is generically reduced. These conditions are equivalent to the dimensional transversality or generic transversality of the intersection of the image of the zero locus of E to the image of σ considered as a map from X to the total space of E (Exercise 7.17).

The the zero locus $V(\sigma)$ of any section σ is locally defined by $\operatorname{rank} E$ equations, so by the Principal Ideal Theorem every component of $V(\sigma)$ has codimension $\leq n$. Thus if σ is dimensionally transverse then $V(\sigma)$ necessarily of pure codimension n in X. If E has a dimensionally transverse section, then we define the *n-th Chern class of E* to be

$$c_n(E) = [V(\sigma)] \in A^n(X).$$

We can immediately see that the definition of $c_n(E)$ is independent of the generically transverse section σ chosen: if σ and τ are two dimensionally transverse sections of E, then we can interpolate between their zero loci $V(\sigma)$ and $V(\tau)$ with the family

$$\Phi = \{\big([s,t],p\big) : (s\sigma + t\tau)(p) = 0\} \subset \mathbb{P}^1 \times X.$$

This gives a rational equivalence between $V(\sigma)$ and $V(\tau)$: since Φ has codimension at most n everywhere, components of Φ intersecting the fibers over 0 or $\infty \in \mathbb{P}^1$ must dominate $\mathbb{P}^1$, and taking the union of these components we get a rational equivalence between the class of the zero locus of σ and that of τ.

In case $n = 1$—that is when E is a line bundle—this agrees with a special case of our previous definition of the first Chern class $c_1(E)$: we defined $c_1(L)$ from a rational section σ of L as the class of the divisor of zeros of σ minus the class of the divisor of poles of σ. Thus whenever L has a nonzero global section, $c_1(L)$ is the Chow class of the zero locus of that section.

We next extend the definition of transversality to a collection of sections. Let $\sigma_1, \ldots, \sigma_k$ be a k-tuple of sections of E, and consider the locus

$$V(\sigma_1 \wedge \cdots \wedge \sigma_k) = \{x \in X : \sigma_1(x) \wedge \cdots \wedge \sigma_k(x) = 0\} \subset X$$

where the values of the σ_i fail to be linearly independent. A version of the Principal Ideal Theorem (Macaulay's "Generalized Unmixedness Theorem", proven in general by Eagon and Northcott—see for example Eisenbud [1995] Exercise 10.9) shows that every component of $V(\sigma_1 \wedge \cdots \wedge \sigma_k)$ has codimension at most $n - k + 1$; this is what we will call the *expected codimension* of the locus $V(\sigma_1 \wedge \cdots \wedge \sigma_k)$. (Macaulay also proved the *Unmixedness Theorem*, saying that when the codimension is $n - k + 1$ then the scheme $V(\sigma_1 \wedge \cdots \wedge \sigma_k)$ has no embedded components either, so if it is generically transverse then it is in fact reduced; but we will not need this refinement.)

We can give a heuristic argument for Macaulay's codimension bound as follows: inductively, if we assume that $V(\sigma_1 \wedge \cdots \wedge \sigma_{k-1})$ has the expected codimension $n-k+2$, the locus $V(\sigma_1 \wedge \cdots \wedge \sigma_k)$ is the union of that locus with the locus in $X \setminus V(\sigma_1 \wedge \cdots \wedge \sigma_{k-1})$ where the section $\overline{s_k}$ of the bundle

$$\frac{E}{\oplus_{i=1}^{k-1} \mathcal{O}_X \sigma_i}$$

is zero. Since this quotient bundle has rank $n-k+1$, we "expect" this vanishing to happen in codimension $n-k+1$. The argument is only heuristic because it can well happen that $V(\sigma_1 \wedge \cdots \wedge \sigma_k)$ has the expected codimension without $V(\sigma_1 \wedge \cdots \wedge \sigma_{k-1})$ having the expected codimension.

We will say that *the collection $\sigma_1, \ldots, \sigma_k$ is dimensionally transverse* if the subscheme $V(\sigma_1 \wedge \cdots \wedge \sigma_k)$ has codimension $n - k + 1$. By Macaulay's

Theorem above, it is then of pure codimension $n-k+1$. We say that the the collection is generically transverse if, in addition, V is generically reduced.

If the bundle E in fact has a dimensionally transverse collection of k sections $\sigma_1, \ldots, \sigma_k$, we define the $(n-k+1)^{\text{st}}$ *Chern class* of E to be the class

$$c_{n-k+1}(E) = [V(\sigma_1 \wedge \cdots \wedge \sigma_k)] \in A^{n-k+1}(X)$$

of the degeneracy locus $V(\sigma_1 \wedge \cdots \wedge \sigma_k)$. Again, this is well-defined: if $\tau_1, \ldots, \tau_k$ is another generically transverse collection of sections, we can interpolate between the corresponding cycles by taking linear combinations of the σ_i and τ_i. Finally, in case E has a dimensionally transverse collection of n sections, we define the *total Chern class*, or often simply the *Chern class* of E, to be the sum

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E) \in A(X).$$

When is it the case that we can find dimensionally transverse or generically transverse collections of k sections of a bundle? To be able to discuss this condition, we give it a name:

Definition 7.1. Let $E \to X$ be a vector bundle of rank n. We say that E has *enough sections* if, for any cycle $A \in Z(X)$, there exists a generically transverse collection of n sections $\sigma_1, \ldots, \sigma_n$ such that the degeneracy loci $V(\sigma_1 \wedge \cdots \wedge \sigma_j)$ are generically transverse to A;

We take a moment out before continuing to elucidate the meaning of this condition.

Proposition 7.2. *Suppose that E is a vector bundle of rank r on X. Let $\sigma_1, \ldots, \sigma_m$ be global sections of E and let $\tau_1, \ldots, \tau_k$ be general linear combinations of the σ_i.*

(a) *If E has is generated everywhere by a set of sections , then (every components of) the dependency locus of $\tau_1, \ldots, \tau_k$ has codimension $r-k+1$. If the characteristic is zero then $\tau_1, \ldots, \tau_k$ are generically transverse .*

(b) *If $E = F \otimes L$ where F is generated by global sections and L is very ample, and if $\sigma_1, \ldots, \sigma_m$ span the global sections of E, then general enough sections, then E is generated by its global sections. The converse is true in characteristic zero; and in general, if E is generated by global sections and L is a very ample line bundle, the $E \otimes L$ has enough sections. In particular, since X is projective, if E is any vector bundle then $E \otimes L$ has enough sections for any sufficiently ample line bundle L.*

********put in an example of an insep map defined by a complete linear series. Note that the map $(s^p, t^p) : \mathbb{P}^1 \to \mathbb{P}^1$ is NOT such an example, since

it's not the complete series. But if you put a cusp in in the curve, then it is..****

Proof. ****the following needs to be filled in.**** For the first statement it's enough to consider the condition of being transverse to a point. For the characteristic zero statement, use the generic smoothness of the induced map to the Grassmannian, and the transversality to this map of the general translate of an appropriate Schubert cycle, which follows from Kleiman's theorem. For the general case with a line bundle, work over an open affine set where the bundle E trivializes, and use the sections from the line bundle, together with Bertini's theorem. □

Remark. Segre Classes ****explain that by taking more and more sections (and taking the determinantal ideals, say) we could define the Segre classes. Say how these are related to the Chern classes (inverse). Say that we'll give a different def that works in a more general case later in the Chapter on Projective Bundles.**

So far we have defined Chern classes only for vector bundles with enough sections. To extend our definition to all vector bundles, we will develop of theory of these classes for bundles with enough sections enough to describe what happens to the Chern classes when we tensor with a line bundle. Using the last statement of Proposition 7.2, this will allow us to relating the Chern class of arbitrary bundles to those for bundles with enough sections. The relations we develop along the way will serve many other purposes as well; they are the basic properties of the Chern classes.

7.2 Basic properties

As we said in the initial section, up to this point we haven't actually done anything; all we've done is give our ignorance a name. Now comes the crucial part: we're going to describe the basic properties of Chern classes, and in particular the relations between the Chern classes of bundles related by linear algebra constructions. This will serve two crucial purposes: first, this is how we will compute Chern classes in practice; and second, this will allow us to extend the definition of Chern classes to all vector bundles, not just those with enough sections.

7.2.1 Pullbacks

****We have deleted the condition in the definition of "enough sections" that this is stable under pullback by a generically smooth morphism, since it follows from the condition given, so it has to be proven here.****

If E is a bundle on a smooth projective variety X with enough sections and $f : Y \to X$ is a generically smooth morphism with Y also smooth and projective, we can find a generically transverse collection of sections $\sigma_1, \ldots, \sigma_k$ of E such that the pullbacks $f^*\sigma_1, \ldots, f^*\sigma_k$ are again generically transverse by taking ensuring that zero locus $V(\sigma_1 \wedge \cdots \wedge \sigma_k)$ is generically transverse to the image of f and to all the loci where the fiber dimensions of f jump ****should we say more?****; in this case

$$V(f^*\sigma_1 \wedge \cdots \wedge f^*\sigma_k) = f^{-1}V(\sigma_1 \wedge \cdots \wedge \sigma_k)$$

and so we have

$$c(f^*E) = [f^{-1}V(\sigma_1 \wedge \cdots \wedge \sigma_k)] = f^*c(E).$$

7.2.2 Line bundles

If L is an arbitrary line bundle, then we have defined its Chern class to be

$$c_1(L) = [D - E]$$

where D is the divisor of zeroes of σ and E its polar divisor. From this definition it follows that

$$c_1(L^*) = -c_1(L),$$

and also that for any two line bundles L and M,

$$c_1(L \otimes M) = c_1(L) + c_1(M)$$

(in fact, the first follows from the second by taking $M = L^*$). These relations form the cornerstone of the calculus of Chern classes: we'll use them, in combination with the splitting principle and the Whitney sum formula below, to derive all our relations among Chern classes of bundles in general.

Note that the pullback formula $c(f^*E) = f^*c(E)$ holds for all line bundles, not just those with enough sections: we can always find a rational section such that the map is generically transverse to the support of its divisor.

7.2.3 Whitney sum

Let X be as usual a smooth variety, and E and F vector bundles on X of ranks m and n respectively. There is a fundamental formula relating the

Chern classes of E and F to the Chern class of the direct sum $E \oplus F$: this is the *Whitney sum formula*, most conveniently expressed in terms of the total Chern classes:

$$c(E \oplus F) = c(E)c(F).$$

The formula actually holds more generally:

Theorem 7.3 (Whitney sum formula)**.** *If*

$$0 \longrightarrow E \longrightarrow G \longrightarrow F \longrightarrow 0$$

is an exact sequence of vector bundles, then

$$c(G) = c(E \oplus F) = c(E)c(F);$$

that is,

$$c_k(G) \sum_{0 \le l \le k} \sigma_l(E)\sigma_{k-l}(F).$$

for all $k \ge 0$.

In the proof we will first reduce to the case $G = E \oplus F$ by a deformation argument. Then we will choose sections σ_i of E and τ_i of F, and look at the degeneracy locus of pairs (σ_i, τ_i) as sections of $E \oplus F$. Our claim will be that if the sections σ_i and τ_i are general enough, then the sections (σ_i, τ_i) will be generically transverse and have the given class.

This works easily for c_1. Suppose the ranks of E and F are e and f, respectively. We have

$$c_1(E \oplus F) = c_1(\wedge^{e+f}(E \oplus F)) = c_1(\wedge^e(E) \otimes \wedge^f(F)).$$

The section $\sigma_1 \wedge \cdots \wedge \sigma_e \otimes \tau_1 \wedge \cdots \wedge \tau_f$ of $\wedge^e(E) \otimes \wedge^f(F)$ vanishes on the union of $V(\sigma_1 \wedge \cdots \wedge \sigma_e)$ and $V(\tau_1 \wedge \cdots \wedge \tau_f)$; that is $c_1(E \oplus F) = c_1(E) + c_1(F)$ as required.

A very slightly more involved argument proves that $c_{e+f}(E \oplus F) = c_e(E)c_f(F)$: since the zero locus of the section (σ_1, τ_1) is $V(\sigma_1) \cap V(\tau_1)$, it suffices to be sure that σ_1 is generically transverse , and that τ_1 is generically transverse *and* that $V(\tau_1)$ is generically transverse to $V(\sigma_1)$.

However if we want to compute the class of the dependency locus of several pairs, say $(\sigma_1, \tau_1), (\sigma_2, \tau_2), (\sigma_3, \tau_3)$ the situation is complicated by the fact that, at some points the sections $\sigma_1, \sigma_2, \sigma_3$ might not only be dependent but even have two linear dependency relations. To avoid this problem, we will resort, as usual, to an incidence correspondence argument.

With these ideas in mind, we are ready for the proof itself.

Proof. We begin by reducing the problem to the case where $G \cong E \oplus F$. Suppose that we are given the exact sequence of vector bundles on X:

$$0 \longrightarrow E \xrightarrow{\alpha} G \xrightarrow{\beta} F \longrightarrow 0.$$

We define a family over $\mathbb{A}^1 = \operatorname{Spec} K[t]$ of vector bundles $\mathcal{G}$ on X by

$$\mathcal{G} := \frac{E \oplus G}{(t, \alpha)(E)}.$$

where (t, α) denotes the map from E to $E \oplus G$ that is multiplication by t from E to E and acts by α from E to G. This family is trivial on the open set $t \neq 0$, while the fiber at $t = 0$ is the direct sum $E \oplus F$. (Those familiar with the Yoneda construction of Ext will recognize this family as the middle vector bundle in the extension that is t times the extension class of the given exact sequence.)

By Lemma **??******to be written********, a general choice of $k+1$ sections $\tilde{\tau}_i$ of G will map to a collection of $k+1$ sections τ_i of F such that each subset (up to size equal to $\operatorname{rank} F$) is at least dimensionally transverse . Further, if we assume that E has enough sections, then by the same Lemma a general collection of sections σ_i of E will have degeneracy locus of every subset (up to size equal to $\operatorname{rank} E$) generically transverse , and meeting components of dependency loci of subsets of the τ_i only at generic points, so that by Lemma **??******to be written******** the collection of pairs (σ_i, τ_i) is dimensionally transverse in $E \oplus F$.

For each i we have a section of $\mathcal{G}$ that is the image of $\mu_i = (\sigma_i, \tilde{\tau}_i)$. This section may be identified with $t^{-1}\alpha(\sigma_i) + \tilde{\tau}_i$ in fibers where $t \neq 0$, and with (σ_i, τ_i) in the 0 fiber. The family $V(\mu_0 \wedge \cdots \wedge \mu_k)$ is then a rational equivalence demonstrating that $c_{e+f-k}(G) = c_{e+f-k}(E \oplus F)$.

To finish the proof we must treat the case when $G = E \oplus F$. Fixing k, we take $\sigma_0, \ldots, \sigma_k$ and $\tau_0, \ldots, \tau_k$ to be general sections of E and F respectively; we want to describe the (class of the) degeneracy locus of the sections

$$(\sigma_0, \tau_0), (\sigma_1, \tau_1), \ldots, (\sigma_k, \tau_k)$$

of $E \oplus F$. Let Σ be the incidence correspondence

$$\Sigma := \{(p, a, b) \in X \times \mathbb{P}^k \times \mathbb{P}^k \mid \sum a_i \sigma_i(p) = \sum b_i \tau_i(p) = 0\},$$

where a_i, b_i are the coordinates of a and b, and and $\pi : \Sigma \to X$ and $\gamma : X \times \mathbb{P}^k \times \mathbb{P}^k \to \mathbb{P}^k \times \mathbb{P}^k$ are the projections.

Let $\Delta \subset \mathbb{P}^k \times \mathbb{P}^k$ be the diagonal. The intersection of Σ with the preimage of the diagonal $\gamma^{-1}(\Delta)$ projects via π onto the degeneracy locus of the collection $(\sigma_0, \tau_0), \ldots, (\sigma_k, \tau_k)$.

We claim that the projection is generically finite on $\Sigma \cap \gamma^{-1}(\Delta)$. Since the fibers of this projection are linear spaces, it will follow that the projection

is generically one-to-one, and we can conclude that

$$c_{e+f-k}(E \oplus F) = \pi_*([\Sigma] \cdot \gamma^* \Delta).$$

To show that π is generically finite on $\Sigma \cap \gamma^{-1}(\Delta)$ ...****to add****.

Let α and $\beta \in A^1(\mathbb{P}^k \times \mathbb{P}^k)$ be the pullbacks of the hyperplane classes from the two factors. By Theorem **??** we have

$$[\Delta] = \sum_{i+j=k} \alpha^i \beta^j \in A(\mathbb{P}^k \times \mathbb{P}^k$$

The class $\alpha^i \beta^j$ is represented by the cycle

$$\Gamma_{i,j} = \{(a, b) \in \mathbb{P}^k \times \mathbb{P}^k \mid a_{k-i+1} = \cdots = a_k = b_{k-j+1} = \cdots = b_k = 0\}$$

from which we see that

$$\pi\big(\Sigma \cap \gamma^{-1}(\Gamma_{i,j})\big) = V(\sigma_0 \wedge \cdots \wedge \sigma_{k-i}) \cap V(\tau_0 \wedge \cdots \wedge \tau_{k-j}).$$

We claim that Σ is dimensionally transverse to the smooth subvariety $\gamma^{-1}(\Gamma_{i,j})$. From this it will follow that

$$\pi_*\big([\Sigma] \cdot \gamma^*(\alpha^i \beta^j)\big) = c_{m-k+i}(E) c_{n-k+j}(F).$$

****make the rest of the arg into a proof of the claim**** On the other hand, the degeneracy locus of the sections $(\sigma_0, \tau_0), \ldots, (\sigma_k, \tau_k)$ of $E \oplus F$ is just the image of the locus

$$\Phi = \big\{\big(p; [a_0, \ldots, a_k], [b_0, \ldots, b_k]\big) : a_0 = b_0, \ldots, a_k = b_k\big\} \subset \Sigma;$$

that is, if $\Delta \subset \mathbb{P}^k \times \mathbb{P}^k$ is the diagonal,

$$V\big((\sigma_0, \tau_0) \wedge \cdots \wedge (\sigma_k, \tau_k)\big) = \pi\big(\gamma^{-1}(\Delta)\big).$$

Thus, writing e and f for the ranks of E and F,

$$c_{e+f-k}(E \oplus F) = \pi_*\big(\gamma^*[\Delta]\big);$$

and applying the formula

$$[\Delta] = \sum_{i=0}^{k} \alpha^i \beta^{k-i} \in A^k(\mathbb{P}^k \times \mathbb{P}^k)$$

for the class of the diagonal in $\mathbb{P}^k \times \mathbb{P}^k$, we arrive at

$$\begin{aligned} c_{e+f-k}(E \oplus F) &= \sum_{i=0}^{k} \pi_*\big(\gamma^*(\alpha^i \beta^{k-i})\big) \\ &= \sum c_{m-k+i}(E) c_{n-i}(F), \end{aligned}$$

which is the Whitney formula. □

7.2.4 The splitting principle

A natural complement to the Whitney sum formula is the *splitting principle*, which allows us to relate an arbitrary bundle to a direct sum of line bundles:

Theorem 7.4 (Splitting principle)**.** *Let X be any smooth variety, E a vector bundle of rank n on X. Then there exists a smooth variety Y and a morphism $\varphi : Y \to X$ with two properties:*

(a) *The pullback map $\varphi^* : A^*(X) \to A^*(Y)$ is injective; and*

(b) *The pullback bundle $\varphi^* E$ on Y admits a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = \varphi^* E$$

by vector subbundles $E_i \subset \varphi^* E$ with successive quotients E_i/E_{i-1} locally free of rank 1.

********smoothness is irrelevant here. Also in the Cor below. Shouldn't we admit this?********

Given this, we can use the Whitney sum formula and our a priori definition of the Chern class of a line bundle to describe the Chern class of the pullback:

$$c(\varphi^* E) = \prod_{i=1}^{n} c(E_i/E_{i-1});$$

and by the first part of the Lemma this determines the Chern classes of E.

Proof. To start, let $\pi : \mathbb{P}E \to X$ be the projectivization of the bundle E, that is, the variety

$$\mathbb{P}E = \{(x, \xi) : x \in X \text{ and } \xi \subset E_x \text{ a 1-dimensional subspace}\}$$

with the map $\pi : (x, \xi) \mapsto x$. We observe that the pullback bundle $\pi^* E$ has a tautological sub-line bundle S, defined by specifying its fiber over a point $(x, \xi) \in \mathbb{P}E$ as

$$S_{(x,\xi)} = \xi \subset E_x.$$

Now let Q be the quotient bundle $\pi^* E/S$ on $\mathbb{P}E$ so that we have an exact sequence

$$0 \to S \to \pi^* E \to Q \to 0.$$

We can repeat this process, taking the projectivization $\mathbb{P}Q$ and introducing the tautological sub-line bundle of the pullback of Q to $\mathbb{P}Q$. Indeed, iterating this $n-1$ times we arrive at the *flag bundle* $\varphi : \mathbb{F}E \to X$, consisting of all flags in the fibers of the original bundle E:

$$\mathbb{F}E = \{(x, \xi_1, \xi_2, \ldots, \xi_{n-1}) : x \in X \text{ and } \xi_1 \subset \xi_2 \subset \cdots \subset \xi_{n-1} \subset E_x\};$$

and on $\mathbb{F}E$ we have the desired filtration

$$0 \subset S_1 \subset S_2 \subset \cdots \subset S_{n-1} \subset \varphi^* E$$

where S_k is the rank k subbundle of $\varphi^* E$ defined by

$$(S_k)_{(x,\xi_1,\xi_2,\ldots,\xi_{n-1})} = \xi_k.$$

It remains now to see that the pullback map $\varphi^* : A^*(X) \to A^*(\mathbb{F}E)$ is injective. This in turn follows from a direct calculation:

Lemma 7.5. *With X, E, $\pi : \mathbb{P}E \to X$ and S as above,*

$$\pi_*\big(c_1(S^*)^{n-1}\big) = [X].$$

Proof. The class on the left has dimension equal to $\dim X$, so a priori we have

$$\pi_*\big(c_1(S^*)^{n-1}\big) = c[X]$$

for some integer c; we just need to determine c (and in fact for our application we need only $c \neq 0$). We do this as follows: let $x \in X$ be any point, and $\mathbb{P}^{n-1} = \mathbb{P}E_x$ the fiber of $\mathbb{P}E$ over x. By the push-pull formula and the functoriality of the first Chern class,

$$\begin{aligned} c &= \big([x] \cdot \pi_*\big(c_1(S^*)^{n-1}\big)\big) \\ &= \big(\pi^*[x] \cdot c_1(S^*)^{n-1}\big) \\ &= \big(c_1\left(S^*|_{\mathbb{P}^{n-1}}\right)\big)^{n-1}. \end{aligned}$$

But the restriction of S^* to $\mathbb{P}^{n-1}$ is the bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$, whose first Chern class is just the class ζ of a hyperplane in $\mathbb{P}^{n-1}$; thus

$$c = \zeta^{n-1} = 1.$$

□

Given this, Theorem 7.4 follows: by the push-pull formula in general we have for any class $\alpha \in A^*(X)$

$$\alpha = \pi_*\left(\pi^*\alpha \cdot c_1(S^*)^{n-1}\right)$$

and so in particular $\pi^*\alpha \neq 0$ if $\alpha \neq 0$. □

7.2.5 Tensor products with line bundles

As an application of the splitting principle, we'll derive the relation between the Chern classes of a vector bundle E of rank n on a variety X and the Chern classes of the tensor product $E \otimes L$ of the tensor product of E with a line bundle L.

To do this, let $\varphi : Y \to X$ be a morphism such that the pullback $\varphi^* : A^*(X) \to A^*(Y)$ is injective, and such that we have a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = \varphi^* E;$$

let $M_k = E_k/E_{k-1},\ k = 1, \ldots, n$ be the successive quotients in the filtration. Let $\alpha_k = c_1(M_k) \in A^1(Y)$ be the first Chern class of M_k, and let $\beta = c_1(\varphi^* L) = \varphi^* c_1(L)$ be the first Chern class of the pullback of L. By the Whitney sum formula, we have

$$\prod_{k=1}^{n} (1 + \alpha_k) = c(\varphi^* E) = \varphi^* c(E),$$

so that the elementary symmetric polynomials in the α_k are the pullbacks of the Chern classes of E:

$$\begin{aligned} \alpha_1 + \alpha_2 + \cdots + \alpha_n &= \varphi^* c_1(E) \\ \sum_{1 \le i < j \le n} \alpha_i \alpha_j &= \varphi^* c_2(E) \\ &\vdots \\ \alpha_1 \alpha_2 \ldots \alpha_n &= \varphi^* c_n(E). \end{aligned}$$

On the other hand, tensoring the filtration above with $\varphi^* L$ we arrive at a filtration

$$0 = E_0 \otimes \varphi^* L \subset E_1 \otimes \varphi^* L \subset \cdots \subset E_{n-1} \otimes \varphi^* L \subset E_n \otimes \varphi^* L = \varphi^*(E \otimes L)$$

with successive quotients $M_k \otimes \varphi^* L$; since $c_1(M_k \otimes \varphi^* L) = \alpha_k + \beta$, we have

$$\varphi^* c(E \otimes L) = \prod_{k=1}^{n} (1 + \alpha_k + \beta).$$

Now, we can express the product on the right as a polynomial in the elementary symmetric polynomials: for example, we have

$$c_1(E \otimes L) = \sum_{i=1}^{n} \alpha_i + \beta = c_1(E) + n c_1(L)$$

and likewise

$$\begin{aligned} c_2(E \otimes L) &= \sum_{1 \le i < j \le n} (\alpha_i + \beta)(\alpha_j + \beta) \\ &= \sum_{1 \le i < j \le n} \alpha_i \alpha_j + (n-1)\beta \sum_{i=1}^{n} \alpha_i + \binom{n}{2} \beta^2 \\ &= c_2(E) + (n-1) c_1(E) c_1(L) + \binom{n}{2} c_1(L)^2. \end{aligned}$$

The formula for $c_k(E)$ in general is obtained as follows: for each i, the Chern class $c_i(E)$ is the sum of all the products of i distinct α_k. These products each occur in the expression for $\varphi^* c(E \otimes L)$ with $n - i$ other factors, each equal to 1 or β. To get a class of codimension k, we much choose exactly k of these factors equal to β. We have proven:

Proposition 7.6. *If E is a vector bundle of rank n, and L is a line bundle, then*

$$c_k(E \otimes L) = \sum_{i=0}^{k} \binom{n-i}{k-i} c_1(L)^{k-i} c_i(E).$$

This is our first application of the splitting principle, and it is a template for all that follow: to relate the Chern classes of tensor products of vector bundles, or symmetric or exterior powers of a given bundle, we pull them back to a suitable variety $Y \to X$ where they all split (or at any rate admit filtrations with one-dimensional quotients), use the relations for the Chern classes of sums and products of line bundles to derive a relation there and deduce the relation on X. A colloquial formulation of the splitting principle is thus:

Any general relation among the Chern classes of bundles that can be proved for direct sums of line bundles holds true in general.

We will often apply this argument without explicitly invoking the map $Y \to X$.

7.2.6 Definition of Chern Classes in General

We can now give the definition of the Chern classes for bundles that may not have enough sections:

Proposition-Definition 7.7. Suppose that F is a vector bundle on X, and L is a line bundle such that $E := F \otimes L^*$ has enough sections. Note that $F = E \otimes L$. The classes

$$c_k := \sum_{i=0}^{k} \binom{n-i}{k-i} c_1(L)^{k-i} c_i(E)$$

are independent of the choice of L. We define $c_k(E)$ to be the class c_k computed for any such choice.

Proof. Given line bundles L^* and M^* so that $F \otimes L^*$ and $F \otimes M^*$ both have enough sections, we may write $F \otimes L^* = (F \otimes M^*) \otimes (M \otimes L^*)$.

Thus it suffices to show that the map taking a Chern class $1+c_1(E)+\cdots$ to the class

$$\sum_{i=0}^{k} \binom{n-i}{k-i} c_1(L)^{k-i} c_i(E)$$

is an action of the Picard group $\mathrm{Pic}(X)$ on the set of Chern classes of bundles—that is, that the formula is multiplicative in the appropriate sense. This could be derived by direct computation, but it also follows from the splitting principle: if we pull back to a variety on which E has a filtration by line bundles with Chern classes $1+\alpha_k$, and write the Chern class of L as $1+\beta$, then from the way the formula was derived we see that the action is

$$\prod_{k=1}^{n}(1+\alpha_k) \xrightarrow{L} \prod_{k=1}^{n}(1+\alpha_k+\beta).$$

Since the Chern class of the product of L with a line bundle M having Chern class $1+\gamma$ is $1+\beta+\gamma$, the multiplicativity follows. □

Proposition 7.8. *If $f : Y \to X$ is a map of smooth projective varieties and E is a vector bundle on X, then*

$$c_k(f^*E) = f^*c_k(E)$$

for any k.

Proof. First suppose that E is generated by global sections σ_i. In this case f^*E is generated by the pullbacks $f^*\sigma_i$, so the dependency schemes, of rank $E-k+1$ general linear combinations of these define $c_k(E)$, and $c_k(f^*E)$ on X and on Y, respectively. The dependency scheme on X pulls back to that on Y, completing the proof in this case.

If L is a line bundle on X, then we may choose a rational section of L that is regular on an open subset of the image of Y, and thus pulls back to a rational section on Y. Since the schemes of zeros and poles pull back to their counterparts on Y, this shows that the formula of the Proposition holds for arbitrary line bundles.

The case of an arbitrary vector bundle E follows from the two cases above, using Proposition-Definition 7.7. □

In the applications of Chern classes we will generally have a collection of sections whose dependency locus has the right dimension, and we will want to compute class of the locus as a Chern class. The theory above guarantees that we can do this if the bundle has enough sections, and it is true in general:

Theorem 7.9. *Suppose that $\sigma_1, \ldots \sigma_k$ are global sections of a vector bundle E of rank r on a smooth projective variety X. If the dependency locus Y has codimension $r-k+1$ (or is empty), then $c_{r-k+1}(E) = [Y]$.*

A proof will be found in Section 14.1.

7.2.7 Dual Bundles

As an immediate application of the splitting principle, we can express the Chern classes of the dual E^* of a vector bundle E in terms of those of E. Briefly, if E were the direct sum

$$E = \bigoplus_{i=1}^{n} L_i \quad \text{with} \quad c_1(L_i) = \alpha_i,$$

we would have

$$c(E) = \prod_{i=1}^{n} (1 + \alpha_i).$$

We would also have, by the same token,

$$E^* = \bigoplus_{i=1}^{n} L_i^*$$

and hence

$$c(E^*) = \prod_{i=1}^{n} (1 - \alpha_i);$$

thus

$$c_k(E^*) = (-1)^k c_k(E).$$

The splitting principle then tells us that this formula holds for all vector bundles, not just split ones.

7.2.8 Determinant of a Bundle

By the *determinant* $\det E$ of a bundle E we mean the line bundle that is the highest exterior power $\det E := \wedge^{\operatorname{rank} E} E$.

Proposition 7.10. *If E is a vector bunde on a smooth variety then $c_1(E) = c_1(\det E)$.*

********smoothness is irrelevant. Should we be mentioning this?******** The formula is immediate from the definition in the case when E has enough sections. Here is an easy general proof illustrating the use of the Splitting Principle.

Proof. By the Splitting Principle we may assume that E has a filtration $0 \subset E_1 \subset \cdots \subset E$ such that each E_{i+1}/E_i is a line bundle. It follows that $\det E = \prod_i E_{i+1}/E_i$, and thus $c_1(E) = \sum c_1(E_{i+1}/E_i) = c_1(\prod_i E_{i+1}/E_i) = c_1(\det E)$. □

7.3 Application: A strong Bertini Theorem

****This is a natural consequence of the vanishing of Chern classes above the rank. The proof given needs to be reworked from this point of view. It's used in the excess intersection chapter.****

Proposition 7.11. *Let X be a smooth, n-dimensional projective variety and $\mathcal{D}$ a linear system of divisors on X whose (scheme-theoretic) base locus is a smooth k-dimensional subscheme $Z \subset X$. If $k \leq n/2$, then the general member of the linear system $\mathcal{D}$ is smooth.*

Before proving this, note that the inequality $k \leq n/2$ is necessary, and indeed if we were to replace it by any strictly weaker inequality the statement of the Lemma would be false. The simplest nontrivial example of this would be to take $X = \mathbb{P}^4$ and $Z = \mathbb{P}^2$ a 2-plane in $\mathbb{P}^4$: if $Y = V(F) \subset \mathbb{P}^4$ is any hypersurface of degree $d > 1$ containing Z, the partial derivatives of F must have a common zero somewhere along Z and so Y must be singular. For some extensions of this example see Exercises 7.27 and 7.28.

Proof of Lemma 7.11. We start by passing to the blow-up $\pi : \tilde{X} = Bl_Z X$ of X along Z, with exceptional divisor $E = \pi^{-1}(Z)$. Let $\tilde{\mathcal{D}}$ be the proper transform in $\tilde{X}$ of the linear system $\mathcal{D}$; that is,

$$\tilde{\mathcal{D}} = \{\pi^* D - E : D \in \mathcal{D}\}.$$

(Note that a general $D \in \mathcal{D}$ will have multiplicity 1 along Z, so that the corresponding element $\tilde{D} = \pi^* D - E \in \tilde{\mathcal{D}}$ will indeed be its proper transform, but not every member of $\tilde{\mathcal{D}}$ need be the proper transform of a member of $\mathcal{D}$.)

Now, by our hypothesis that the base locus of $\mathcal{D}$ is the smooth scheme Z, the linear system $\tilde{\mathcal{D}}$ has no base locus, and so by the classical Bertini theorem the general member $\tilde{D} \in \tilde{\mathcal{D}}$ will be smooth. The question then is, does the map $\pi : \tilde{D} \to X$ introduce any singularities? Clearly it doesn't away from E.

As for what goes on along $\tilde{D} \cap E$, we observe first that *the intersection $\tilde{D} \cap \pi^{-1}(p)$ of $\tilde{D}$ with any fiber of π over a point $p \in Z$ is a reduced linear subspace of $\pi^{-1}(p) \cong \mathbb{P}^{n-k-1}$*, either a hyperplane in $\pi^{-1}(p)$ or all of $\pi^{-1}(p)$. To see this, let $D \in \mathcal{D}$ be any divisor of the original linear series,

$L = \mathcal{O}_X(D)$ the line bundle associated to D and $\sigma \in H^0(L)$ the section of L defining D. The restriction to Z of the section σ then gives a section τ of the tensor product $(\mathcal{I}_Z/\mathcal{I}_Z^2) \otimes L = N^*_{Z/X} \otimes L$ of the conormal bundle of Z in X with L; and the intersection $\tilde{D} \cap \pi^{-1}(p)$ is the zero locus, in $\pi^{-1}(p) = \mathbb{P}(N_{Z/X})_p = \mathbb{P}(N_{Z/X} \otimes L)_p$, of $\tau(p)$—either a hyperplane in $\pi^{-1}(p)$ if $\tau(p) \neq 0$, or all of $\pi^{-1}(p)$ if $\tau(p) = 0$. Moreover, we see from this that D will be smooth at p exactly when the former is the case, that is, $\tau(p) \neq 0$ and $\tilde{D} \cap \pi^{-1}(p)$ is correspondingly a proper subspace of $\pi^{-1}(p)$.

Now let $V \subset H^0(L)$ be the vector space of sections of L corresponding to the linear system $\mathcal{D}$, and let $W \subset H^0(N^*_{Z/X} \otimes L)$ be the associated vector space of sections of $N^*_{Z/X} \otimes L$. We know that, as linear forms on $N_{Z/X} \otimes L$, the sections $\tau \in W$ have no common zeroes outside the zero section, which is to say W generates $N^*_{Z/X} \otimes L$ everywhere; we claim that a general element $\tau \in W$ is nowhere zero. This in turn follows from the

Lemma 7.12. *Let E be a vector bundle of rank r on a variety Z, and $W \subset H^0(E)$ a vector space of sections generating E. If $r > \dim Z$, then a general element $\tau \in W$ is nowhere zero.*

Proof. Since W generates E—that is, the evaluation map $W \to E_p$ is surjective for all $p \in Z$—for each $p \in Z$ the subspace $W_p \subset W$ of sections zero at p has codimension r in W. Let $\Sigma \subset \mathbb{P}W \times Z$ be the incidence correspondence

$$\Sigma = \{(\tau, p) : \tau(p) = 0\};$$

since the fibers of the projection $\Sigma \to Z$ have codimension r in $\mathbb{P}W$, we have

$$\dim \Sigma = \dim \mathbb{P}W - r + \dim Z;$$

if $\dim Z < r$, then, the image of Σ in $\mathbb{P}W$ must be a proper subvariety. □

This concludes the proof of Lemma 7.11: we see that for general $D \in \mathcal{D}$, the corresponding divisor $\tilde{D} = \pi^* D - E$ is smooth and contains no fibers $\pi^{-1}(p)$ of E over Z, and so D is smooth. □

7.4 Chern Classes of Some Interesting Bundles

There are a lot of different vector bundles, but some come up again and again in geometric applications: the tautological bundles on projective spaces and Grassmannians; tangent bundles; and normal bundles of subvarieties. We'll analyze some examples of the first two types here (with generalizations in chapters to come). Normal bundles will be used, for example, in Chapter8.

7.4.1 Universal bundles on projective space

We start with the most basic of all bundles: the bundle $\mathcal{O}_{\mathbb{P}^r}(1)$ on projective space $\mathbb{P}^r$. We have

$$c_1(\mathcal{O}_{\mathbb{P}^r}(1)) = \zeta \in A^1(\mathbb{P}^r)$$

where ζ is the hyperplane class; and similarly

$$c_1(\mathcal{O}_{\mathbb{P}^r}(n)) = n \cdot \zeta \in A^1(\mathbb{P}^r)$$

for any $n \in \mathbb{Z}$.

This in turn allows us to compute the Chern class of the universal quotient bundle Q on $\mathbb{P}^r$: if $\mathbb{P}^r = \mathbb{P}V$, from the exact sequence

$$0 \to S = \mathcal{O}_{\mathbb{P}^r}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^r} \to Q \to 0$$

we have

$$\begin{aligned} c(Q) &= \frac{1}{c(\mathcal{O}_{\mathbb{P}^r}(-1))} \\ &= \frac{1}{1-\zeta} \\ &= 1 + \zeta + \zeta^2 + \cdots + \zeta^r. \end{aligned}$$

Note that we could also arrive at this directly from the description of Chern classes as degeneracy loci of sections: an element $v \in V$ gives rise to a global section σ of the bundle Q; given k elements $v_1, \ldots, v_k \in V$ the corresponding sections $\sigma_1, \ldots, \sigma_k$ of Q will be linearly dependent at a point $x \in \mathbb{P}^r$ exactly when x lies in the $\mathbb{P}^{k-1}$ corresponding to the subspace $W = \langle v_1, \ldots, v_k \rangle \subset V$ spanned by the v_i. Thus

$$c_{r-k+1}(Q) = [\mathbb{P}^{k-1}] = \zeta^{r-k+1} \in A^{r-k+1}(\mathbb{P}^r).$$

7.4.2 Chern Classes of Tangent Bundles

Chern classes in general are invariants of a pair (X, E) consisting of a variety X and a vector bundle E on X. Since the tangent and cotangent bundles of a smooth variety (or, more generally the sheaf of differentials on any scheme) is intrinsically associated to that variety, their Chern classes are intrinsic invariants of the variety alone, and they are indeed important ones.

We see this first for curves, where the genus of a (smooth, projective) curve C can be characterized as

$$g = \frac{\deg c_1(T_C)}{2} + 1.$$

More generally, the Hirzebruch Riemann-Roch theorem (explained in Chapter 17) says that the Euler characteristic of the structure sheaf of any

smooth projective variety can be expressed in terms of the Chern classes of its tangent bundle; for example, If S is a smooth surface then

$$\chi(\mathcal{O}_S) = \frac{c_1(T_S)^2 + c_2(T_S)}{12}.$$

As with surfaces, the Hirzebruch Riemann-Roch Theorem gives an expression for the Euler characteristic of any line bundle in terms of intersection theory and the Chern classes of the Tangent bundle.

When we study a smooth subvariety Y of a variety X, much about the relationship is encoded in the *normal bundle*, defined $N_{Y/X}$, which is defined to be the quotient of the map of tangent bundles $N_{Y/X} = \operatorname{coker}\big(T_Y \to T_X|_Y\big)$ induced by the inclusion of $Y \subset X$.

The *canonical bundle* ω_X of a smooth variety X is by definition the determinant of the cotangent bundle, and thus $c_1(\omega_X) = -c_1(T_X)$. From the definition of the normal bundle of a smooth subvariety $Y \subset X$, and the Whitney Sum Formula, we see further that $c(N_{Y/X}) = c(T_X)/c(T_Y)$, and in particular $c_1(N_{Y/X}) = c_1(\omega_Y) - c_1(\omega_X)$. This is often called the *Adjunction Formula*, and it is valid more generally for the canonical bundles of locally complete intersection subvarieties ****give a reference****.

On the whole, perhaps the most important things we can know about a variety, after a description of its Chow or cohomology ring, are the Chern classes of its tangent bundle. We will make some computations of these in the next few sections.

7.4.3 Tangent Bundles of Projective Spaces

We can also calculate the Chern classes of the tangent bundle to projective space. First describe the tangent bundle in a simple geometric way, and use this description to determine its classes. The description of tangent bundles to Grassmannians in Chapter 2 gives a different approach.

To begin with, it's natural to try to relate the tangent space to $\mathbb{P}^r = \mathbb{P}V$ at a point x to the tangent space to V at a point $v \in V$ lying over x. Indeed the differential of the map

$$\pi : V \setminus \{0\} \to \mathbb{P}V$$

gives a surjection

$$V = T_v V \to T_x \mathbb{P}V$$

with kernel the one-dimensional subspace $\langle v \rangle \subset V$ spanned by v, and hence an identification

$$\overline{\pi}_v : T_x \mathbb{P}V \cong V/\langle v \rangle.$$

The problem is that *this identification depends on the choice of* v: if we multiply v by a scalar λ, the map π_v is divided by the same λ.

****Insert Figure****

To specify a vector field on an open subset $U \subset \mathbb{P}V$, accordingly, we can give a vector field on the preimage $\pi^{-1}(U) \subset V \setminus \{0\}$ that is homogeneous linear on the fibers of π, that is, in terms of coordinates $X_0, \ldots, X_r$ on V, an expression of the form

$$\sum_{i=0}^{r} H_i(X) \frac{\partial}{\partial X_i}$$

where H_i is homogeneous of degree 1, that is,

$$H_i = \frac{F_i}{G_i}$$

with F_i and G_i homogeneous, $\deg(F_i) = \deg(G_i) + 1$ and $G_i \neq 0$ on U. Moreover, the vector field so described will be zero if and only if the vector field $\sum_{i=0}^{r} H_i(X) \frac{\partial}{\partial X_i}$ points in the direction of $\langle v \rangle$ everywhere, that is, if it is a scalar multiple of $\sum X_i \frac{\partial}{\partial X_i}$. In sum, there is an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r} \xrightarrow{\begin{pmatrix} X_0 \\ \vdots \\ X_r \end{pmatrix}} \mathcal{O}_{\mathbb{P}^r}(1)^{r+1} \xrightarrow{\left(\frac{\partial}{\partial X_0}, \ldots, \frac{\partial}{\partial X_r}\right)} T_{\mathbb{P}^r} \to 0$$

Using this sequence and the Whitney sum formula we see that the Chern class of the tangent bundle of projective space is

$$\begin{aligned} c(T_{\mathbb{P}^r}) &= (1+\zeta)^{r+1} \\ &= 1 + (r+1)\zeta + \binom{r+1}{2}\zeta^2 + \cdots + (r+1)\zeta^r. \end{aligned}$$

7.4.4 Tangent bundles to hypersurfaces

We can combine the formula above for the Chern classes of the tangent bundle to projective space $\mathbb{P}^r$ and the Whitney formula to calculate the Chern classes of the tangent bundle to a smooth hypersurface $X \subset \mathbb{P}^r$ of degree d.

To do this, we use the standard normal bundle sequence

$$0 \to T_X \to T_{\mathbb{P}^r}|_X \to N_{X/\mathbb{P}^r} \to 0$$

and the identification

$$N_{X/\mathbb{P}^r} = \mathcal{O}_{\mathbb{P}^r}(X)|_X = \mathcal{O}_X(d).$$

Letting ζ_X denote the restriction to X of the hyperplane class on $\mathbb{P}^r$, we can write

$$\begin{aligned} c(T_X) &= \frac{c(T_{\mathbb{P}^r})|X}{N_{X/\mathbb{P}^r}} \\ &= \frac{(1+\zeta_X)^{r+1}}{1+d\zeta_X} \\ &= \left(1+(r+1)\zeta_X+\binom{r+1}{2}\zeta_X^2+\dots\right)\left(1-d\zeta_X+d^2\zeta_X^2+\dots\right). \end{aligned}$$

There is a beautiful consequence for the topology of a smooth hypersurface over the complex numbers. In the classical, or analytic topology on X, the Hopf Index Theorem **ref** says that the Euler characteristic of X is equal to the degree of the top Chern class of its tangent bundle. Applying the formula above, then, and remembering that $\deg(\zeta_X^{r-1}) = d$, we have

$$\chi(X) = c_{r-1}(T_X) = \sum_{k=0}^{r-1}(-1)^k\binom{r+1}{k+2}d^{k+1}.$$

In fact, we know more: the Lefschetz Hyperplane Theorem **ref** tells us that the integral cohomology groups of X, except for the middle one $H^{r-1}(X)$, are all equal to the corresponding cohomology groups of projective space: in particular, the Betti numbers are 1 in even dimensions and 0 in odd. (In fact, the analogous statement is true for any smooth complete intersection: all the cohomology groups except the middle are equal to those of projective space.) Given that, knowing the Euler characteristic determines the middle betti number as well. In Table 7.1 we give the results of this calculation in a few of the cases where it is actually used.

variety	χ	middle Betti number
quadric surface	4	2
cubic surface	9	7
quartic surface	24	22
quintic surface	55	53
quadric threefold	4	0
cubic threefold	−6	10
quartic threefold	−56	60
quintic threefold	−200	204
quadric fourfold	6	2
cubic fourfold	27	23

TABLE 7.1. Euler Characteristics of Favorite Hypersurfaces

Note that some of these we already knew: the quadric surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, from which we can see directly both the Euler characteristic and the second Betti number; the quadric fourfold may also be viewed as the Plücker embedding of the Grassmannian $\mathbb{G}(1,3)$, whose cohomology has as basis its six Schubert cycles, and whose middle cohomology in particular has basis given by the two Schubert cycles $\sigma_{1,1}$ and σ_2.

7.4.5 Bundles on Grassmannians

Let's consider next the case of the Grassmannian $G = G(k,n)$ of k-planes in an n-dimensional vector space V, and its universal sub- and quotient bundles S and Q.

We'll start with Q, since this bundle has enough sections to determine its Chern classes directly from their degeneracy loci. Specifically, elements $v \in V$ give rise to sections σ of Q simply by taking their images in each quotient of V: that is, for a k-plane $\Lambda \subset V$, we set

$$\sigma(\Lambda) = \overline{v} \in V/\Lambda.$$

Now, given a collection $v_1, \ldots, v_m \in V$, the corresponding sections will fail to be independent at a point $\Lambda \in G$ exactly when the corresponding $\overline{v}_i \in V/\Lambda$ are dependent, which is to say when Λ intersects the span $W = \langle v_1, \ldots, v_m \rangle \subset V$ in a nonzero subspace—that is, when

$$\mathbb{P}\Lambda \cap \mathbb{P}W \neq \varnothing.$$

We may recognize this locus as the Schubert cycle $\Sigma_{n-k-m+1}(W)$, from which we conclude that the Chern class of Q is the sum

$$c(Q) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_{n-k}.$$

Unlike Q, the universal subbundle S doesn't have nonzero global sections, so we can't use the characterization of Chern classes as degeneracy loci. But the dual bundle S^* does: if $l \in V^*$ is a linear form, we can define a section τ of S^* by restricting l to each k-plane $\Lambda \subset V$ in turn; in other words, we set

$$\tau(\Lambda) = l|_\Lambda.$$

Now, if we have m independent linear forms $l_1, \ldots, l_m \in V^*$, the corresponding sections of S^* will fail to be independent at the point $\Lambda \in G$—that is, some linear combination of the l_i will vanish identically on Λ—exactly when Λ fails to intersect the common zero locus U of the l_i properly; that is, when

$$\dim(\mathbb{P}\Lambda \cap \mathbb{P}U) \geq k - m.$$

Again, this locus is a Schubert cycle in G, specifically the cycle $\Sigma_{1,1,\ldots,1}(U)$; and we conclude that

$$c(S^*) = 1 + \sigma_1 + \sigma_{1,1} + \cdots + \sigma_{1,1,\ldots,1};$$

from this we can deduce in turn that

$$c(S) = 1 - \sigma_1 + \sigma_{1,1} + \cdots + (-1)^k \sigma_{1,1,\ldots,1}.$$

We have seen in the Grassmannian chapter ****make a precise reference**** that the tangent bundle on the Grassmannian is $T_{\mathbb{G}(k,n)} = S^* \otimes Q$. As the reader may guess from the discussion above, the formula for the Chern class of the tensor product of two bundles is complicated, but at least the splitting principle makes it easy to find the first Chern class:

Corollary 7.13. *If E, F are vector bundles on a smooth variety then $c_1(E \otimes F) = (\operatorname{rank} E)c_1(F) + (\operatorname{rank} F)c_1(E)$. In particular, $c_1(T_{\mathbb{G}(k,n)}) = (n+1)\sigma_1$ for every k, n.*

Proof. We already know the the first formula in the special case where both E and F are line bundles. Theorem 7.4, the splitting principle, reduces the first formula to the case where $E = \oplus E_i$ and $F = \oplus F_i$ are sums of line bundles. In this setting $E \otimes F = \oplus_{i,j} E_i \otimes F_j$, and we can apply Theorem 7.3. The second formula follows from the first together with $T_{\mathbb{G}(k,n)} = S^* \otimes Q$ and $c_1(S^*) = c_1(Q) = \sigma_1$. □

7.5 Generators and Relations for $A^*(G(k,n))$

We have seen in Corollary 3.4 that the Chow ring of the Grassmannian is a free abelian group generated by the Schubert cycles. We will now show that it is generated multiplicatively by just the the *special Schubert cycles*, which are the Chern classes of the universal subbundle. At the same time, we will see that the Whitney Sum formula and the fact that the Chern classes of a bundle vanish above the rank of the bundle provide a complete description of the relations among the special Schubert cycles, and that these form a complete intersection.

Theorem 7.14. *The Chow ring of the Grassmannian $G(k,n)$ has the form*

$$A^*(G(k,n)) = \mathbb{Z}[c_1, \ldots, c_k]/I$$

where $c_i \in A^i(G(k,n))$ is the i-th Chern class of the universal subbundle S and the ideal I is generated by the terms of total degree $n-k+1, \ldots, n$ in the power series expansion

$$\frac{1}{1 + c_1 + \cdots + c_k} = 1 - (c_1 + \cdots + c_k) + (c_1 + \cdots + c_k)^2 - \cdots \in \mathbb{Z}[[c_1, \ldots, c_k]].$$

Moreover, I is a complete intersection.

For example, the Chow ring of $G(3,7)$ is $\mathbb{Z}[c_1,\dots,c_3]/I$ where I is generated by the elements

$$c_1^5 + 4c_1^3c_2 + 3c_1c_2^2 + 3c_1^2c_3 + 2c_2c_3,$$
$$c_1^6 + 5c_1^4c_2 + 6c_1^2c_2^2 + c_2^3 + 4c_1^3c_3 + 6c_1c_2c_3 + c_3^2,$$
$$c_1^7 + 6c_1^5c_2 + 10c_1^3c_2^2 + 4c_1c_2^3 + 5c_1^4c_3 + 12c_1^2c_2c_3 + 3c_2^2c_3 + 3c_1c_3^2,$$

and these elements form a regular sequence.

The proof uses two results from commutative algebra, Proposition 7.15 and Lemma 7.16, which are variations on some frequently used results; the reader may wish to familiarize himself with them before reading the proof of Theorem 7.14. Recall that the *socle* of a finite dimensional graded algebra T is the submodule of elements annihilated by all elements of positive degree. In particular, if d is the largest degree such that $T_d \neq 0$, then the socle of T contains T_d. For a somewhat different proof, and the generalization to Flag bundles of arbitrary vector bundles, see Grayson and Stillman [$\geq$ 2011].

Proof. Set $A = A^*(G(k,n))$ and write t_i for the degree i part of the power series expansion of $1/(1 + c_1 + \cdots + c_k)$, so that $t_0 = 1,\ t_1 = -c_1,\ t_2 = c_1^2 - c_2, \dots$. Let $J = (t_{n-k+1}, \dots, t_n)$ and let $R = \mathbb{Z}[c_1,\dots,c_k]/J$.

The Whitney sum formula Theorem **??** applied to the tautological sequence of vector bundles

$$0 \to S \to \mathcal{O}^n_{G(k,n)} \to Q \to 0$$

on $G(k,n)$ shows that $c(Q) = 1/c(S)$. Since Q has rank $n-k$, the classes $c_i(Q)$ vanish for all $i > n-k$, and it follows that there is a ring homomorphism

$$\varphi : R \to A; \quad t_i \mapsto c_i(Q).$$

Under this homomorphism the class c_i goes to the Schubert cycle $c_i(S) = \sigma_{1^i}$ (where the subscript denotes a sequence of i ones.) Recall from Corollary 3.2 that $\sigma_{1^k}^{n-k}$ is the class of a point.

We will show that for any field F the sequence $t_{n-k+1}, \dots, t_n$ is a regular sequence in $R \otimes_{\mathbb{Z}} F$, and the induced map

$$R' := R \otimes_{\mathbb{Z}} F \xrightarrow{\varphi' := \varphi \otimes_{\mathbb{Z}} F} A' := A \otimes_{\mathbb{Z}} F$$

is an isomorphism. Since A is a finitely generated abelian group, the surjectivity of φ follows from this result using Nakayama's Lemma and the two cases $F = \mathbb{Z}/(p)$ and $F = \mathbb{Q}$. On the other hand, by Corollary 3.4, A is a free abelian group so, as an abelian group, $\varphi(R)$ is free. Thus the kernel of φ is a summand of R, so the injectivity of φ follows from the injectivity,

for every choice of F, of φ'. Using Lemma 7.16 inductively, it also follows that $t_{n-k+1}, \ldots, t_n$ is a regular sequence, proving the Theorem.

To show that $t_{n-k+1}, \ldots, t_n$ is a regular sequence in R' it suffices, since the t_i have positive degree, to show that

$$F[c_1, \ldots, c_k]/J$$

has Krull dimension zero. Since F was arbitrary it suffices, by the Nullstellensatz, to show that if $f_i \in F$ for the c_i in such a way that $t_{n-k+1} = \cdots = t_n = 0$, then all the f_i are zero.

Indeed, after such a substitution we see that $1/(1 + f_1 x + f_2 x^2 \cdots + f_k x^k) = p(x) + q(x)$ where $p(x)$ is a polynomial of degree $\leq n - k$ and $q(x)$ is a rational function vanishing to order at least $n+1$ at 0. We may rewrite this as

$$\frac{1 - p(x)(1 + f_1 x + f_2 x^2 \cdots + f_k x^k)}{1 + f_1 x + \cdots + f_k x^k} = q(x).$$

However, the denominator of the left hand side is nonzero at the origin, and the numerator has degree at most n. Since $q(x)$ vanishes to order at least $n + 1$ at the origin, both sides must be identically zero; that is $p(x) = 1, q(x) = 0$, and thus all $f_i = 0$ as required. (See Exercise 7.26 for a more explicit version.)

Combining this information with Proposition **??** we get:

- The dimension of R' (as a vector space over F) is $\binom{n}{k}$;
- The highest degree d such that $R'_d \neq 0$ is $k(n-k)$.
- Since a complete intersection is Gorenstein (Eisenbud [1995] Theorem ****) every nonzero ideal of R' contains $R'_{k(n-k)}$.

.

We now return to the map φ'. By Corollary 3.6, the rank of $A^*(G(k,n))$ is also $\binom{n}{k}$; thus to show that φ' is an isomorphism, it suffices to show that its kernel is zero. We know that $(\sigma_{1^k})^{n-k}$ is in the image of φ', so $\operatorname{Ker} \varphi'$ does not contain $R_{k(n-k)}$. Since $R'_{k(n-k)}$ is the socle of R', the kernel of φ' must be zero. □

We have used the following two results from commutative algebra:

Proposition 7.15. *Suppose that F is a field and that*

$$T = F[x_1, \ldots, x_k]/(g_1, \ldots, g_k)$$

is a zero-dimensional graded complete intersection with $deg x_i = \delta_i > 0$ and $\deg g_i = \epsilon_i > 0$. *The Hilbert series of T is*

$$H_T(d) := \sum_{u=0}^{\infty} \dim_F T_u d^u = \frac{\prod_{i=1}^{k}(1 - d^{\epsilon_i})}{\prod_{i=1}^{k}(1 - d^{\delta_i})}$$

The degree of the socle of T is $\sum_{i=0}^{k} \epsilon_i - \sum_{i=0}^{k} \delta_i$, and the dimension of T is

$$\dim_F T = \frac{\prod_{i=1}^{k}(\epsilon_i - 1)}{\prod_{i=1}^{k}(\delta_i - 1)}.$$

Proof. We begin with the Hilbert series. The polynomial ring $F[x_1, \ldots, x_k]$ is the tensor product of the polynomial rings in 1 variable $F[x_i]$ so

$$H_{F[x_1,\ldots,x_k]}(d) := \frac{1}{\prod_{i=1}^{k}(1 - d_i^{\delta})}.$$

We can put in the relations one by one using the exact sequences

$$\begin{aligned} 0 \longrightarrow F[x_1, \ldots, x_k]/(g_1, \ldots, g_i)(-\epsilon_i) &\xrightarrow{g_{i+1}} F[x_1, \ldots, x_k]/(g_1, \ldots, g_i) \\ &\longrightarrow F[x_1, \ldots, x_k]/(g_1, \ldots, g_{i+1}) \longrightarrow 0, \end{aligned}$$

and using induction we see that

$$H_T(d) = H_{F[x_1,\ldots,x_k]/(g_1,\ldots,g_k)}(d) = \frac{\prod_{i=1}^{k}(1 - d^{\epsilon_i})}{\prod_{i=1}^{k}(1 - d^{\delta_i})}.$$

A priori this is a rational function of degree $s := \sum_{i=1}^{k} \epsilon_i - \sum_{i=1}^{k} \delta_i$. Since we know from the computation above that T is a finite dimensional vector space over F, the Hilbert series must be a polynomial. Thus it is a polynomial of degree s, so the largest degree in which T is nonzero is s.

The dimension of T is the value of $H_T(d)$ at $d = 1$. The product $(1-d)^k$ obviously divides both numerator and denominator of the expression for the Hilbert series above. After dividing, we get:

$$H_T(d) = \frac{\prod_{i=1}^{k} \sum_{j=0}^{\epsilon_i - 1} d^j}{\prod_{i=1}^{k} \sum_{j=0}^{\delta_i - 1} d^j}.$$

Setting $d = 1$ in this expression gives us the desired result. □

The other result from commutative algebra that we used is a version of the fact that regular sequences in a local ring can be permuted (Eisenbud [1995], Corollary 17.2). The same result holds in the local case when every element of the regular sequence has positive degree, but the case we need is slightly different, since one element of the regular sequence is an integer. The result may also be viewed as a variation on the local criterion of flatness (Eisenbud [1995]Section 6.4).

Lemma 7.16. *Suppose that R is a finitely generated graded algebra over $\mathbb{Z}$, with algebra generators in positive degrees, and that $f \in R$ is a homogeneous element. If R is free as a $\mathbb{Z}$-module and $f \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ is a monomorphism for every prime p, then f is a monomorphism and $R/(f)$ is free as a $\mathbb{Z}$-module as well.*

Proof. Since R is free, so is every submodule; in particular fR is free, and the kernel K of multiplication by f is a free summand of R. It follows that $K \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \subset R \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$. Since this ideal is obviously contained in the kernel of multplication by f on $R \otimes \mathbb{Z}/(p)$, we see that $K \otimes_{\mathbb{Z}} \mathbb{Z}/(p) = 0$. Since K is free, this implies that $K = 0$ as well; that is, f is a nonzerodivisor on R, and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R(-1) & \xrightarrow{f} & R & \longrightarrow & R/fR & \longrightarrow & 0 \\ & & \downarrow p & & \downarrow p & & \downarrow p & & \\ 0 & \longrightarrow & R(-1) & \xrightarrow{f} & R & \longrightarrow & R/fR & \longrightarrow & 0 \end{array}$$

has exact rows. A diagram chase (the *Snake Lemma*) shows that p is a nonzerodivisor on R/fR. Since p was an arbitrary prime, R/fR is a torsion free abelian group. Since R is finitely generated and f is homogenous, R/fR is a direct sum of finitely generated generated abelian groups, and torsion-freeness implies freeness. □

7.6 Exercises

Exercise 7.17. Suppose that $E = U \times K^n$ is a trivial vector bundle on a smooth affine variety U, and that σ is a section given by functions

$$f_1, \ldots, f_n \in \mathcal{O}_X(U)$$

Show that σ is generically transverse to the zero section of E in the sense defined in the text if and only if the differentials df_i are independent at a general point of any component of the zero locus of σ in U.

Exercise 7.18. Show that that the collection $\sigma_1, \ldots, \sigma_k$ is generically transverse if and only if, (after replacing the σ_i by k general linear combinations of them), the section σ_j is generically transverse as section of the bundle $E/\langle \sigma_1, \ldots, \sigma_{j-1} \rangle$ on the complement $X \setminus V(\sigma_1 \wedge \cdots \wedge \sigma_{j-1})$.

Exercise 7.19. We saw in Chapter2 that the tangent bundle of the Grassmannian $G = G(k, V)$ is given by

$$T_G \cong S^* \otimes Q,$$

where S and Q are the universal sub- and quotient bundles on G. In case $k = 1$—that is, $G = \mathbb{P}V$—use this and the formula ** above for the Chern class of a tensor product with a line bundle to derive the Chern classes of $T_{\mathbb{P}^r}$ again.

Exercise 7.20. Find the Betti numbers of the smooth intersection of a quadric and a cubic hypersurface in $\mathbb{P}^4$, and of the intersection of three

quadrics in $\mathbb{P}^5$. (Both of these are examples of *K3 surfaces*, which are diffeomorphic to a smooth quartic surface in $\mathbb{P}^3$.)

Exercise 7.21. Find the Betti numbers of the smooth intersection of two quadrics in $\mathbb{P}^5$. This is the famous *quadric line complex*, about which you can read more in [GH], Chapter 6.

Exercise 7.22. Find the Euler characteristic of a smooth hypersurface of bidegree (a, b) in $\mathbb{P}^m \times \mathbb{P}^n$.

Exercise 7.23. Show that the cohomology groups of a smooth quadric threefold $Q \subset \mathbb{P}^4$ are isomorphic to those of $\mathbb{P}^3$ ($\mathbb{Z}$ in even dimensions, 0 in odd), but its cohomology ring is different (the square of the generator of $H^2(Q, \mathbb{Z})$ is twice the generator of $H^4(Q, \mathbb{Z})$). (This is a useful example if you're ever teaching a course in algebraic topology.)

Exercise 7.24. Prove that $\deg c_{\dim \mathbb{G}(k,n)} T_{\mathbb{G}(k,n)} = \binom{n+1}{k+1}$. ****what's an easy way to see this?****

Exercise 7.25. Calculate all the Chern classes of the tangent bundle of $\mathbb{G}(1, 3)$:

(a) by applying the splitting sprinciple; and

(b) by using the Pl"ucker embedding of G as a quadric hypersurface in $\mathbb{P}^5$.

Check that these two results agree!

Exercise 7.26. With notation as in Theorem 7.14 and the convention that $t_i = 0$ for $i < 0$, show that for $i \geq 1$

$$t_i = \sum_{j=1}^{k} (-1)^j c_j t_{i-j} \quad \text{when } i \geq 1.$$

Deduce that J contains all t_j with $j > n - k$ (this is a little stronger than the argument given in the text, which shows that these t_i are in the radical of J. Use this to show that if we substitute elements f_i for the c_i making $t_{n-k+1} = \cdots = t_n = 0$ then the inverse of the polynomial in one variable $p(x) := 1 + f_1 x + \cdots + f_k x^k \in F[x]$ is itself a polynomial. It follows that $p(x)$ is constant, so all f_i are zero.

Exercise 7.27. Let $S \subset \mathbb{P}^4$ be a smooth complete intersection of hypersurfaces of degrees d and e, and let $Y \subset \mathbb{P}^4$ be any hypersurface of degree f containing S. Show that if f is not equal to either d or e, then Y is necessarily singular.

(Hint: Assume Y is smooth, and apply the Whitney formula to the sequence

$$0 \to N_{S/Y} \to N_{S/\mathbb{P}^4} \to N_{Y/\mathbb{P}^4}|_S \to 0$$

to arrive at a contradiction.)

Exercise 7.28. Let $S \subset \mathbb{P}^n$ be a smooth k-dimensional complete intersection of hypersurfaces of degrees $d_1, \ldots d_{n-k}$, and let $Y \subset \mathbb{P}^n$ be any hypersurface of degree e containing S. Show that if $k > n/2$ and $e \neq d_i$ for any i, then Y is necessarily singular.

8

Lines on Hypersurfaces

Keynote Questions

(a) Let $X \subset \mathbb{P}^4$ be a general quintic hypersurface. How many lines $L \subset \mathbb{P}^4$ does X contain?

(b) Let $\{X_t \subset \mathbb{P}^3\}_{t\in\mathbb{P}^1}$ be a general pencil of quartic surfaces—that is, let F and G be general homogeneous polynomials of degree 4 in four variables, and set $X_t = V(t_0F + t_1G) \subset \mathbb{P}^3$. How many of the surfaces X_t contain a line?

(c) Now let $\{X_t \subset \mathbb{P}^3\}_{t\in\mathbb{P}^1}$ be a general pencil of cubic surfaces, and consider the locus of all lines contained in some member of this pencil. What is the degree of the surface swept out by these lines? What is the genus of the curve parametrizing them?

8.1 What to Expect

The sort of questions we will consider in this chapter are: when would we expect a general hypersurface $X \subset \mathbb{P}^n$ of degree d to contain lines? What dimensional family of lines we would expect it to contain? When the dimension is zero, how many lines will there be?

It's easy to see how one might try to answer the first of these questions. First, by way of notation, for any subscheme $X \subset \mathbb{P}^n$ let $F_k(X) \subset \mathbb{G}(k, n)$

be the locus of k-planes $\Lambda \subset \mathbb{P}^n$ contained in X. That is is a closed subset of $\mathbb{G}(k,n)$ is elementary (see for example Harris [1992]); shortly we'll see how to endow it with a natural scheme structure. Let $\mathbb{P}^N$ be the projective space parametrizing hypersurfaces of degree d in $\mathbb{P}^n$ (here $N = \binom{n+d}{n} - 1$) and introduce the incidence correspondence

$$\Phi = \{(X, L) \in \mathbb{P}^N \times \mathbb{G}(1,n) \mid L \subset X\}.$$

Consider the line $L \subset \mathbb{P}^n$ given as the locus $\{X_2 = \cdots = X_n = 0\}$. The set of hypersurfaces containing L is a linear subspace of $\mathbb{P}^N$ of codimension $d+1$, since it corresponds to the linear space of those forms of degree d that do not involve the $d+1$ monomials $X_0^i X_1^{d-i}$. Since any two lines differ by a linear automorphism of $\mathbb{P}^n$, the fibers of the projection map $\Phi \to \mathbb{G}(1,n)$ are all projective spaces of dimension $N-d-1$. Since Φ is projective, and $\mathbb{G}(1,n)$ is irreducible of dimension $2n-2$, it follows that Φ is irreducible of dimension $(2n-2)+(N-d-1) = N+2n-3-d$.

This implies immediately that *in case $d > 2n-3$, the general hypersurface of degree d in $\mathbb{P}^n$ contains no lines* and that, if the general hypersurface of degree $d \leq 2n-3$ has any lines at all, then the family of lines it contains will have dimension $2n-3-d$. (One of the things we'll see in this Chapter is that it is indeed the case that if $d \leq 2n-3$ then every hypersurface $X \subset \mathbb{P}^n$ of degree will contain lines; this is Corollary 8.12 below.) Some examples—the fact that a smooth quadric surface in $\mathbb{P}^3$ has a one-parameter family of lines, for examples, or that a general cubic surface has a finite number of them—may already be familiar.

Assuming for a moment that the locus $F_1(X) \subset \mathbb{G}(1,n)$ of lines on a general $X \subset \mathbb{P}^n$ has the expected dimension, our next question will be to determine the class of the locus $F_1(X)$ in the Chow ring of the Grassmannian. In particular, in case $d = 2n-3$—so that we will have a finite number of lines on a general such hypersurface X—this amounts to asking how many lines X will contain. The first case of this is of course the famous count of the 27 lines on a cubic surface in $\mathbb{P}^3$, and we will carry out the calculation in this case and many others.

As we have already shown, the general surface in $\mathbb{P}^3$ of degree $d \geq 4$ contains no lines. But we can say more, using the same sort of incidence correspondence argument made above. For example, we'll see in Exercise 8.27 that a general surface $S \subset \mathbb{P}^3$ of degree $d \geq 4$ containing a line contains only one; we can conclude in particular that the locus $\Sigma \subset \mathbb{P}^N$ of surfaces that do contain a line has codimension $d-3$.

We can make an analogous estimate of the dimension of the variety of k-planes on a hypersurface $X \subset \mathbb{P}^n$ of degree d. Let

$$\Phi = \{(X, \Lambda) \in \mathbb{P}^N \times \mathbb{G}(k,n) \mid \Lambda \subset X\}$$

be the incidence correspondence. By the same argument as in the case of lines, the fibers of $\Phi \to \mathbb{G}(k,n)$ have dimension $N - \binom{k+d}{k}$, and hence Φ is

irreducible of dimension

$$\dim(\Phi) = N + (k+1)(n-k) - \binom{k+d}{k}.$$

By the Principal Ideal Theorem **??** the dimension of any component of the family of k-planes on a hypersurface of degree d in $\mathbb{P}^n$ is at least

$$\varphi(n,d,k) := (k+1)(n-k) - \binom{k+d}{k},$$

and, by analogy with the case of lines, we might hope that equality holds for a general such hypersurface.

However, this is false in cases when $\varphi(n,d,k) > 0$ but $n \leq 2k$, for example when $n = 4, k = 2$ and $d = 2$. In such cases any hypersurface of degree > 1 in $\mathbb{P}^n$ containing a k-plane is singular! This is an elementary fact, but worth establishing: we have

Proposition 8.1. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 1$. If $L \cong \mathbb{P}^k \subset X$ is a k-plane on X, then*

$$k \leq \frac{n-1}{2}.$$

Proof. If F is a polynomial of degree d that vanishes on the k-plane L defined by $Z_{k+1} = \cdots = Z_n = 0$, then the derivatives $\partial F/\partial Z_0, \ldots, \partial F/\partial Z_k$ vanish identically on L. Thus the hypersurface $X = V(F)$ is singular wherever the $n-k$ derivatives $\partial F/\partial Z_{k+1}, \ldots, \partial F/\partial Z_n$ vanish on L. But when $n \leq 2k$ (so that $n-k \leq k$) and $d > 1$, these derivatives must have a common zero on L, so $F = 0$ is singular at some point of L. Put another way: the Gauss map $\mathcal{G}_X : X \to \mathbb{P}^{n*}$ carries L to its annihilator $L^\perp \cong \mathbb{P}^{n-k-1} \subset \mathbb{P}^{n*}$; but there are no nonconstant regular maps from a projective space to a strictly lower-dimensional projective space. □

Thus, for example, a smooth quadric hypersurface in $\mathbb{P}^4$ cannot contain a 2-plane, even though the "expected" dimension of $F_2(X)$ is $3 \cdot 2 - \binom{4}{2} \geq 0$.

A parenthetical note: it's not immediately relevant, but we should mention at least that much more is true: by the Lefschetz hyperplane theorem (see Appendix 22), if $X \subset \mathbb{P}^n$ is smooth and $Y \subset X$ is any subvariety of dimension $k > (n-1)/2$, then in fact

$$\deg(X) | \deg(Y).$$

8.2 Fano Schemes and Chern Classes

To formalize the arguments above, we give the set of k-planes on a projective scheme $X \subset \mathbb{P}^n$ the structure of a subscheme of the Grassmannian

$\mathbb{G}(k,n)$, called the *Fano scheme* $F_k(X)$. In the case where X is a hypersurface, as in the case of cubic surfaces treated in Section **??**, the scheme structure is easy to define. It would be difficult to write explicitly the equations defining $F_k(X) \subset \mathbb{G}(k,n)$ globally, because it's given, in the case of X a hypersurface, as the zero locus of a section of a certain vector bundle. (The general case follows by writing an arbitrary scheme X as the intersection of hypersurfaces Z_i and taking $F_1(X) = \cap F_1(Z_i)$.) We'll see in Section 8.6, however, how to write down local equations for it.

First, the condition that the hypersurface X given as the zero locus of a homogeneous polynomial F contain a particular k-dimensional linear space L is that, under the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_L(d))$$

the image of F is 0. To describe the scheme $F_k(X) \subset \mathbb{G}(k,n)$ of k-planes on X, we will realize the family of vector spaces $H^0(\mathcal{O}_L(d))$, with varying k-planes L, as a vector bundle E of rank $\binom{k+d}{k}$ on the Grassmannian $\mathbb{G}(k,n)$ in such a way that the images of F in these vector spaces are the values of a section σ_F of the bundle. The scheme $F_k(X)$ is defined to be the zero locus $V(\tau_F)$ of this section, and when the codimension of $F_k(X)$ is equal to the dimension of this bundle, the class $[F_k(X)] \in A(\mathbb{G}(k,n))$ is thus the top Chern class of the bundle.

To carry this out, recall first our convention that for a linear subspace $\Lambda \cong \mathbb{P}^k \subset \mathbb{P}V$ we denote the corresponding $(k+1)$-dimensional vector subspace of V by $\tilde{\Lambda}$. We have a tautological sequence of bundles on $G = \mathbb{G}(k,n)$ of the form

$$0 \to S \to V \otimes \mathcal{O}_G \to Q \to 0,$$

where $V \otimes \mathcal{O}_G$ is the trivial bundle on G with fiber V. The fiber of S over the point in $\mathbb{G}(k,n)$ representing the subspace $L \subset \mathbb{P}V$ is the corresponding space $\tilde{L} \subset V$. The fiber of the dual bundle S^* at L is thus the space of linear forms on $\tilde{L}$, that is to say $H^0(\mathcal{O}_{\mathbb{P}\tilde{L}}(1))$; and the dual map $V^* \otimes \mathcal{O}_G \to S^*$ evaluated at a point L, takes a linear form $\varphi \in V^*$, thought of as a constant section of the trivial bundle $V^* \otimes \mathcal{O}_G$, to the restriction of φ to $\tilde{L}$. The vector space of forms of degree d is $\operatorname{Sym}_d V^* = H^0(\mathcal{O}_{\mathbb{P}V}(d))$, and the induced map on symmetric powers

$$\operatorname{Sym}_d V^* \to \operatorname{Sym}_d S^*$$

evaluated at L takes a form F of degree d to its restriction to $\tilde{L}$, as required. We write $\tau_F \in H^0(\operatorname{Sym}_d S^*)$ for the global section of $\operatorname{Sym}_d S^*$ that is the image of F, and we define $F_k(X) \subset \mathbb{G}(k,n)$ to be the zero locus of this section, as promised.

In particular, we see from the definition that $F_k(X)$ is defined locally by $\operatorname{rank}(\operatorname{Sym}_d S^*) = \binom{k+d}{k}$ functions; thus, by the Principal Ideal Theorem,

the dimension of any component of $F_k(X)$ in $\mathbb{G}(k,n)$ is at least

$$\dim \mathbb{G}(k,n) - \operatorname{rank}(\operatorname{Sym}_d S^*) = (k+1)(n-k) - \binom{k+d}{k} = \varphi(n,k,d),$$

recovering the computation made in the section above.

8.2.1 Counting lines on cubics

Let's see how this will work for the case of lines on a cubic surface $X \subset \mathbb{P}^3$. In the language above, we want to compute something about the Fano scheme $F_1(X)$ in the Grassmannian $G = \mathbb{G}(1,3)$. As we've seen, the Chern class of S^* is given by

$$c(S^*) = 1 + \sigma_1 + \sigma_{1,1}.$$

Since S has rank 2, the rank of $\operatorname{Sym}_3 S^*$ is 4, so we want to compute $c_4(\operatorname{Sym}_3 S^*)$. To do this, we will apply the splitting principle. We pretend that S^* splits into a direct sum of two line bundles L and M, with Chern classes

$$c(L) = 1 + \alpha \qquad \text{and} \qquad c(M) = 1 + \beta;$$

by the Whitney sum formula, we must then have

$$c(S^*) = (1+\alpha)(1+\beta)$$

so that

$$\alpha + \beta = \sigma_1 \qquad \text{and} \qquad \alpha \cdot \beta = \sigma_{1,1}.$$

In these circumstances, the bundle $\operatorname{Sym}_3 S^*$ will likewise split:

$$\operatorname{Sym}_3 S^* = L^3 \oplus (L^2 \otimes M) \oplus (L \otimes M^2) \oplus M^3$$

so that we have

$$c(\operatorname{Sym}_3 S^*) = (1+3\alpha)(1+2\alpha+\beta)(1+\alpha+2\beta)(1+3\beta).$$

In particular, the top Chern class can be written

$$\begin{aligned} c_4((\operatorname{Sym}_3 S^*) &= 3\alpha(2\alpha+\beta)(\alpha+2\beta)3\beta \\ &= 9\alpha\beta(2\alpha^2 + 5\alpha\beta + 2\beta^2) \\ &= 9\alpha\beta(2(\alpha+\beta)^2 + \alpha\beta) \\ &= 9\sigma_{1,1}(2\sigma_1^2 + \sigma_{1,1}) \\ &= 27\sigma_{2,2}; \end{aligned}$$

that is,

$$\deg\left(c_4((\operatorname{Sym}_3 S^*)\right) = 27.$$

The whole Chern class of $\operatorname{Sym}_3 S^*$ can likewise be computed by hand, or with the following commands in Macaulay2:

```
loadPackage "Schubert2"
G = flagBundle({2,2}, VariableNames=>{s,q})
-- sets G to be the Grassmannian of 2-planes in 4-space,
-- and gives the names $s_i$ and $q_i$ to the Chern classes
-- of the sub and quotient bundles, respectively.
(S,Q)=G.Bundles
-- names the sub and quotient bundles on G
chern symmetricPower(3,dual S)
```

which returns the output

```
                    2                          2
o4 = 1 + 6q  + (21q  - 10q ) + 42q q  + 27q
           1       1      2       1 2      2

                QQ[][s , s , q , q ]
                      1   2   1   2
o4 : ----------------------------------------------
     (s  + q , s  + s q  + q , s q  + s q , s q )
       1    1   2    1 1    2   2 1    1 2   2 2
```

The answer, on the first output line "o4" is written in terms of the Chern classes $q_i := c_i(Q)$, which generate the (rational) Chow ring of the Grassmannian, described on the second output line "o4".

This computation is only useful if the dimension of the family of lines on X is equal to the dimension of $\mathbb{G}(1,3)$ minus 4—that is, zero—and we deduce that if a cubic surface $X \subset \mathbb{P}^3$ contains only finitely many lines, then the number of these lines, counted with the appropriate multiplicity (that is, the degree of the corresponding component of the zero-scheme of τ_F), is 27. In fact, it turn out (as we will see shortly) that the Fano scheme $F_1(X)$ of a smooth cubic surface X is necessarily of dimension zero and smooth, so the actual number of lines is always 27. In the next section we will develop a general technique that will allow us to prove this statement and much more.

8.3 Tangent spaces to Fano schemes

As in the example of cubic surfaces, in order to translate the Chern class calculation into an actual count of lines we need to know something about the local structure of the Fano scheme of lines on a hypersurface—in particular, when it's smooth and/or reduced. Happily, we can give a simple description of the tangent space $T_L F_1(X)$ to the Fano scheme of X a point $L \in F_1(X)$ in terms of the local geometry of X around $L \subset X$, which we'll

do in this section. This will be sufficient to see, for example, that for a general $X \subset \mathbb{P}^n$ of degree $d = 2n - 3$ the Fano scheme $F_1(X)$ is reduced, from which we can deduce the exact number of lines on X. In section 8.6 below, we'll go a little deeper and show how to calculate, for example, the multiplicity of $F_1(X)$ at L by writing down explicit local equations for $F_1(X) \subset \mathbb{G}(1, n)$.

One note: the approach taken here, based on the universal property of the Fano scheme and the correspondence between first-order deformations of a scheme and global sections of its normal bundle, yields strong and very general results, but may seem somewhat abstract when first encountered. The reader who feels this way can look first at Section 8.6 below, where many of these results are derived (in a far more limited setting) by calculations in local coordinates.

8.3.1 Normal Bundles and the Smoothness of the Fano Scheme

We have defined a scheme structure on the set of planes in a projective variety X, and it is a good one, in the sense that we can compute something about it using Chern classes. But there is a much stronger sense in which it is the "right" scheme structure: as defined, the Fano scheme $F_k(X)$ is the Hilbert scheme of k-planes on X, as we'll now prove.

Recall first the fact that the Grassmannian $\mathbb{G}(k, n)$ is the Hilbert scheme of k-planes in $\mathbb{P}^n$: that is, given any morphism of schemes $B \to \mathbb{G}(k, n)$, we can pull back the universal family of k-planes

$$\begin{array}{ccc} \mathcal{L}_0 := \mathbb{P}(S) & \hookrightarrow & \mathbb{P}(\widetilde{V}) = \mathbb{G}(k,n) \times \mathbb{P}^n \\ & \searrow \quad \swarrow & \\ & \mathbb{G}(k,n) & \end{array}$$

to a family of k-planes over B

$$\begin{array}{ccc} \mathcal{L} & \hookrightarrow & B \times \mathbb{P}^n \\ & \searrow \quad \swarrow & \\ & B, & \end{array}$$

and $\mathbb{G}(k, n)$ with the family $\mathcal{L}_0$ *represents the functor of k-planes in* $\mathbb{P}^n$ in the sense that the correspondence between morphisms of a scheme B and families of k-planes over B is one-to-one. In the same sense, we have:

Proposition 8.2. *If $X \subset \mathbb{P}^n$ is a subscheme, then the scheme $F_k(X)$ represents the functor of k-planes on X, in the sense that the correspondence above induces a one-to-one correspondence between morphisms of schemes $B \to F_k(X) \subset \mathbb{G}(k,n)$ and flat families of k-planes $\mathcal{L} \subset B \times X \subset B \times \mathbb{P}^n$.*

It is not hard to show that when B is reduced, any family whose closed fibers are all k-planes is flat; we will not need this. When B is not reduced (as in the next section!) then by a flat family of k-planes we mean a flat family of subschemes of $\mathbb{P}^n$ whose closed fibers are all k-planes.

Proof. This is a tautology: Suppose that X is defined by some homogeneous forms G_i of degrees d_i. Writing τ_{G_i} for the section of $\mathrm{Sym}_{d_i} Q$ that is the image of the form G_i, we see that G_i vanishes on the k-plane L if and only if the sections τ_{G_i} vanish at the point corresponding to L. □

We will make use of this universal property to give a geometric condition for the smoothness of $F_k(X)$ at a given point. Recall that if $Y \subset X$ is a smooth subvariety of the smooth variety X then the *normal bundle* $N_{Y/X}$ of Y in X is the quotient of the map of tangent bundles $N_{Y/X} = \mathrm{coker}\big(T_Y \to T_X|_Y\big)$ induced by the inclusion of $Y \subset X$. Recall also that the Zariski tangent space of a scheme F at a point p is by definition $\mathrm{Hom}_{\mathcal{O}_p}(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathcal{O}_p/\mathfrak{m}_p)$, where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_p$ of F at p.

Theorem 8.3. *Suppose that $L \subset X$ is a k-plane in a smooth variety $X \subset \mathbb{P}^n$, and let $[L] \in F_k(X)$ be the corresponding point. The Zariski tangent space of $F_k(X)$ at $[L]$ is $H^0(N_{L/X})$.*

The result is intuitively plausible if we think of a section of $N_{L/X}$ as providing an infinitesimal normal vector at each point in X, with a corresponding infinitesimal motion of X.

****insert picture****

We postpone the proof that this intuition is correct until the next section and proceed now to explain how the result can be used.

Corollary 8.4. *Suppose that $L \subset X$ is a k-plane in a smooth variety $X \subset \mathbb{P}^n$, and let $[L] \in F_k(X)$ be the corresponding point. Let d be the dimension of $F_k(X)$ at $[L]$; that is, d is the dimension of the largest irreducible family of k-planes on X that includes L. We have $d \le h^0(N_{L/X})$, and $F_k(X)$ is smooth at $[L]$ if and only if $d = h^0(N_{L/X})$.*

Proof of Corollary 8.4. By the Principal Ideal Theorem the dimension of the Zariski tangent space of a local ring is always at least the dimension of the ring; and equality holds if and only if the ring is regular. See Eisenbud [1995]. □

To apply it we need to be able to compute normal bundles, and this is often easy. For example, we have:

Proposition 8.5. *Suppose that $Y \subset X$ is a smooth subvariety of a smooth variety X.*

(a) $N_{Y/X} = Hom_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$.

(b) *If $Y \subset X \subset W$ are all smooth varieties, then there is an induced map $\alpha : N_{Y/W} \to N_{X/W}|_Y$ such that $N_{Y/X} = \operatorname{Ker} \alpha$.*

(c) *If Y is a divisor on X then $N_{Y/X} = \mathcal{O}_X(Y)$. More generally, if Y is the zero locus of a section of a bundle E of rank e on X, and Y has codimension e in X, then*

$$N_{Y/X} = E|_Y.$$

(d) *Suppose that $Y \subset X \subset \mathbb{P}^n$ are complete intersections of hypersurfaces with homogeneous ideals*

$$I_X = (g_1, \dots g_s) \subset I_Y = (f_1, \dots, f_t); \qquad g_i = \sum_j a_{i,j} f_j.$$

If $\deg f_i = \varphi_i$ and $\deg g_i = \gamma_i$, then

$$N_{Y/\mathbb{P}^n} = \bigoplus_{i=1}^{t} \mathcal{O}_Y(\varphi_i); \qquad N_{X/\mathbb{P}^n} = \bigoplus_{i=1}^{s} \mathcal{O}_X(\gamma_i)$$

and the induced map $\alpha : N_{Y/\mathbb{P}^n} \to N_{X/\mathbb{P}^n}|_Y$ is given by the matrix $(a_{j,i})$.

We remark that in case $Y \subset X$ is a smooth subvariety, the normal bundle can be used to compute the canonical bundle of Y from that of X: in fact

$$\omega_Y = \omega_X \otimes_{\mathcal{O}_X} \bigwedge^{\operatorname{codim} Y} N^*_{Y/X}.$$

One might call this the general adjunction formula, since it generalizes the classical adjunction formula for the case when Y is a divisor: $\omega_Y = \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(Y) = \omega_X(Y)|_Y$, or, in the more usual form for divisors, $K_Y = (K_X + Y)|_Y$. The general adjunction formula follows from the exact sequences in the proof of Part (a) below. The result holds more generally for locally complete intersection subvarieties of Cohen-Macaulay varieties. See also Hartshorne [1977] Proposition 8.20.

In light of the formula in Part (a), we define the *normal sheaf* of any subscheme Y of a scheme X to be $N_{Y/X} = Hom_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$.

Proof. **(a):** For any inclusion of subschemes $Y \subset X$ there is a right exact sequence involving the cotangent sheaves of X and Y:

$$\mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2 \xrightarrow{d} \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0,$$

where d is the map taking the class of a (locally defined) function $f \in \mathcal{I}_{Y/X}$ to its differential $df \in \Omega_X|_Y$; see for example Eisenbud [1995] Proposition 16.12. Since X and Y are smooth, Y is locally a complete intersection in X, so $\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}$ is a locally free sheaf on Y of rank equal to $\dim X - \dim Y = \operatorname{rank} \Omega_X|_Y - \operatorname{rank} \Omega_Y$. Thus the left hand map is a monomorphism, and we actually have an exact sequence

$$0 \longrightarrow \mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X} \xrightarrow{d} \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0,$$

of bundles. Since Y is smooth Ω_Y is locally free, so dualizing preserves exactness, and we get an exact sequence

$$0 \longleftarrow Hom_{\mathcal{O}_Y}(\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}, \mathcal{O}_Y) \longleftarrow T_X|_Y \longleftarrow T_Y \longleftarrow 0,$$

where the right hand map is the differential of the inclusion $Y \subset X$, proving that $N_{Y/X} = Hom_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}^2_Y, \mathcal{O}_Y)$.

(b): Apply the snake lemma applied to the diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & \operatorname{Ker}\alpha & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_Y & \longrightarrow & T_W|_Y & \longrightarrow & N_{Y/W} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & T_X|_Y & \longrightarrow & T_W|_Y & \longrightarrow & N_{X/W}|_Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & N_{Y/X} & & 0 & & & &
\end{array}$$

(c) The first formula follows at once from Part (a), since in that case $\mathcal{I}_{Y/X} = \mathcal{O}_X(-Y)$, and taking the dual of a bundle commutes with restriction.

We will give both a geometric and a (more general) algebraic argument for the second statement Part(c). Suppose that Y is the zero locus of a section τ of E.

Geometry: Let Z be the total space of the bundle E. The tangent bundle to Z restricted to the zero section $X \subset Z$ is $T_X \oplus E$.

****picture here****

Along the zero locus Y of τ, the derivative $D\tau$ of τ is thus a map $T_X|_Y \to T_X|_Y \oplus E_Y$. Since the vertical component of $D\tau$ is zero along Y, the composite

$$T_Y \longrightarrow T_X|_Y \xrightarrow{D\tau} T_X|_Y \oplus E_Y \longrightarrow E_Y$$

is zero. Locally at each point $y \in Y$ the image of $(T_X)_y$ in $(T_X)_y \oplus E_y$ is the tangent space to $\tau(X) \subset Z$. Since Y is smooth of codimension equal to the rank of E the manifold $\tau(X)$ meets the zero locus $X \subset Z$ transversely. This means that $(T_X)_y$ projects onto E_y, and tells us that the composite map of bundles

$$T_X|_Y \xrightarrow{D\tau} T_X|_Y \oplus E_Y \to E_Y$$

is surjective. Considering the ranks, it follows that the sequence

$$0 \longrightarrow T_Y \longrightarrow T_X|_Y \longrightarrow E_Y \longrightarrow 0$$

is exact; that is, $N_{Y/X} = E_Y$.

Algebra: We may think of τ as defining the map $\mathcal{O}_X \to E$ that sends $1 \in \mathcal{O}_X$ to $\tau \in E$. Dualizing, the statement that Y is the zero locus of τ means that the ideal sheaf $\mathcal{I}_{Y/X}$ is the image of the map $\tau^* : E^* \to \mathcal{O}_X$. Since the codimension of Y is e we see that Y is locally a complete intersection. Thus the kernel of τ^* is generated by the Koszul relations; that is, the sequence

$$\cdots \wedge^2 E^* \xrightarrow{\kappa} E^* \longrightarrow \mathcal{I}_{Y/X} \longrightarrow 0$$

is exact, where $\kappa(e \wedge f) = \tau^*(e)f - \tau^*(f)e$. Because the coefficients in the map κ lie in $\mathcal{I}_{Y/X}$ they become zero on tensoring with $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}_Y$, so we we get the right exact sequence

$$\cdots \wedge^2 E^*|_Y \xrightarrow{0} E^*|_Y \longrightarrow \mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X} \longrightarrow 0.$$

This shows that $E^*|_Y \cong \mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}$.

(d) The complete intersection X is the zero locus of the section $(g_1, \ldots, g_s)$ of the bundle $\mathcal{O}_{\mathbb{P}^n}(\gamma_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(\gamma_s)$ and similarly for Y. Using the formula of Part(c) we see that

$$N_{X/\mathbb{P}^n} = \bigoplus_{i=1}^{s} \mathcal{O}_X(\gamma_i),$$

and similarly for Y. The identification of α follows at once from Part (a). □

To illustrate the power of these results we return to the question of lines on smooth surfaces in $\mathbb{P}^3$:

Corollary 8.6. *If $X \subset \mathbb{P}^3$ is a smooth surface of degree ≥ 3 then $F_1(X)$ is smooth and zero-dimensional. In particular, X contains only finitely many lines, and if $d = 3$ then X contains exactly 27 distinct lines.*

Proof. Suppose $L \subset X$ is a line. By Corollary 20.5, the self-intersection number of L on X is negative, so the normal bundle $N_{L/X}$ is a line bundle of negative degree. It follows that $h^0(N_{L/X}) = 0$, and Corollary 8.4 now implies that L is isolated and $F_1(X)$ is smooth at L. □

In particular, in the case of the cubic surface, the fact that the class of the Fano scheme is 27 points implies, with this result, that the Fano scheme actually consists of 27 reduced points.

We can actually strengthen this a little; see Exercise 8.35 for a statement.

8.3.2 First-order Deformations

The proof of Theorem 8.3 involves the very important idea of *first order infinitesimal deformations*, which is the main content of this section. First, however, we explain the connection with the Zariski tangent space. Although the material in this section can be made very general, we will stick with the case of schemes of finite type over an algebraically closed field K, and our points will all be closed points. We write $\mathfrak{m}_{F,l}$ for the maximal ideal of the local ring $\mathcal{O}_{F,l}$ of F at l. *Throughout this section we let R_n denote the ring $R = K[\epsilon_1, \ldots, \epsilon_n]/(\epsilon_1, \ldots, \epsilon_n)^2$ and set $T_n = \operatorname{Spec} R_n$, a scheme with one closed point and n-dimensional Zariski tangent space.* To simplify notation we set $\epsilon := \epsilon_1, \ldots, \epsilon_n$. The reader will lose nothing essential by focusing on the case $n = 1$—as the notation suggests—though the case $n = 2$ is briefly needed in the proof of Theorem 8.3.

Lemma 8.7. *If $l \in F$ is a point on a scheme, then the Zariski tangent space $Hom_{\mathcal{O}_{F,l}}(\mathfrak{m}_{F,l}/\mathfrak{m}_{F,l}^2, \mathcal{O}_{F,l}/\mathfrak{m}_{F,l})$ of F at l may be identified with the set of morphisms $T_1 \to F$ that take the closed point to l.*

Proof. A giving a morphism $a : T \to F$ as in the Lemma is is equivalent to giving the local map of K-algebras $a^* : \mathcal{O}_{F,l} \to K[\epsilon]/(\epsilon^2)$. Since $a^*(\mathfrak{m}_{F,l} \subset (\epsilon)$ we see that $a^*(\mathfrak{m}_{F,l}^2) = 0$. Since the residue of $\mathcal{O}_{F,l}$ is K the map a^* is determined by the induced map of vector spaces $\mathfrak{m}_{F,l}/\mathfrak{m}_{F,l}^2 \to (\epsilon)$, which we regard as an element of the Zariski tangent space to F at l. Conversely, any map $\mathfrak{m}_{F,l}/\mathfrak{m}_{F,l}^2 \to (\epsilon)$ extends to a local algebra homomorphism $a^* : \mathcal{O}_{F,l} \to K[\epsilon]/(\epsilon^2)$. □

Now the universal property of $F_k(X)$ shows that morphisms $T_n \to F_k(X)$ sending the closed point to a point l are the same as flat families of k-planes in X over the base T_n whose central fiber is the plane L corresponding to l, that is diagrams

$$\begin{array}{ccccc} L & \hookrightarrow & L_{T_n} & \hookrightarrow & X \times T_n \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & \swarrow{\scriptstyle \text{projection}} & \\ \operatorname{Spec} K & \hookrightarrow & T_n & & \end{array}$$

with β flat and α the pull-back of β. It turns out to be easy to describe such families. Here is the general result:

Theorem 8.8. *Suppose that $Y \subset X$ are schemes. There is a one-to-one correspondence between flat families of subschemes of X over the base T_n with central fiber Y and homomorphisms of $\mathcal{O}_Y$-modules $\mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y^n$. In particular, flat families of deformations of Y in X over T_1 correspond to sections of the normal sheaf of Y in X.*

We will use the following easy characterization of flatness over T:

Lemma 8.9. *If M is a (not necessarily finitely generated) module over the ring R_n, then M is flat if and only if the map*

$$M^n \xrightarrow{(\epsilon_1,\dots,\epsilon_n)} M$$

induces an isomorphism $(M/(\epsilon_1,\dots,\epsilon_n)M)^n \cong (\epsilon_1,\dots,\epsilon_n)M$.

Proof. The general criterion of Eisenbud [1995] Proposition 6.1 says that M is flat if and only if the multiplication map $\mu_I : I \otimes_R M \to IM$ is an isomorphism for all ideals I. But every nontrivial ideal of R is a summand of $(\epsilon_1,\dots,\epsilon_n) = (\epsilon)$; and since $(R/(\epsilon))^n \cong (\epsilon)$, the map $\mu_{(\epsilon)}$ may be identified with the given map $(M/(\epsilon)M)^n \to (\epsilon)M$. □

Proof of Theorem 8.8. The problem is local, so we may assume that X and Y are affine. Since any homomorphism $\mathcal{I}_Y \to \mathcal{O}_Y^n$ must annihilate $\mathcal{I}_Y^2$, we may identify a homomorphism $\varphi : \mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y^n$ with the composition $\mathcal{I}_Y \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y^n$. Given this, we choose generators $\{g_i\}$ for $\mathcal{I}_Y$ and associate to φ the ideal $\mathcal{I}_\varphi \subset \mathcal{O}_{X\times T_n} = \mathcal{O}_X[\epsilon]/(\epsilon)^2$ generated by the elements $g_i + \sum_j \tilde{\varphi}_j(g_i)\epsilon_j$, where φ_j is the j-th component of φ and $\tilde{\varphi}_j(g_i)$ is any lift of $\varphi_j(g_i)$ to $\mathcal{O}_X$. Since $(\epsilon)\mathcal{I}_\varphi = (\epsilon)\mathcal{I}$, the ideal $\mathcal{I}_\varphi$ is independent of the lifting chosen.

From $\mathcal{I}_\varphi$ we construct the family

$$\begin{array}{ccccc} Y & \hookrightarrow & Y_{T_n} & \hookrightarrow & X \times T_n \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & \swarrow{\scriptstyle\text{projection}} & \\ \operatorname{Spec} K & \hookrightarrow & T_n & & \end{array}$$

where Y_{T_n} is defined by $\mathcal{I}_\varphi$. If we set all $\epsilon_j = 0$, then $\mathcal{I}_\varphi$ becomes equal to $\mathcal{I}_Y$, so α is indeed the pullback of β.

We may identify $\mathcal{I}_\varphi/(\mathcal{I}_Y(\epsilon))$ with the graph of $\varphi : \mathcal{I}_Y \to \mathcal{O}_Y^n$ in

$$\mathcal{I}_Y \oplus \mathcal{O}_Y^n = \mathcal{I}_Y \oplus \bigoplus \mathcal{O}_Y\epsilon_j \subset \mathcal{O}_X \oplus \bigoplus \mathcal{O}_Y\epsilon_j = \mathcal{O}_X[\epsilon]/((\epsilon)^2 + \mathcal{I}_Y\epsilon).$$

Thus $\mathcal{I}_\varphi \cap \mathcal{O}_X(\epsilon) = \mathcal{I}_Y(\epsilon)$, and it follows that

$$\begin{aligned}(\epsilon)(\mathcal{O}_X/\mathcal{I}_\varphi) &= \mathcal{O}_X(\epsilon)/\mathcal{I}_\varphi \cap \mathcal{O}_X(\epsilon) \\ &= \mathcal{O}_X(\epsilon)/\mathcal{I}_Y(\epsilon) \\ &\cong (\mathcal{O}_X/\mathcal{I}_Y)^n \cong \mathcal{O}_Y^n\end{aligned}$$

proving that $\mathcal{O}_X/\mathcal{I}_\varphi$ is flat over R, by Lemma 8.9.

Conversely, given an R-algebra of the form

$$S := \mathcal{O}_X[\epsilon]/((\epsilon)^2 + \mathcal{I}),$$

the statement that $Y_{T_n} := \operatorname{Spec} S$ has Y as its pullback over $\operatorname{Spec} K \subset T_n$ means that $\mathcal{I}$ is congruent to $\mathcal{I}_Y$ modulo (ϵ). Multiplying by (ϵ) and using $(\epsilon)^2 = 0$ we see that $\mathcal{I} \supset (\epsilon)\mathcal{I}_Y$. If S is flat over R then by Lemma 8.9 we must have $\mathcal{I} \cap (\epsilon) = (\epsilon)\mathcal{I}_Y$. Putting these facts together, we see that $\mathcal{I}/\mathcal{I}_Y(\epsilon)$ is the graph of a homomorphism $\mathcal{I}_Y \to \mathcal{O}_X(\epsilon)/\mathcal{I}_Y(\epsilon) \cong \mathcal{O}_Y^n$, and this is the inverse of the construction above. □

Completion of the Proof of Theorem 8.3. Together Lemma 8.7 and Theorem 8.8 define a one-to-one correspondence between the elements of the Zariski tangent space to $F_k(X)$ at l and the flat families of k-planes in X over T. It remains to show that this correspondence is an isomorphism of vector spaces. The necessary computations for addition and for scalar multiplication are similar, and the one for addition is more complicated, so we will prove additivity and leave the scalar multiplication to the reader.

Let $R_2 = K[\epsilon_1, \epsilon_2]/(\epsilon_1, \epsilon_2)^2$ and set $T_2 = \operatorname{Spec} R_2$. A morphism of schemes $\Psi : T_2 \to F$ sending the closed point to l corresponds to a homomorphism $\psi : \mathfrak{m}_{F,l} \to K\epsilon_1 \oplus K\epsilon_2$ or, equivalently, a pair of homomorphisms $\psi_1, \psi_2 : \mathfrak{m}_{F,l} \to K$, or a pair of morphisms $\Psi_1, \Psi_2 : T \to F$ (one might say: T_2 is the coproduct of T with itself in the category of pointed schemes). Moreover the embedding

$$T \xrightarrow{+} T_2; \qquad \epsilon_i \mapsto \epsilon,$$

which corresponds to factoring out $\epsilon_1 - \epsilon_2$ from R_2, has the property that $\Psi \circ +$ is the morphism corresponding to the sum $\psi_1 + \psi_2 : \mathfrak{m}_{F,l} \to K$.

Let L_{φ_i} be the family obtained by pulling back the universal family along Ψ_i, and let $\varphi_i : \mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y$ be the homomorphism corresponding to this flat family. We have a pullback diagram

$$\begin{array}{ccccc} L_{\varphi_1} & \longrightarrow & L_2 & \longleftarrow & L_{\varphi_2} \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & T_2 & \longleftarrow & T \end{array}$$

of flat families, where $L_2 \to T_2$ is the family obtained by pulling back along Ψ. It suffices to show that the pullback of L_2 along the map $+ : T \to T_2$ is the family $L_{\varphi_1+\varphi_2}$.

Let $\varphi_i : \mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y^2$ be the homomorphism corresponding to L_2, so that the ideal of L_2 is the graph of the corresponding homomorphism $\mathcal{I}_Y \to \mathcal{O}_Y\epsilon_1 \oplus \mathcal{O}_Y\epsilon_2$. Reducing mod ϵ_2 gives us the map corresponding to

φ_1, and similarly for ϵ_1 and φ_2, so φ is in fact the map

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \xrightarrow{\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}} \mathcal{O}_Y^2.$$

Thus if we pull back along the map $+$, that is, factor out $\epsilon_1 - \epsilon_2$, the resulting algebra corresponds to the the map $\varphi = \varphi_1 + \varphi_2$ as required. □

8.3.3 Normal bundles of lines on hypersurfaces

In order to apply the description of the tangent space to a Fano scheme $F_1(X)$ of lines on a hypersurface, we need to know what are the possible normal bundles of lines L on smooth hypersurfaces $X \subset \mathbb{P}^n$ of degree d. Happily, Proposition 8.5 allows us to do exactly this.

Suppose that $L \subset X \subset \mathbb{P}^n$ is a line on a hypersurface of degree d in $\mathbb{P}^n$; for simplicity we'll assume that the characteristic of the ground field does not divide d. Choose coordinates so that the ideal of the line is $I_L = (x_2, \ldots, x_n)$, and let $I_X = (g)$. Since $g \in I_L$, we may write

$$g \equiv \sum_{i=2}^{n} x_i h_i(x_0, x_1) \bmod I_L^2 \quad \text{with } h_i \equiv \frac{1}{d}\partial g/\partial x_i \bmod (I_L).$$

By Part (c) of Proposition 8.5 we see that

$$N_{L/X} = \operatorname{Ker}\big(\mathcal{O}_L^{n-1}(1) \xrightarrow{\alpha=(h_2,\ldots,h_n)} \mathcal{O}_L(d)\big).$$

From the description of the h_i in terms of derivatives, we see that X is smooth along L if and only if α is surjective. On the other hand, Sard's Theorem **??** tells us that if the general member of a linear series is not multiple, then it can only be singular along the base locus of the series, and it follows that the general g with a given map α is smooth except possibly along L. Thus if $\alpha : \mathcal{O}_L^{n-2}(1) \to \mathcal{O}_L(d)$ is any surjective map of sheaves, there is smooth hypersurface X containing L such that $N_{L/X} = \operatorname{Ker}\alpha$.

For example when $d = 2n - 3$ so that $h^0(\mathcal{O}_L^{n-1}(1)) = h^0(\mathcal{O}_L(d))$, the choice

$$\alpha = (x_0^{d-1}, x_0^{d-3}x_1^2, \ldots, x_1^{d-1})$$

makes the map $H^0(\alpha) : H^0(\mathcal{O}_L^{n-1}(1)) \to H^0(\mathcal{O}_L(d))$ an isomorphism, and it follows that for hypersurfaces inducing this map α we have $H^0(N_{L/X}) = 0$. Since the rank of a linear transformation is upper semicontinuous as the transformation varies, this will also be true for the general hypersurface of degree d containing L.

In fact, for any $n \geq 3$ and $d \geq 1$, it is not too hard to find a matrix of monomials $\alpha = (m_1, \ldots, m_{n-1})$ such the map $\alpha : \mathcal{O}_L^{n-2}(1) \to \mathcal{O}_L(d)$ is

a surjection of sheaves and is either an injection or a surjection on global sections, depending on the sign of $2n-3-d$.

Here is a sharp result that gives an overview of the possibilities at the expense of requiring a little more technique. The following result represents the crucial step in the proof of Theorem 8.15 below.

Proposition 8.10. *Suppose that $n \geq 3$ and $d \geq 1$. There exists a smooth hypersurface X in $\mathbb{P}^n$ of degree d, and a line $\mathbb{P}^1 \cong L \subset X$ such that $N_{L/X} = \oplus \mathcal{O}_{\mathbb{P}^n}(e_i)$ if and only if*

$$e_i \leq 1 \text{ for all } i \quad \text{and} \quad \sum_{i=1}^{n-2} e_i = n-1-d.$$

Proof. If the normal bundle is $N_{L/X} = \oplus \mathcal{O}_{\mathbb{P}^1}(e_i)$ then from the fact that there is an inclusion $N_{L/X} \to N_{L/\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^1}^{n-1}(1)$ it follows that $e_i \leq 1$ for all i. Computing Chern classes from the exact sequence of sheaves on $\mathbb{P}^1$

$$0 \to \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^1}(e_i) \to \mathcal{O}_{\mathbb{P}^1}^{n-1}(1) \to \mathcal{O}_{\mathbb{P}^1}(d) \to 0$$

we get $\sum_i e_i = n-1-d$ as well.

Conversely, suppose the e_i satisfy the given conditions. To simplify the notation, let $F = \oplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^1}(e_i)$ and let $G = \mathcal{O}_{\mathbb{P}^1}^{n-1}(1)$. Let $\beta : F \to G$ be any map, and let α be the map $G \to \mathcal{O}_{\mathbb{P}^1}(d)$ given by the matrix of $(n-2) \times (n-2)$ minors of the matrix of β. [1] The composition $\alpha\beta$ is zero because the i-th entry of the composite matrix is the Cauchy expansion of the determinant of a matrix obtained from β by repeating the i-th column.

Now if we take β of the form

$$\beta = \begin{pmatrix} x_0^{1-e_1} & 0 & \cdots & 0 & 0 \\ x_1^{1-e_2} & x_0^{1-e_2} & \cdots & 0 & 0 \\ 0 & x_1^{1-e_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_1^{1-e_{n-3}} & x_0^{1-e_{n-3}} \\ 0 & \cdots & 0 & 0 & x_1^{1-e_{n-2}} \end{pmatrix}$$

********this matrix needs cosmetic work to make the diagonals line up!******** then the top $(n-2) \times (n-2)$ minor will be x_0^{d-1} and the bottom $(n-$

[1] More formally and invariantly, α is the composite map

$$G \cong \mathcal{O}_{\mathbb{P}^1}(n-1) \otimes \wedge^{n-2} G^* \xrightarrow{\mathcal{O}_{\mathbb{P}^1}(n-1) \otimes \wedge^{n-2}\beta^*} \mathcal{O}_{\mathbb{P}^1}(n-1) \otimes \wedge^{n-2} F^* \cong \mathcal{O}_{\mathbb{P}^1}(d).$$

where we have used an identification of G with $\mathcal{O}_{\mathbb{P}^1}(n-1) \otimes \wedge^{n-2} G^*$ corresponding to a global section of $\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-n+1) \otimes \wedge^{n-1} G$.

2) $\times$ $(n-2)$ minor will be x_1^{d-1}. This shows that the map α will be an epimorphism of sheaves, so that the general hyperplane X containing L induced map $\alpha : N_L \to N_X$ will be smooth; and also, by Eisenbud [1995] Theorem 20.9, that the sequence

$$0 \longrightarrow F \xrightarrow{\beta} G \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0$$

is exact, so that $N_{L/X} \cong F$.

□

Corollary 8.11. *If $d \leq 2n-3$, then there exists a pair (X, L) with $X \subset \mathbb{P}^n$ a smooth hypersurface of degree d and $L \subset X$ a line such that $F_1(X)$ is smooth of dimension $2n-3-d$ in a neighborhood of $[L]$.*

Proof. This is really a special case of the Proposition 8.10: we just observe that if $d \leq 2n-3$, then we can choose all the $e_i \geq -1$, so that $h^0(\mathcal{O}_{\mathbb{P}^1}(e_i)) = e_i + 1$ for all i and hence $h^0(N_{L/X}) = 2n-3-d$. Since $F_1(X)$ has dimension at least $2n-3-d$ everywhere, the Corollary follows. □

Corollary 8.12. *If $d \leq 2n-3$, then every hypersurface of degree d in $\mathbb{P}^n$ contains a line.*

Proof. Let

$$\Phi = \{(X, L) \in \mathbb{P}^N \times \mathbb{G}(1, n) \mid L \subset X\}.$$

be the standard incidence correspondence. We've see that Φ is irreducible of dimension $N-d+2n-3$; since Corollary 8.11 asserts that at some point $(X, L) \in \Phi$ the fiber dimension of the projection $\Phi \to \mathbb{P}^N$ is $2n-d-3$, it follows that this projection is surjective. □

We remarked that the analog of Corollaries 8.11 and 8.12 for k-planes in general—that is, that if $\binom{k+d}{d} \leq (k+1)(n-k)$, then there exists a pair (X, Λ) with $X \subset \mathbb{P}^n$ a smooth hypersurface of degree d and $\Lambda \subset X$ a k-plane such that $F_k(X)$ is smooth of dimension $(k+1)(n-k) - \binom{k+d}{d}$ in a neighborhood of Λ—is false in general, citing the example of 2-planes on quadric hypersurfaces $X \subset \mathbb{P}^4$. In fact, it is true for $d \geq 3$: we have

Theorem 8.13. *If $d \geq 3$ and $(k+1)(n-k) \geq \binom{k+d}{k}$, then every hypersurface of degree d in $\mathbb{P}^n$ contains a k-plane.*

It is, however, much harder to prove, possibly because we lack an analogous classification of vector bundles on $\mathbb{P}^k$ in general. See Exercise 8.37 for an example that can be worked out explicitly, and Exercises 8.39 and 8.40 for the proof.

We have seen above that the Fano scheme of any smooth cubic surface in $\mathbb{P}^3$ is reduced and of the correct dimension. We can now say something about the higher-dimensional case as well:

Corollary 8.14. *The Fano scheme of lines on any smooth hypersurface of degree $d \leq 3$ is smooth and of dimension $2n-3-d$. But if $n \geq 4$ and $d \geq 4$ then there exist smooth hypersurfaces of degree d in $\mathbb{P}^n$ whose Fano schemes are singular or of dimension $> 2n-3-d$.*

Proof. We follow the notation of Proposition 8.10. If $d \leq 3$ then for any $e_1, \ldots, e_{n-2}$ allowed by the conditions of the Proposition we will have all $e_i \geq -1$, and thus $h^0(N_{L/X}) = \chi(N_{L/X}) = 2n-3$, proving that the Fano scheme is smooth and of expected dimension at L.

On the other hand, if $n \geq 4$ and $d \geq 4$ then we can take $e_1 = \cdots = e_{n-3} = 1$ and $e_{n-2} = 2-d \leq -2$. In this case $h^0(N_{L/X}) = 2n-6 > 2n-3-d$, so the Fano scheme is singular or of "too large" dimension at L. □

8.4 Lines on Quintic Threefolds and Beyond

We can now answer the first of the keynote questions of this Chapter: how many lines are contained in a general quintic threefold $X \subset \mathbb{P}^4$? More generally, we can now compute the number of distinct lines on a general hypersurface X of degree $d = 2n-3$ in $\mathbb{P}^n$, the case in which the expected dimension of the family of lines is zero.

The set-up is the same as that for the lines on a cubic surface: the defining equation F of the hypersurface X gives a section σ_F of the bundle $\mathrm{Sym}_d S^*$ on the Grassmannian $\mathbb{G}(1,n)$; the zero locus of σ_F is then the Fano scheme $F_1(X)$ of lines on X; and (assuming $F_1(X)$ has the expected dimension 0) the degree D of this scheme is the degree of the top Chern class $c_{d+1}(\mathrm{Sym}_d S^*) \in A^{d+1}(\mathbb{G}(1,n))$. If we can show in addition that $H^0(N_{L/X}) = 0$ for each line $L \subset X$, then it follows as in the previous section that the Fano scheme is zero-dimensional and reduced, so the actual number of distinct lines on X is exactly D.

To calculate the Chern class we could use the splitting principle. The computation is reasonable for $n = 4, d = 5$, the case of the quintic 3-fold, but becomes successively more complicated for larger n and d. Schubert2 (in Macaulay2) instead deduces it from a Gröbner basis for the Chow ring. Here is a Schubert2 script that computes the numbers for $n = 3, \ldots, 20$, followed by its output:

```
loadPackage "Schubert2"
grassmannian = (m,n) -> flagBundle({m+1, n-m})
time for n from 3 to 10 do(
     G=grassmannian(1,n);
     (S,Q) = G.Bundles;
```

```
     d = 2*n-3;
     print integral chern symmetricPower(d, dual S))

27
2875
698005
305093061
210480374951
210776836330775
289139638632755625
520764738758073845321
1192221463356102320754899
338192976632054635615064019
116439626640205162647858259911 65
478377865020631950883110323925781 25
231191601420598135249236900564098773215
129845157720179659258999916179526414353 1439
838662602951244072557173626577304717228992 2129
617308443705084878177983281890389233971812803846 57
513687287764790207960329434065844597978401438841796875
479849240965383456367278060519107076039364076181726998 5515
     -- used 119.123 seconds
```

The following result gives a geometric meaning to these numbers beyond the fact that they are degrees of certain Chern classes.

Theorem 8.15. *If $X \subset \mathbb{P}^n$ is a general hypersurface of degree $d \geq 1$, then the Fano scheme $F_1(X)$ of lines on X is reduced and has the expected dimension $2n - d - 3$.*

We now have the definitive answer to the Keynote question (a)

Corollary 8.16. *A general quintic threefold $X \subset \mathbb{P}^4$ contains exactly 2875 lines. More generally, the numbers in the table above are equal to the number of distinct lines on general hypersurfaces of degrees $3, 5, \ldots, 37$ and dimensions $2, 3, \ldots, 19$.*

We have seen that *every* smooth cubic surface has exactly 27 distinct lines. By contrast, the hypothesis "general" in the preceding Corollary is really necessary for quintic threefolds: by Corollary 8.14, the Fano scheme of a smooth quintic threefold may be singular or positive-dimensional (we'll see in Exercises 8.46 and 8.52 that both possibilities actually occur).

The 2875 lines on a quintic threefold have played a significant role in algebraic geometry, and even show up in physics. For example, the Lefschetz Hyperplane Theorem (**ref**; also cited in Chapter 3) implies that

all $2,875$ are homologous to each other, but one can show that they are linearly independent in the group of cycles mod algebraic equivalence. On the other hand, the number of rational curves of degree d on a general quintic threefold, of which the $2,875$ lines are the first example, is one of the first predictions of *mirror symmetry* (see for example Cox and Katz [1999].)

Proof of Theorem 8.15. First of all, in case $d > 2n-3$ we already know that for general X the Fano scheme $F_1(X)$ is empty, so we can assume $d \leq 2n-3$. We have already seen in Corollary 8.11 that there exists a pair (X, L) with $X \subset \mathbb{P}^n$ a smooth hypersurface of degree d and $L \subset X$ a line such that $\dim T_L F_1(X) = 2n-3-d$—that is, $F_1(X)$ is smooth of the expected dimension in a neighborhood of L. We now use a standard incidence correspondence trick to deduce that for general X, an open dense set of $L \in F_1(X)$ have this property (and in particular, if $d = 2n-3$ then X will contain a finite number of lines, every one of which will be a reduced point of $F_1(X)$).

We will return to the notation introduced at the beginning of Section 8.1: Let $\mathbb{P}^N$ be the projective space of forms of degree d in $n+1$ variables, whose points we think of as hypersurfaces in $\mathbb{P}^n$. Let

$$\begin{array}{ccc} & \Phi = \{(X, L) \in \mathbb{P}^N \times \mathbb{G}(1,n) \mid L \subset X\} & \\ \swarrow^{\varphi} & & \searrow^{\gamma} \\ \mathbb{P}^N & & \mathbb{G}(1,n) \end{array}$$

where $\varphi : \Phi \to \mathbb{P}^N$and $\gamma : \Phi \to \mathbb{G}(1,n)$ are the projections, so that the fiber of φ over the point X of $\mathbb{P}^N$ is the Fano scheme of X. As we have seen, Φ is smooth and irreducible of dimension $N + 2n - 3 - d$. It follows that the fiber of φ through any point of Φ has dimension $\geq 2n-3-d$.

The set of points of Φ where the fiber dimension is equal to $2n-3-d$ is open; and within that the set of points where the fiber is smooth is also open. We will show that this open set is nonempty; given this, the following elementary result shows that if X is a generic hypersurface of degree d, then any component of $F_1(X)$ is generically smooth of dimension D. Since $F_1(X)$ is defined by the vanishing of a section of a bundle of rank $d+1$, it is locally a complete intersection in the case when $2n-3-d > 0$. Thus $F_1(X)$ cannot have embedded components, and the fact that it is generically smooth implies that it is reduced. □

Lemma 8.17. *Suppose that $\varphi : X \to P$ is a dominant morphism of (not necessarily projective) varieties. If $B \subset X$ is a closed subset that contains a component of each fiber of φ over an open subset of P, then $B = X$. In particular, the generic fiber is equidimensional.*

Proof. We may suppose for simplicity that $\varphi(X) = P$. Let $X' \subset X$ be the subset of points x such that the dimension of $\varphi^{-1}(\varphi(x))$ is minimal. By the semicontinuity of fiber dimension X' is open, and for $x \in X'$ we have $\dim X = \dim P + \dim \varphi^{-1}(\varphi(x))$. Since B contains a component of the generic fiber, a similar computation shows that $\dim B \geq \dim X$, proving that $B = X$. □

8.5 The Universal Fano Scheme and the Geometry of Families of Lines

We turn now to Keynote Question (c): what is the degree of the surface in $\mathbb{P}^3$ swept out by the lines on a cubic surface as the cubic surface moves in a general pencil? As we remarked in Chapter 2, the degree of this surface is the number of its intersections with a general line; and since a general point of the surface lies on only one of the lines in question (Reason: a general point that lay on two lines would have to lie on lines from different surfaces in the pencil, and thus would lie in the base locus of the pencil, contradicting the assumption that it was a general point), this is the same as the degree in the Plücker embedding of the curve $C \subset \mathbb{G}(1,3)$ consisting of the points corresponding to lines on the various elements of the pencil of cubic surfaces. Our tools allow us to do more: we will calculate both the degree and the genus of the curve C. The same tool will give the answer to Keynote Question (b).

The incidence correspondence

$$\Phi = \Phi[n,d] = \{(X, L) \in \mathbb{P}^N \times \mathbb{G}(1,n) \mid L \subset X\}.$$

which we may call the *universal* or *relative* Fano scheme of lines on cubic surfaces, was introduced in Section 8.1. We can learn about its global geometry by realizing it as the zero locus of a section of a bundle, just as with the case of the Fano scheme of a given hypersurface.

We have seen that the maps of vector spaces

$$\left\{\begin{array}{l}\text{homogeneous cubic}\\ \text{polynomials on } \mathbb{P}^3\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{homogeneous cubic}\\ \text{polynomials on } L\end{array}\right\}$$

for different $L \in \mathbb{G}(1,3)$ fit together to form a bundle map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \otimes \mathcal{O}_{\mathbb{G}(1,3)} \to \mathrm{Sym}_3 S^*$$

on the Grassmannian $\mathbb{G}(1,3)$. Also, if $V = H^0(\mathcal{O}_{\mathbb{P}^3}(3))$ is the 20-dimensional vector space of all cubic polynomials, then the inclusions

$$\langle F \rangle \hookrightarrow V$$

fit together to form a map of vector bundles on $\mathbb{P}V \cong \mathbb{P}^{19}$

$$T = \mathcal{O}_{\mathbb{P}^{19}}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^{19}}$$

where T is the universal subbundle on $\mathbb{P}^{19}$. We will put these two constructions together.

More generally, consider for any n and d the universal Fano scheme

$$\Phi = \{(t, L) : L \subset X_t\} \subset \mathbb{P}^{\binom{n+d}{d}-1} \times \mathbb{G}(1, n).$$

of lines on hypersurfaces of degree d in $\mathbb{P}^n = \mathbb{P}(V)$. On the product of $\mathbb{P}(\operatorname{Sym}_d V^*)$ and the Grassmannian $\mathbb{G}(1, n)$ we have both a universal rank 1 sub-bundle $\mathcal{O}_{\mathbb{P}(\operatorname{Sym}_d V^*)}(-1) \subset V \times \mathbb{P}(\operatorname{Sym}_d V^*)$ and a universal k-subbundle $S \subset V$. Writing π_1, π_2 for the projections

$$\mathbb{P}(\operatorname{Sym}_d V^*) \xleftarrow{\pi_1} \mathbb{P}(\operatorname{Sym}_d V^*) \times \mathbb{G}(1, n) \xrightarrow{\pi_2} \mathbb{G}(1, n),$$

we consider the maps

$$\pi_1^* \mathcal{O}_{\mathbb{P}(\operatorname{Sym}_d V^*)}(-1) \longrightarrow \pi_1^* \operatorname{Sym}_d V^* \cong \pi_2^* \operatorname{Sym}_d V^* \longrightarrow \pi_2^* \operatorname{Sym}_d S^*.$$

Restricted to the fiber over the point of $\mathbb{P}(\operatorname{Sym}_d V^*)$ corresponding to f the composite map takes a generator of $\pi_1^* \mathcal{O}_{\mathbb{P}(\operatorname{Sym}_d V^*)}(-1)|_{\langle f \rangle}$ to τ_f. Thus the zero locus of the composite map is the incidence correspondence Φ. Tensoring with $\pi_1^* \mathcal{O}_{\mathbb{P}(\operatorname{Sym}_d V^*)}(1)$ we get a map $\mathcal{O}_{\mathbb{P}(\operatorname{Sym}_d V^*)} \longrightarrow \pi_2^* \operatorname{Sym}_d S^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^n}(1)$. Let τ be the global section of the bundle

$$E := \pi_2^* \operatorname{Sym}_d S^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^n}(1)$$

that is the image of $1 \in \mathcal{O}_{\mathbb{P}(\operatorname{Sym}_d V^*)}$. The zero locus of the composite map is the same as the zero locus of τ, that is, Φ. We will show that this locus is reduced and of the expected codimension $\operatorname{rank} E = d + 1$, so we can use this fact to analyze Φ; for example, this will show that the class of Φ is

$$[\Phi] = c_{d+1}(E) \in A^{d+1}(\mathbb{P}(\operatorname{Sym}_d V^*) \times \mathbb{G}(1, n)).$$

We developed deformation theory to prove that the Fano scheme of a general hypersurface is reduced. Fortunately we don't need a new theory for the universal Fano scheme; the reducedness of the zero locus of τ follows from our previous result.

Proposition 8.18. *If M is a general m-plane in the projective space of hypersurfaces of degree d in $\mathbb{P}^n$, then the universal Fano scheme of lines in hypersurfaces that belong to M, defined as the preimage of M in Φ, is reduced and of the expected dimension.*

Proof. In characteristic zero this can be deduced from Sard's Theorem and the smoothness of Φ (proved by writing Φ as a bundle over the Grassmannian). Here is a different argument, that works in any characteristic.

Let Φ_M be the preimage of M in Φ. If we add the m linear equations of a general point $X \in M$ to the equations corresponding to the vanishing of τ, we get the equations of the Fano scheme $F_1(X)$ of lines on the (general) hypersurface X, and we already know that this is reduced. Choose a point $p \in F_1(X)$. We will show that Φ_M has the expected dimension at p, and that if p is a general—and therefore smooth—point of $F_1(X)$, then p is a smooth point of Φ_M. Thus Φ_M is locally a complete intersection, and is generically reduced. It follows that Φ_M cannot have embedded associated subvarieties, so it is everywhere reduced.

To prove the necessary facts about Φ_M, we compare the local ring R of p on $F_1(X)$ with the local ring S of p on Φ_M. The ring R is obtained from S by factoring out m elements. Writing $\mathfrak{m}_R$ and $\mathfrak{m}_S$ for the maximal ideals we see that

$$\dim_K(\mathfrak{m}_R/\mathfrak{m}_R^2) \geq \dim_K(\mathfrak{m}_S/\mathfrak{m}_S^2) - m.$$

By the Principal Ideal Theorem, $\dim R \geq \dim S - m$. On the other hand, since Φ_M is defined as the zero locus of a vector bundle whose rank is the expected codimension, the Prinicipal Ideal Theorem shows that $\dim S$ is at least the expected dimension; and of course that $\dim S \leq \dim_K(\mathfrak{m}_S/\mathfrak{m}_S^2)$. Finally, since R is regular, $\dim R = \dim_K(\mathfrak{m}_R/\mathfrak{m}_R^2)$. Putting these inequalities together we get

$$\dim_K \mathfrak{m}_S/\mathfrak{m}_S^2 \leq m + \dim_K \mathfrak{m}_R/\mathfrak{m}_R^2 = m + \dim R \leq \dim S$$

so $\dim S = \dim_K \mathfrak{m}_S/\mathfrak{m}_S^2 = \dim R + m$, whence S is regular of the expected dimension. □

The following statement summarizes the results of these computations and answers Keynote Questions (b) and (c).

Theorem 8.19. *The universal Fano scheme of lines on a general m-dimensional linear family $M = \mathbb{P}^m$ of hypersurfaces of degree d in $\mathbb{P}^n$ is reduced and of codimension $d+1$ in the $(2n-2+m)$-dimensional space $\mathbb{P}^m \times \mathbb{G}(2,n)$. It is the zero locus of a section of the rank $d+1$ vector bundle $E = \pi_2^* Sym_d S^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1)$ on that space, so its class is $c_{d+1}(E)$.*

For example, the class of the universal Fano scheme $\Phi(3,3)$ of lines on cubic surfaces in $\mathbb{P}^3$ is

$$\begin{aligned}[\Phi(3,3)] &= c_4\big(\pi_2^* Sym_3 S^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1)\big) \\ &= 27\sigma_{2,2} + 42\sigma_{2,1}\zeta + (11\sigma_2 + 21\sigma_{1,1})\zeta^2 + 6\sigma_1\zeta^3 + \zeta^4.\end{aligned}$$

while the class of the universal Fano scheme $\Phi(4,3)$ of lines on quartic surfaces in $\mathbb{P}^3$ is

$$\begin{aligned}[\Phi(3,4)] &= c_5\big(\pi_2^* Sym_4 S^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1)\big) \\ &= 320\sigma_{2,2}\zeta + 220\sigma_{2,1}\zeta^2 + (30\sigma_2 + 55\sigma_{1,1})\zeta^3 + 10\sigma_1\zeta^4 + \zeta^5.\end{aligned}$$

If C is the curve of lines on a general pencil of cubic surfaces then the degree of C is 42 and the genus of C is 70. The number of quartic surfaces that contain a line in a general pencil of quartic surfaces is 320.

Note that restricting to a point $t \in \mathbb{P}^1$, we see again that a particular cubic surface X_t will contain

$$[\Phi(3,3)] \cdot \zeta^{19} = 27$$

lines.

Proof. The first statements have already been proven. For the explicit computations of the Chern classes one can use the splitting principle or appeal to Schubert2. Here is the computation, via the splitting principle, for the case of $\Phi(3,3)$, the 4-th Chern class of the bundle E on $\mathbb{P}^{19} \times \mathbb{G}(1,3)$. We will use use the symbol ζ for the pullback to $\mathbb{P}^{19} \times \mathbb{G}(1,3)$ of the hyperplane class on $\mathbb{P}^{19}$; and we'll use the symbols $\sigma_{i,j}$ for the pullbacks to $\mathbb{P}^{19} \times \mathbb{G}(1,3)$ of the corresponding classes in $A^*(\mathbb{G}(1,3))$.

Formally factoring the Chern class of $\nu^* S^*$ as

$$c(\nu^* S^*) = 1 + \sigma_1 + \sigma_{1,1} = (1+\alpha)(1+\beta),$$

we can write

$$c\big(\mu^* \mathcal{O}_{\mathbb{P}^{19}}(1) \otimes \nu^* \operatorname{Sym}_3 S^*\big) = \\ (1+3\alpha+\zeta)(1+2\alpha+\beta+\zeta)(1+\alpha+2\beta+\zeta)(1+3\beta+\zeta)$$

and in particular the top Chern class is given by

$$c_4(\mu^* \mathcal{O}_{\mathbb{P}^{19}}(1) \otimes \nu^* \operatorname{Sym}_3 S^*) = \\ (3\alpha+\zeta)(2\alpha+\beta+\zeta)(\alpha+2\beta+\zeta)(3\beta+\zeta) \in A^4(\mathbb{P}^1 \times \mathbb{G}(1,3)).$$

Evaluating, we first have

$$(3\alpha+\zeta)(3\beta+\zeta) = 9\sigma_{1,1} + 3\sigma_1\zeta + \zeta^2;$$

and then

$$(2\alpha+\beta+\zeta)(\alpha+2\beta+\zeta) = 2\sigma_1^2 + \sigma_{1,1} + 3\sigma_1\zeta + \zeta^2.$$

Multiplying out, we have

$$[\Phi] = 27\sigma_{2,2} + 42\sigma_{2,1}\zeta + (11\sigma_2 + 21\sigma_{1,1})\zeta^2 + 6\sigma_1\zeta^3 + \zeta^4.$$

Here is the corresponding Schubert2 code:

```
n=3
d=3
m=19
```

```
P = flagBundle({1,m}, VariableNames=>{z,q1})
(Z,Q1)=P.Bundles
V = abstractSheaf(P,Rank =>n+1)
G = flagBundle({2,n-1},V,VariableNames=>{s,q})
(S,Q) = G.Bundles
p = G.StructureMap
ZG = p^*(dual Z)
chern_4 (ZG**symmetricPower_d dual S)
```

Replacing the line $d = 3$ with the line $d = 4$ we get the corresponding result for $\Phi(3, 4)$.

From the computation of $[\Phi(3, 3)]$ we see that the number of lines on members of our pencil of cubics meeting a given line is

$$[\Phi] \cdot \sigma_1 \cdot \zeta^{18} = 42$$

from which we deduce that the degree of C, which is equal to the degree of the surface swept out by the lines on our pencil of cubics, is 42. For the genus $g(C)$ of C we use Part (c) of Proposition 8.5 to conclude that the normal bundle of C is the bundle $E|_C$, where E is the bundle on $\mathbb{P}^1 \times \mathbb{G}(1, n)$ whose section defines $\Phi(3, 3)_{\mathbb{P}^1}$, as above. From the exact sequence

$$0 \to T_C \to T_{\mathbb{P}^1 \times \mathbb{G}(1,n)}|_C \to N_{C/\mathbb{P}^1 \times \mathbb{G}(1,n)} \to 0$$

we deduce that the degree of T_C, which is $2 - 2g(C)$, is

$$\begin{aligned} \deg T_C = \deg c_1(T_C) &= \deg\big([C]c_1(T_{\mathbb{P}^1 \times \mathbb{G}(1,n)})\big) - \deg c_1(N_{C/\mathbb{P}^1 \times \mathbb{G}(1,n)}) \\ &= c_4(E)(4\sigma_1 + 2\zeta) - c_4(E)c_1(E), \end{aligned}$$

where we have used the computation $c_1(T_{\mathbb{G}(1,n)}) = (n + 1)\sigma_1$ from Corollary 7.13.****in Ch 5****. We can compute $c_1(E)$ by the splitting principle or by calling

```
chern_1 (ZG**symmetricPower_d dual S)
```

and we get $c_1(E) = 6\sigma_1 + 4\zeta$. Using the fact that ζ^2 restricts to zero on the preimage of a line in $\mathbb{P}^1 9$, this gives

$$\begin{aligned} 2 - 2g(C) &= \deg(27\sigma_{2,2} + 42\sigma_{2,1}\zeta)\big(4\sigma_1 + 2\zeta - 6\sigma_1 - 4\zeta\big) \\ &= \deg(-138\sigma_{2,2}\zeta) = -138, \end{aligned}$$

whence $g = 70$.

Finally, the number of quartic surfaces that contain a line in a general pencil of quartic surfaces is the number of lines that lie on some quartic surface in the pencil, that is, the degree of $\Phi(3, 4) \cap \mathbb{P}^1$. (If a line lay on more than one element of the pencil, it would be a component of the base locus—but the pencil being general, the base locus is smooth and connected.)

Writing τ again for the section of $\pi_2^* \operatorname{Sym}_4 S^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1)$ defined above, this is

$$\deg \zeta^{18} c_5\big(\pi_2^* \operatorname{Sym}_4 S^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1)\big).$$

By the compuation of $[\Phi(3,4)]$ above, this is 320. □

The coefficients of higher powers of ζ in the class of $\Phi(3,3)$ computed above have to do with the geometry of larger linear systems of cubics: for example, we'll see how to answer questions about lines on a net of cubics in Exercise 8.34.

8.5.1 Lines on the quartic surfaces in a pencil

Here is a slightly different approach to Keynote Question (b). Given that the set of quartic surfaces that contain a line is a hypersurface Γ in the projective space $\mathbb{P}^{34}$ of quartic surfaces, we are asking for the degree of Γ.

To find that number we look again at the bundle E on the Grassmannian $\mathbb{G}(1,3)$ whose fiber over a point $L \in \mathbb{G}(1,3)$ is the vector space

$$E_L = H^0(\mathcal{O}_L(4))$$

—that is, the fourth symmetric power $\operatorname{Sym}_4 S^*$ of the dual of the universal subbundle on $\mathbb{G}(1,3)$. As before, the polynomials F and G generating the pencil define sections τ_F and τ_G of the bundle E. The locus of lines $L \subset \mathbb{P}^3$ that lie on some element of the pencil is the locus where the values of the sections τ_F and τ_G are dependent so the degree of this locus is the fourth Chern class $c_4(E) \in A^4(\mathbb{G}(1,3)) \cong \mathbb{Z}$. As before this can be computed either with the splitting principle or with a system such as Schubert2, and one finds again the number 320.

We'll see another way of calculating the genus of the curve Φ in the following Chapter (after we've determined the number of singular cubic surfaces in a general pencil), by expressing Φ as a 27-sheeted cover of $\mathbb{P}^1$ and using Riemann-Hurwitz.

8.6 Local Equations for $F_1(X)$

As we've seen, the fact that the Fano scheme $F_1(X)$ is the Hilbert scheme of lines on X tells us many things about its geometry. It allows us to describe its tangent space at a point L, for example, in terms of the normal bundle of the line L in X. In particular, it gives us necessary and sufficient conditions for L to be a smooth point of $F_1(X)$, which was instrumental in proving

that the number of lines on a general hypersurface of degree $d = 2n-3$ in $\mathbb{P}^n$ is equal to the degree of the Chern class $c_{2n-2}(\mathrm{Sym}^d S^*)$.

But there are other aspects of the geometry of $F_1(X)$ that we can't get at in this way, such as its multiplicity at a point L. We might want to know, for example, if we can find a smooth hypersurface $X \subset \mathbb{P}^n$ of degree $2n-3$ whose Fano scheme of lines includes a point of multiplicity exactly 2, as in Harris [1979]; or we might ask, if X has an ordinary double point, how does this affect the number of lines it will contain?

To answer these sorts of questions, we need to have more detailed knowledge of $F_1(X)$, which can only come from looking at the equations defining $F_1(X)$ in local coordinates on $\mathbb{G}(1,n)$. This is largely straightforward, but it's worth going through the steps at least once; so in this section, we'll carry out these calculations and show how to write out the defining equations of $F_1(X)$ in local coordinates. Doing this, we will also recover some of the statements proved earlier in this chapter about the tangent spaces to $F_1(X)$, such as Theorem 8.3 and Corollary 8.11, but in a more concrete form.

8.6.1 Tangent spaces to Fano schemes via local equations

Let $X \subset \mathbb{P}^n$ be the hypersurface given as the zero locus of the homogeneous polynomial $F(Z_0, \ldots, Z_n)$, and let $L \subset X$ be a line. We may choose our homogeneous coordinates on $\mathbb{P}^n$ so that

$$L = V(Z_2, \ldots, Z_n);$$

we can then write the polynomial F as

$$F(Z_0, \ldots, Z_n) = Z_2 \cdot G_2(Z) + \cdots + Z_n \cdot G_n(Z)$$

where $G_2, \ldots, G_n$ are homogeneous polynomials of degree $d-1$ (note that the G_i are not uniquely determined here; a term like $Z_2 Z_3$ could go in either $Z_2 \cdot G_2(Z)$ or $Z_3 \cdot G_3(Z)$. In certain situations, such as the calculation of the tangent spaces to $F_1(X)$, terms vanishing to order 2 along the line won't matter; in these cases we can write

$$F(Z_0, \ldots, Z_n) = Z_2 \cdot \overline{G}_2(Z_0, Z_1) + \cdots + Z_n \cdot \overline{G}_n(Z_0, Z_1) + H \tag{8.1}$$

with $H \in (Z_2, \ldots, Z_n)^2$. (Here the polynomials $\overline{G}_i(Z_0, Z_1)$ are unique.)

We next need to introduce local coordinates on the Grassmannian $\mathbb{G}(1,n)$ in a neighborhood U of $L \in \mathbb{G}(1,n)$. We'll take as our open set U the locus of lines in $\mathbb{P}^n$ that don't meet the $(n-2)$-plane $Z_0 = Z_1 = 0$, so that any line $M \in U$ can be described as the row space of a unique $2 \times (n+1)$ matrix of the form

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 & \ldots & a_n \\ 0 & 1 & b_2 & b_3 & \ldots & b_n \end{pmatrix}.$$

The $2n-2$ entries $a_2, \dots, a_n, b_2, \dots, b_n$ of this matrix are then affine coordinates on $U \cong \mathbb{A}^{2n-2} \subset \mathbb{G}(1,n)$.

The line $L_{a,b}$ given as the row space of the matrix above may be represented parametrically as the image of the map

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^n \\ (s,t) &\mapsto (s, t, a_2 s + b_2 t, a_3 s + b_3 t, \dots, a_n s + b_n t). \end{aligned}$$

The condition that the line $L_{a,b}$ lie in X is thus that the restriction to $L_{a,b}$ of F vanish identically; that is, that the polynomial

$$F_{a,b}(s,t) = F(s, t, a_2 s + b_2 t, a_3 s + b_3 t, \dots, a_n s + b_n t)$$

be zero as a homogeneous polynomial in s and t. To express this condition, we can write

$$F(s, t, a_2 s + b_2 t, a_3 s + b_3 t, \dots, a_n s + b_n t) = \sum_{i=0}^{d} c_i(a,b) s^{t-i} t^i,$$

where the c_i are polynomials in the coordinates $a_2, \dots, a_n, b_2, \dots, b_n$ on U; their simultaneous vanishing then describes the locus $F_1(X) \cap U$. Put another way, the parametrization above of the lines $L_{a,b} \in U$ gives a trivialization of the bundle S^* over the open subset $U \subset \mathbb{G}(1,n)$, and hence of $\operatorname{Sym}_d S^*$; and in terms of this trivialization the section σ_F of $\operatorname{Sym}_d S^*$ is given over U by the $d+1$ coefficients c_i of the polynomial $F_{a,b}(s,t)$.

The tangent space $T_L F_1(X)$ to $F_1(X)$ at the point L is thus given by the vanishing of the differentials of the c_i at the origin in the (a,b)-plane $U \subset \mathbb{G}(1,n)$. In particular, in writing this out we may ignore any term in any of the $c_i(a,b)$ that is divisible more than once by the coordinates $a_2, \dots, a_n, b_2, \dots, b_n$; thus we may ignore the term H in the expression (8.1) for F altogether. Setting $H = 0$, we can write

$$\begin{aligned} F_{a,b}(s,t) &= \overline{G}_2(s,t)(a_2 s + b_2 t) + \cdots + \overline{G}_n(s,t)(a_n s + b_n t) \\ &= \overline{c}_0 s^d + \overline{c}_1 s^{d-1} t + \cdots + \overline{c}_d t^d \end{aligned}$$

where the $\overline{c}_i$ are simply the linear parts of the polynomials $c_i(a,b)$; the zero locus of the $\overline{c}_i$ is then the tangent space to $F_1(X)$ at L.

For example, to deduce Corollary 8.11 from this description, we can just choose

$$\begin{aligned} G_2(s,t) &= s^{d-1} \\ G_3(s,t) &= s^{d-3} t^2 \\ G_4(s,t) &= s^{d-5} t^4 \end{aligned}$$

and continue in this way. We have then

$$\begin{aligned} \overline{c}_0 &= a_2 & \overline{c}_1 &= b_2 \\ \overline{c}_2 &= a_3 & \overline{c}_3 &= b_3 \\ \overline{c}_4 &= a_4 & \overline{c}_5 &= b_4 \end{aligned}$$

and so on. In particular, as long as $d \leq 2n-3$, the functions $\overline{c}_i$ will themselves be independent linear functions, and their common zero locus a vector space of the expected dimension $2n-3-d$.

The calculation above tells us more in some special cases. For example, consider again the case $d = n = 3$ of lines on a cubic surface $X \subset \mathbb{P}^3$. As before, setting $H = 0$ in (8.1) we may write

$$F_{a,b}(s,t) = \overline{G}_2(s,t)(a_2 s + b_2 t) + \overline{G}_3(s,t)(a_3 s + b_3 t)$$

and describe the tangent space to $F_1(X)$ at L as the locus where the four coefficients of this polynomial in s and t simultaneously vanish. But here we observe that if $\overline{G}_2$ and $\overline{G}_3$ are any two homogeneous quadric polynomials in s and t, *the four homogeneous cubic polynomials* $s\overline{G}_2$, $t\overline{G}_2$, $s\overline{G}_3$ *and* $t\overline{G}_3$ *will be linearly dependent—that is, will fail to span the space of cubic polynomials—if and only if* $\overline{G}_2$ *and* $\overline{G}_3$ *have a common root*: if $\overline{G}_2$ and $\overline{G}_3$ do have a common root, all four will likewise vanish at that point; and if not no relation of the form $(as+bt)\overline{G}_2 = (cs+dt)\overline{G}_3$ can hold. Now, a common root of $\overline{G}_2$ and $\overline{G}_3$ is a singular point of S along L; and so we conclude the

Proposition 8.20. *If $X \subset \mathbb{P}^3$ is a cubic surface and $L \subset X$ a line, the tangent space $T_L(F_1(X))$ to the scheme of lines on X will be zero-dimensional if and only if X is smooth along L.*

We can apply these methods more generally to describe Fano schemes $F_k(X)$ of k-planes; an example is given in Exercise 8.37 and the general case in Exercise 8.39.

8.6.2 Lines on singular cubics

Up to this point we have used the local equations for $F_1(X)$ to rederive results established earlier. As we indicated, though, it is possible to get more out of the local equations than just the tangent spaces to $F_1(X)$; for example, we can calculate its multiplicity at a point L. We will do one example of this, describing the lines on a cubic surface with one ordinary double point. Other examples can be found in Exercise **??**, where we'll consider the case of cubic surfaces with more than one double point; in Exercise 8.46, where we'll show that it's possible to find a smooth quintic hypersurface

$X \subset \mathbb{P}^4$ whose Fano scheme contains an isolated double point, and in Exercise 8.49, where we'll look at a case of lines on higher-dimensional singular hypersurfaces.

To set this up, we suppose that $S \subset \mathbb{P}^3$ is a cubic surface, $L \subset S$ a line, and $p \in L$ an ordinary double point of S (if S is smooth along L, we've already seen in Proposition 8.20 that $F_1(X)$ is smooth at L); we assume that S has no other singularities along L. Choose coordinates on $\mathbb{P}^3$ so that (as above) L is given by $Z_2 = Z_3 = 0$, and the point $p = (1, 0, 0, 0)$. Finally, since we're assuming that p is an ordinary double point of S—that is, the tangent cone to S at p is the cone over a smooth conic curve—we can take the tangent cone to S at p to be $Z_1 Z_3 + Z_2^2 = 0$. The defining equation $F(Z)$ of S can thus be written

$$F(Z) = Z_0 Z_1 Z_3 + Z_0 Z_2^2 + \alpha Z_1^2 Z_2 + \beta Z_1^2 Z_3 + \gamma Z_1 Z_2^2 + \delta Z_1 Z_2 Z_3 + \epsilon Z_1 Z_3^2 + H$$

where $H \in (Z_2, Z_3)^3$. (Unfortunately, unlike the tangent space calculation, terms vanishing to order 2 along L cannot be discarded; fortunately, we can at least ignore terms vanishing to order 3, as we'll see.) The condition that S be smooth along L except at p says that $\alpha \neq 0$; otherwise the coefficients $\alpha, \ldots, \epsilon$ are arbitrary.

We can now write down the local equations of $F_1(X)$ near L as before: if we write

$$F(s, t, a_2 s + b_2 t, a_3 s + b_3 t) = c_0 s^3 + c_1 s^2 t + c_2 s t^2 + c_3 t^3$$

we find that

$$\begin{aligned}
c_0 &= a_2^2 \\
c_1 &= a_3 + 2a_2 b_2 + \gamma a_2^2 + \delta a_2 a_3 + \epsilon a_3^2 \\
c_2 &= b_3 + \alpha a_2 + \beta a_3 + b_2^2 + 2\gamma a_2 b_2 + \delta(a_2 b_3 + a_3 b_2) + 2\epsilon a_3 b_3 \\
c_3 &= \alpha b_2 + \beta b_3 + \gamma b_2^2 + \delta b_2 b_3 + \epsilon b_3^2
\end{aligned}$$

Examining these equations, we see that c_1, c_2 and c_3 have independent differentials at the origin; in other words, in a neighborhood of the origin the zero locus of these three is a smooth curve. Moreover, the tangent line to this curve is not contained in the plane $a_2 = 0$, so that $c_0 = a_2^2$ vanishes to order exactly 2 on this curve. Thus the component of $F_1(X)$ supported at L is isomorphic to $\operatorname{Spec} K[\epsilon]/(\epsilon^2)$, and in particular has multiplicity 2.

Having come this far, we might as well go ahead and answer the question: if $X \subset \mathbb{P}^3$ is a cubic surface with one ordinary double point (and otherwise smooth), how many lines will X contain? We have seen that the lines $L \subset X$ passing through p count with multiplicity 2, and those not passing through p with multiplicity one, so the only question is, how many lines on X pass through p?

To answer this, take $p = (1, 0, 0, 0)$ as above and expand the defining equation $F(Z)$ of X around p. Since p is a double point of X, we can write

$$F(Z_0, Z_1, Z_2, Z_3) = Z_0 A(Z_1, Z_2, Z_3) + B(Z_1, Z_2, Z_3)$$

where A is homogeneous of degree 2 and B homogeneous of degree 3. The lines on X through p then correspond to the common zeroes of A and B. Moreover, if we write a line L through p as the span $L = \overline{p, q}$ with $q = (0, Z_1, Z_2, Z_3)$, then *the condition that X be smooth along $L \setminus \{p\}$ is exactly the condition that the zero loci of A and B intersect transversely at* (Z_1, Z_2, Z_3), as we'll ask you to verify in Exercise 8.41. Thus there will be exactly 6 lines on X through p, and we have the

Proposition 8.21. *Let $X \subset \mathbb{P}^3$ be a cubic surface with an ordinary double point p. If X is smooth away from p, it will contain exactly 21 lines: six through p and 15 not passing through p.*

(Compare this with the discussion in Chapter 4 of **?**].)

In Exercises 8.42-8.45 we'll take up the case of cubics with more than one singularity, arriving ultimately at the statement that *a cubic surface $X \subset \mathbb{P}^3$ can have at most four isolated singular points.*

Lastly, we should remark that, while we have so far used the local equations of the Fano scheme to describe the locus of lines on a single hypersurface, we can use a similar approach to say something about the lines on a linear system of hypersurfaces. As a sample application, we'll see in Exercises 8.50 and 8.51 how to describe the singular locus of and tangent spaces to the locus $\Sigma \subset \mathbb{P}^{34}$ of quartic surfaces in $\mathbb{P}^3$ that contain a line.

8.7 The Debarre/de Jong Conjecture

We have focussed for the most part in this Chapter on the enumerative geometry of lines on hypersurfaces: we've calculated the class $[F_1(X)] \in A(\mathbb{G}(1, n))$ of the Fano scheme of lines on a hypersurface $X \subset \mathbb{P}^n$, subject to the condition that it have the expected dimension $2n-3-d$, and in order to give meaning to that class we've also established that for general such X the Fano scheme does indeed have the expected dimension. Nonetheless, while it may be tangential to the main thread of our discussion, it raises the question: what may be the behavior of $F_1(X)$ (and, more generally, $F_k(X)$) for arbitrary smooth X?

For example, it was shown in Harris et al. [1998] that, for $n \gg d, k$, the variety of k-planes on an arbitrary smooth hypersurface $X \subset \mathbb{P}^n$ of degree d has the expected dimension $\binom{k+d}{k}$ in the Grassmannian $\mathbb{G}(k, n)$, but the

sufficient lower bound on n given there is very large. What is the real bound required?

Let's focus for the moment on the case of lines. We will see in Exercises 8.52 and 8.53 that there are many examples of smooth hypersurfaces containing families of lines of dimension strictly greater than the expected $2n-3-d$. But Johan de Jong and Olivier Debarre independently made a striking observation: all the known examples of such hypersurfaces have degree strictly greater than n. (Indeed, the discussion following Exercises 8.52 and 8.53 shows that these constructions work exactly in the range $d > n$.) What's more, for low degrees d it's possible to prove that any smooth hypersurface of degree $d \leq n$ has the right dimensional family of lines (we'll discuss the cases $d \leq 4$ below). They made, accordingly, a conjecture:

Conjecture 8.22 (Debarre/de Jong)**.** *If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d \leq n$, then the Fano scheme $F_1(X)$ of lines on X has dimension $2n-3-d$.*

This conjecture has been proven for $d \leq 5$ by de Jong and Debarre, and for $d = 6$ by Beheshti in Beheshti [2006]; again, by Exercises 8.52 and 8.53, it is sharp if true.

The Debarre/de Jong conjecture is worth a discussion here, especially as it involves many of the deformation-theoretic ideas introduced earlier in this chapter. We'll discuss now the cases $d = 3$ and 4. (Linear spaces on smooth quadric hypersurfaces $Q \subset \mathbb{P}^n$ were described in Proposition 3.7; in particular, we saw there that $\dim F_1(Q) = 2n-5$.)

One observation before we get started: the statement of the conjecture for any d and n implies the statement for d and any $m > n$. The argument for this is as follows: if $X \subset \mathbb{P}^m$ is a smooth hypersurface and $L \subset X$ a line, let $\Lambda \cong \mathbb{P}^n$ be a general n-plane in $\mathbb{P}^m$ containing L and let $Y = \Lambda \cap X$ be the corresponding section of X. At this point, we need a Bertini-type lemma, which we state in somewhat greater generality:

Lemma 8.23. *Let $X \subset \mathbb{P}^m$ be a smooth hypersurface and $L \cong \mathbb{P}^k \subset X$ a k-plane contained in X. If $\Lambda \cong \mathbb{P}^n \subset \mathbb{P}^m$ is a general n-plane containing L, then the intersection $Y = X \cap \Lambda$ is smooth if and only if $n - 1 \geq 2k$.*

Proof. Note that one implication—that if $n - 1 < 2k$ then Y must be singular—is just Proposition 8.1.

For the reverse, it's enough to do it one dimension at a time; that is, we can take $n = m - 1$. Now, by Bertini we know that Y will be smooth away from L. On the other hand, the locus of tangent hyperplanes $\mathbb{T}_p X$ to X at points $p \in L$ is a subvariety of dimension at most k in the dual projective space $\mathbb{P}^{m*}$, while the locus of hyperplanes containing L will be the $(m-k-1)$-plane $L^\perp \subset \mathbb{P}^{m*}$. Thus, if $n - 1 = m - 2 \geq 2k$, so that

$k < m - k - 1$, then not every hyperplane containing L is tangent to X at a point of L; for a general $\Lambda \in L^\perp \subset \mathbb{P}^{m*}$, accordingly, $Y = \Lambda \cap X$ will be smooth. □

Lemma 8.23 tells us that if $L \subset X \subset \mathbb{P}^m$ is a line on a smooth hypersurface and $n \geq 3$ then the intersection $Y = \Lambda \cap X$ of X with a general n-plane containing Λ will be smooth. The Fano scheme $F_1(Y)$ is then just the intersection of $F_1(X)$ with the Schubert cycle $\Sigma_{m-n,m-n}(\Lambda) \subset \mathbb{G}(1,m)$; by subadditivity of codimension, then

$$\dim_L F_1(Y) \geq \dim_L F_1(X) - 2(m-n).$$

Accordingly, if we know $\dim_L F_1(Y) = 2n - 3 - d$, we have $\dim_L F_1(X) \leq 2m - 3 - d$ and hence $\dim_L F_1(X) = 2m - 3 - d$ since every component of $F_1(X)$ has dimension at least $2n - 3 - m$. In conclusion, it's enough to consider only the case $d = n$ of Conjecture 8.22; this will imply the case $d < n$.

The case $d = 3$ of 8.22 thus reduces to the statement that a smooth cubic surface $S \subset \mathbb{P}^3$ contains at most a finite number of lines, which is the case $d = 3$ of Corollary 8.6.

The case $d = 4$ of 8.22 requires a bit more deformation theory (indeed, the difficulty of 8.22 seems to increase quite rapidly with d!); we'll just sketch the argument here. To begin with, we need two lemmas, the first of which is completely elementary.

Lemma 8.24. *If $X \subset \mathbb{P}^n$ is any k-dimensional scheme, then*

$$\dim F_1(X) \leq 2k - 2,$$

with equality if and only if X is a k-plane.

Proof. Let $U = X \times X \setminus \Delta$ be the complement of the diagonal in $X \times X$, and let

$$\Psi = \{(p, q, L) \in U \times F_1(X) \mid p, q \in L\}.$$

Via the projection $\Psi \to U$, we see that $\dim \Psi \leq 2k$; and since the fibers of the projection $\Psi \to F_1(X)$ are 2-dimensional we conclude $\dim F_1(X) \leq 2k - 2$. Moreover, equality can hold only if the projection $\Psi \to U$ is dominant, that is, X contains the line joining any two of its points; in this case, X must be a k-plane. □

Next, we have a key deformation-theoretic lemma. To set this up, let $X \subset \mathbb{P}^n$ be any smooth variety and $F \subset F_k(X) \subset G = \mathbb{G}(k,n)$ a subvariety of the Fano scheme of k-planes on X. Let $Y \subset X$ be the subvariety of X swept out by the k-planes of F; that is, if

$$\Phi = \{(L, p) \in G \times \mathbb{P}^n \mid p \in L\}$$

is the universal k-plane over G and $\alpha : \Phi \to G$ and $\beta : \Phi \to \mathbb{P}^n$ the projections, then

$$Y = \beta\big(\alpha^{-1}(F)\big).$$

Now, let (L, p) be a general point of $\alpha^{-1}(F)$ (so that in particular p is a general point of Y). By Theorem 8.3, we have an inclusion $T_L F \hookrightarrow H^0(N_{L/X})$; evaluating at p gives in turn a map

$$\varphi : T_L F \to (N_{L/X})_p = T_p X / T_p L.$$

We have then

Lemma 8.25. *With X and F as above*

$$T_p Y \subset \operatorname{Im} \varphi + T_p L.$$

Proof. We leave the proof as an exercise. □

We can now prove the next case of Conjecture 8.22.

Theorem 8.26. *If $X \subset \mathbb{P}^4$ is a smooth quartic hypersurface, then the Fano scheme $F_1(X)$ has dimension 1.*

Proof. To begin with, suppose $F \subset F_1(X)$ is an irreducible component with $\dim F \geq 2$, and let $L \in F$ be a general point. By Proposition 8.10, the normal bundle $N = N_{L/X}$ must be either $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$ or $\mathcal{O}_L(1) \oplus \mathcal{O}_L(-2)$. Since $\dim H^0(N) \geq \dim T_L F \geq 2$, it must be the latter, but that doesn't actually matter: either way, the key point is, *N cannot be generated by global sections*; indeed, all global sections of N take values in a sub-line bundle of N.

Now, let $Y \subset X$ be the subvariety swept out by the lines of $F \subset F_1(X)$. By Lemma 8.25, Y can have dimension at most 2. But Y contains a two-dimensional family of lines, and by Lemma 8.24 we conclude that *Y must be a 2-plane*. Proposition 8.1 tells us this is impossible, and we're done. □

One striking aspect of the Debarre/de Jong conjecture is that the inequality $d \leq n$ for a smooth hypersurface $X \subset \mathbb{P}^n$ is exactly equivalent to the condition that the anticanonical bundle ω_X^* is ample, though it's not clear what role this may play in the proof.

8.7.1 Further open problems

The Debarre/de Jong conjecture deals with the dimension of the family of lines on a hypersurface $X \subset \mathbb{P}^n$, but we can also ask further questions about the geometry of $F_1(X)$: for example, whether it's irreducible and/or reduced. Exercises 8.55-8.58, in which we show that the Fano scheme $F_1(X)$

of lines on the Fermat quartic hypersurface $X \subset \mathbb{P}^4$ is neither, shows that the statement of Debarre/de Jong cannot be strengthened for all $d \leq n$. But—based on our knowledge of examples—it does seem to be the case that the smaller d is relative to n, the better behaved $F_1(X)$ is for an arbitrary smooth hypersurface $X \subset \mathbb{P}^n$ of degree d. For example, the following questions are open:

(a) Is it the case that if $d \leq n-1$ and $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d, then $F_1(X)$ is reduced and irreducible? (Note that—assuming it has the expected dimension—$F_1(X)$ is Cohen-Macaulay, so reduced is equivalent to generically reduced.)

(b) The arguments above show that for $d = 3$, the Fano scheme $F_1(X)$ is smooth. In the intermediate range $4 \leq d \leq n-1$, can we bound the dimension of the singular locus of $F_1(X)$ in terms of d?

Next, we should mention that the analogous questions for k-planes in general are completely open. We can ask, for example: given d and k, what is the largest n such that there exists a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d with $\dim F_k(X) > (k+1)(n-k) - \binom{k+d}{d}$? Again, Harris et al. [1998] says that such n are bounded; but the bound given there is—to put it charitably—not likely to be sharp.

Finally, we can ask: why lines? Why not look, for example, at rational curves of any degree on a hypersurface? Here the field is wide open. Specifically, we have an "expected" dimension of the family of rational curves of degree e on such a hypersurface; but it's not known to hold, *even for general hypersurfaces $X \subset \mathbb{P}^n$ of degree d.*

8.8 Exercises

Exercise 8.27. Let $\mathbb{P}^N$ be the space of all surfaces of degree d in $\mathbb{P}^3$. Show that the set of surfaces of degree d containing two or more lines is a subvariety of codimension $2d-6$ in $\mathbb{P}^N$. (Thus a general surface $X \subset \mathbb{P}^3$ of degree $d \geq 4$ containing a line contains only one line, and no surface in a general pencil of such surfaces will contain more than one line.)

Exercise 8.28. Show that the expected number of lines on a hypersurface of degree $2n-3$ in $\mathbb{P}^n$ is always positive, and deduce that *every hypersurface of degree $2n-3$ in $\mathbb{P}^n$ must contain a line.* (This is just a special case of Corollary 8.12; the idea here is to do it without a tangent space calculation.)

Exercise 8.29. Let $X \subset \mathbb{P}^4$ be a general quartic threefold. By Theorem 8.26, X will contain a one-parameter family of lines. Find the class of

the Fano scheme $F_1(X)$, and the degree of the surface $Y \subset \mathbb{P}^4$ swept out by these lines.

Exercise 8.30. Find the class of the scheme $F_2(Q) \subset \mathbb{G}(2,5)$ of 2-planes on a quadric $Q \subset \mathbb{P}^5$. (Do the problem first, then ompare your answer to the result in Proposition 3.7.)

Exercise 8.31. Find the expected number of 2-planes on a general quartic hypersurface $X \subset \mathbb{P}^7$, that is, the degree of $c_{15}(\mathrm{Sym}_4 S^*) \in A(\mathbb{G}(2,7))$.

Exercise 8.32. We can also use the calculation carried out in this Chapter to count lines on complete intersections $X = Z_1 \cap \cdots \cap Z_k \subset \mathbb{P}^n$, simply by finding the classes of the schemes $F_1(Z_i)$ of lines on the hypersurfaces Z_i and multiplying them in $A(\mathbb{G}(1,n))$. Do this to find the number of lines on the intersection $X = Y_1 \cap Y_2 \subset \mathbb{P}^4$ of two cubic hypersurfaces in $\mathbb{P}^5$

Exercise 8.33. Find the Chern class $c_3(\mathrm{Sym}_3 S^*) \in A^3(\mathbb{G}(1,3))$. Why is this the degree of the curve of lines on the cubic surfaces in a pencil? Note that this computation does not use the incidence correspondence Φ.

Exercise 8.34. Let $\{X_t = V(t_0F + t_1G + t_2H)\}$ be a general net of cubic surfaces in $\mathbb{P}^3$.

(a) Let $p \in \mathbb{P}^3$ be a general point. How many lines lying on some member X_t of the net pass through p?

(b) Let $H \subset \mathbb{P}^3$ be a general plane. How many lines lying on some member X_t of the net lie in H?

Compare your answer to the second half of this question to the calculation in Chapter 3 of the degree of the locus of reducible plane cubics!

Exercise 8.35. Let $X \subset \mathbb{P}^3$ be a surface of degree $d \geq 3$. Show that if $F_1(X)$ is positive-dimensional, then either X is a cone or X has a positive-dimensional singular locus.

Exercise 8.36. Prove Theorem 8.3 using the methods of Section 8.6, that is, by writing the local equations of $F_k(X) \subset \mathbb{G}(k,n)$

Exercise 8.37. Using the methods of Section 8.6, show that there exists a pair (X, Λ) with $X \subset \mathbb{P}^7$ a quartic hypersurface and $\Lambda \subset X$ a 2-plane such that Λ is an isolated, reduced point of $F_2(X)$.

Exercise 8.38. Using the result of Exercise 8.37, show that the number of 2-planes on a general quartic hypersurface $X \subset \mathbb{P}^7$ is the number calculated in Exercise 8.31.

The following two exercises use a theorem of Hochster, which says that

****add statement of theorem and reference****

Exercise 8.39. Use Hochster's theorem above to prove the analog of Corollary 8.11 for k-planes in general—that is, that if $\binom{k+d}{d} \leq (k+1)(n-k)$, then there exists a pair (X, Λ) with $X \subset \mathbb{P}^n$ a smooth hypersurface of degree d and $\Lambda \subset X$ a k-plane such that $F_k(X)$ is smooth of dimension $(k+1)(n-k) - \binom{k+d}{d}$ in a neighborhood of Λ—using methods similar to those of Exercise 8.37.

Exercise 8.40. Use the result of the preceding exercise (and the simple estimate on the dimension of the incidence correspondence Φ introduced in Section 8.1) to deduce Theorem 8.13

Exercise 8.41. To complete the proof of Proposition 8.21, let $X \subset \mathbb{P}^3$ is a cubic surface with one ordinary double point $p = (1, 0, 0, 0)$, given as the zero locus of the cubic

$$F(Z_0, Z_1, Z_2, Z_3) = Z_0 A(Z_1, Z_2, Z_3) + B(Z_1, Z_2, Z_3)$$

where A is homogeneous of degree 2 and B homogeneous of degree 3. If we write a line $L \subset X$ through p as the span $L = \overline{p, q}$ with $q = (0, Z_1, Z_2, Z_3)$, show that X is smooth along $L \setminus \{p\}$ if and only if the zero loci of A and B intersect transversely at (Z_1, Z_2, Z_3).

Exercise 8.42. Extending the results of Section 8.6, suppose that X is a general cubic surface having two ordinary double points $p, q \in X$. Describe the scheme structure of $F_1(X)$ at the point corresponding to the line $L = \overline{p, q}$, and in particular determine the multiplicity of $F_1(X)$ at L.

Exercise 8.43. Now let $X \subset \mathbb{P}^3$ be a cubic surface and $P, q \in X$ isolated singular points of X; let $L = \overline{p, q}$. Show that L is an isolated point of $F_1(X)$, and that the multiplicity

$$\operatorname{mult}_L F_1(X) \geq 4$$

Exercise 8.44. Let $X \subset \mathbb{P}^3$ be a cubic surface and $p_1, \ldots, p_\delta$ isolated singular points of X. Show that no three of the points p_i are colinear.

Exercise 8.45. Use the result of the preceding two exercises to deduce the statement that *a cubic surface $X \subset \mathbb{P}^3$ can have at most four isolated singular points.*

Exercise 8.46. Show that there exists a smooth quintic threefold $X \subset \mathbb{P}^4$ whose scheme $F_1(X)$ of lines contains an isolated point of multiplicity 2.

Exercise 8.47. Using the result of the preceding exercise, show that there exists a smooth quintic threefold whose scheme of lines consists of one double point and exactly 2873 simple points (that is, which contains exactly 2874 lines in all).

Exercise 8.48. Let Φ be the incidence correspondence of triples consisting of a hypersurface $X \subset \mathbb{P}^n$ of degree $d = 2n-3$, a line $L \subset X$ and a singular point p of X lying on L: that is,

$$\Phi = \{(X, L, p) \in \mathbb{P}^N \times \mathbb{G}(1, n) \times \mathbb{P}^n \mid p \in L \subset X \text{ and } p \in X_{\text{sing}}\}.$$

Show that Φ is irreducible.

Exercise 8.49. Let Φ be as in the preceding exercise, and suppose that $(X, L, p) \in \Phi$ is a general point. Find the multiplicity of the Fano scheme $F_1(X)$ at L.

Exercise 8.50. Suppose F and G are two quartic polynomials on $\mathbb{P}^3$ and $\{X_t = V(t_0F + t_1G)\}$ the pencil of quartics they generate; let σ_F and σ_G be the sections of $\operatorname{Sym}_4 S^*$ corresponding to F and G. Let X_{t_0} be a member of the pencil containing a line $L \subset \mathbb{P}^3$.

(a) Find the condition on F and G that L is a reduced point of $V(\sigma_F \wedge \sigma_G) \subset \mathbb{G}(1, 3)$.

(b) Show that this is equivalent to the condition that the point $(t_0, L) \in \mathbb{P}^1 \times \mathbb{G}(1, 3)$ is a simple zero of the map $\mu^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \nu^* \operatorname{Sym}_d S^*$.

Exercise 8.51. Let $\Sigma \subset \mathbb{P}^{34}$ be the space of quartic surfaces in $\mathbb{P}^3$. Interpret the condition of the preceding problem in terms of the geometry of the pencil $\mathcal{D}$ around the line L, and use this to answer two questions:

(a) What is the singular locus of Σ?

(b) What is the tangent hyperplane $\mathbb{T}_X\Sigma$ at a smooth point corresponding to a smooth quartic surface X containing a single line?

The following two exercises give constructions of smooth hypersurfaces containing more than the expected families of lines.

Exercise 8.52. Let $Z \subset \mathbb{P}^{n-2}$ be any smooth hypersurface. Show that the cone $\overline{p, Z} \subset \mathbb{P}^{n-1}$ over Z in $\mathbb{P}^{n-1}$ is the hyperplane section of a smooth hypersurface $X \subset \mathbb{P}^n$, and hence that for $d > n$ there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ whose Fano scheme $F_1(X)$ of lines has dimension strictly greater than $2n - 3 - d$.

Exercise 8.53. Take $n = 2m + 1$ odd, and let $\Lambda \subset \mathbb{P}^n$ be an m-plane. Show that there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ of any given degree d containing Λ, and deduce once more that for $d > n$ there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ whose Fano scheme $F_1(X)$ of lines has dimension strictly greater than $2n - 3 - d$.

Note that the construction of Exercise 8.53 cannot be modified to provide counterexamples to the de Jong/Debarre conjecture, since by Proposition 8.1 there do not exist smooth hypersurfaces $X \subset \mathbb{P}^n$ containing linear

spaces of dimension strictly greater than $(n-1)/2$. The following exercise shows that the construction of Exercise 8.52 is similarly extremal, but is harder: it requires use of the Second Fundamental Form of a hypersurface (see Harris [1992])

Exercise 8.54. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 2$. Show that X can have at most finitely many hyperplane sections that are cones.

To see some of the kinds of odd behavior the variety of lines on a smooth hypersurface can exhibit, short of having the wrong dimension, the following series of exercises will look at the Fermat quartic $X \subset \mathbb{P}^4$, that is, the zero locus

$$X = V(Z_0^4 + Z_1^4 + Z_2^4 + Z_3^4 + Z_4^4).$$

The conclusion is that $F_1(X)$ has 40 irreducible components, each of which is everywhere non-reduced! We start with a useful more general fact:

Exercise 8.55. Let $S = \overline{p, C} \subset \mathbb{P}^3$ be the cone with vertex p over a plane curve C of degree $d \geq 2$, and $L \subset S$ any line. Show that the tangent space $T_L F_1(S)$ has dimension at least two, and hence that $F_1(S)$ is everywhere nonreduced.

Exercise 8.56. Show that X has 40 conical hyperplane sections Y_i, each a cone over a quartic Fermat curve in $\mathbb{P}^2$.

Exercise 8.57. Show that the class of the reduced locus $F_1(Y_i)_{\text{red}}$ has class $\sigma_{3,2}$.

Exercise 8.58. Using your answer to Exercise 8.29, conclude that

$$F_1(X) = \bigcup_{i=1}^{40} F_1(Y_i);$$

in other words, $F_1(X)$ is the union of 40 double curves.

9

Singular Elements of Linear Series

Keynote Questions

(a) If F and G are general forms of degree d on $\mathbb{P}^n$, how many hyper surfaces of the form $C_t = V(t_0F + t_1G)$ with $t = (t_0, t_1) \in \mathbb{P}^1$ are singular?

(b) Let H be a third general form of degree d, and consider $\{C_t = V(t_0F + t_1G + t_2H) \mid t = (t_0, t_1, t_2) \in \mathbb{P}^2\}$, a *general net of plane curves of degree d*. What is the degree and genus of the curve $\Gamma \subset \mathbb{P}^2$ traced out by the singular points of members of the net? What is the degree and genus of the discriminant curve $\mathcal{D} = \{t : C_t \text{ is singular}\}$?

(c) If $C \subset \mathbb{P}^2$ is a general (smooth) plane curve of degree d, then C has $3(d-2)d$ flex lines (lines that have order of contact at least 3 with C at some point), since these are the intersections of C with its *Hessian*, a curve of degree $3(d-2)$. What about curves in $\mathbb{P}^n$? For example, is there a formula for the number of hyperplanes that have order of contact at least $n+1$ with C at some point?

In this chapter we introduce the *bundle of principal parts* associated with a line bundle. Its sections represent the Taylor series expansions of sections of the line bundle up to a given order. We will use the techniques we've developed to compute Chern classes of these bundles, and this will enable us to answer many questions about singular points and other special points of varieties in families.

Since quadrics play a special role in our theory (see the definition of ordinary double point, below, and its use in the proof of Proposition9.1 *we will assume throughout this Chapter that the characteristic of our ground field is not 2*; the case of characteristic 2 requires a more subtle treatment. ****maybe only necessary in the first couple of sections. Check whether char 3 is a problem for singularities of order 3, etc.****

9.1 General Singular Hypersurfaces and the Universal Singularity

Before starting on this path, we'll take a moment out to talk about loci of singular plane curves, and more generally hypersurfaces in $\mathbb{P}^n$. Let $\mathbb{P}^N = \mathbb{P}^{\binom{d+n}{n}-1}$ be the projective space parametrizing all hypersurfaces of degree d. Our object of interest is the *discriminant locus* $\mathcal{D} \subset \mathbb{P}^N$, defined as the set of singular hypersurfaces.

To establish some notation, we remind the reader that a hypersurface X, locally defined as the vanishing of a function $f(x)$, is *singular* at a point p if f and all its first derivatives vanish at p. It will be useful for us to think of this in terms of the Taylor expansion f at p. Changing coordinates so that p is the origin 0, this takes the form

$$f(x) = f_0 + f_1(x) + f_2(x) + \cdots ,$$

where f_i is a form of degree i. Of course f vanishes at 0 if $f_0 = 0$, and is then singular at 0 if $f_1(x) = 0$. The *tangent cone* of X at p is the variety in $\mathbb{P}^{n-1}$ defined by the vanishing of the form f_i, where i is the smallest index for which $f_i \neq 0$.

For example, the simplest possible singularity of a hypersurface X, generalizing the case of an ordinary node of a plane curve, is called an *ordinary double point*: This is a point $p \in X$ such that the equation of X can be written in local coordinates with $p = 0$ as above with $f_0 = f_1 = 0$ and where $f_2(x)$ is a *non-degenerate quadratic form*—that is, the tangent cone to X at p is a smooth quadric. (Why we will assume that the characteristic is not 2: A quadric in $\mathbb{P}^{n-1}$ is smooth if the generator f_2 of its ideal, together with the derivatives of f_2, is an irrelevant ideal; when the characteristic is not 2, Euler's formula $2f_2 = \sum x_i \partial f_2/\partial x_i$ shows that it is equivalent to assume that the partial deriviatives of f_2 are linearly independent, and this is the property we will actually use. In characteristic 2 *no* quadratic form in an odd number of variables has this property.) As with nodes of plane curves, we shall see that the general singular hypersurface has only one singularity, which is an ordinary double point.

****picture: node of a plane curve, cone over a nonsingular conic, . . . ****

A central role in this chapter will be played by the *universal singular point* $\Sigma = \Sigma_{n,d}$ of hypersurfaces of degree d in $\mathbb{P}^n$, defined as follows:

$$\begin{array}{ccc} \mathbb{P}^N \times \mathbb{P}^n \supset \Sigma := \{(C,p) \mid p \in C_{\text{sing}}\} & \xrightarrow{\pi_1} & \mathbb{P}^n \\ \Big\downarrow \pi_2 & & \\ \{\text{hypersurfaces of degree } d \text{ in } \mathbb{P}^n\} = \mathbb{P}^N. & & \end{array}$$

If we write the general form of degree d on $\mathbb{P}^n$ as $F = \sum_{|I|=d} a_I x^I$ and think of it as a bihomogeneous form in the coordinates $x_0, \ldots, x_n$ of $\mathbb{P}^n$ and the coordinates a_I of $\mathbb{P}^N$, then Σ is defined by the bihomogeneous equations

$$F(x) = 0; \text{ and } \frac{\partial F}{\partial x_i} = 0 \text{ for } i = 0, \ldots, n,$$

so Σ is an algebraic set. It follows from the definition that the image of Σ in $\mathbb{P}^N$ is the discriminant hypersurface $\mathcal{D}$. We shall see that Σ is a resolution of singularities of $\mathcal{D}$:

Proposition 9.1. *With notation as above, suppose that $d \geq 2$.*

(a) *The variety Σ is smooth and irreducible of dimension $N-1$; in fact, is a $\mathbb{P}^{N-n-1}$-bundle over $\mathbb{P}^n$.*

(b) *The general singular hypersurface of degree d has a unique singularity, which is an ordinary double point. In particular, Σ is birational to the discriminant $\mathcal{D}$, which is its image in $\mathbb{P}^N$.*

(c) *$\mathcal{D}$ is an irreducible hypersurface in $\mathbb{P}^N$.*

Proof. Let $p \in \mathbb{P}^n$ be a point, and let $x_0, \ldots, x_n$ of $\mathbb{P}^n$ be homogeneous coordinates of $\mathbb{P}^n$ such that $p = (1, 0 \ldots, 0)$. Let $f(x_1/x_0, \ldots, x_n/x_0) = x_0^{-d} F(x_0, \ldots, x_n) = 0$ be the affine equation of the hypersurface $F = 0$. For $d \geq 1$ the $n+1$ coefficients of the constant and linear terms f_0 and f_1 in the Taylor expansion of f at p are independent linear functions of the coefficients of F, so the fiber of Σ over p is a projective subspace of $\mathbb{P}^N$ of codimension $n+1$. The first part of the Proposition follows from this, and implies that the discriminant $\mathcal{D} = \pi_2(\Sigma)$ is irreducible.

To prove the statements in the second part of the Proposition, note that the fiber of Σ over a point $p \in \mathbb{P}^n$ contains the hypersurface that is the union of $d-2$ hyperplanes not containing p with a cone over a nonsingular quadric in $\mathbb{P}^{n-1}$ with vertex p. This hypersurface has an ordinary double point at p, and is generically smooth. By the previous argument, the hypersurfaces corresponding to points of the fiber of Σ over p form a linear system of hypersurfaces, with no base points other than p. Bertini's Theorem **?? ****** we need to have a general statement somewhere******** shows that a general member of this system is smooth away from p. Thus the fiber of

the map $\pi_2 : \Sigma \to \mathbb{P}^N$ over a general point of $\mathbb{P}^N$ consists of just one point, showing that the map is birational onto its image, $\mathcal{D}$. ****Do we have to worry about inseparability**** Since smoothness is an open condition on a quadratic form, the the general member has only an ordinary double point at p.

The fact that Σ, which has dimension $N-1$, is birational to $\mathcal{D}$ shows that $\mathcal{D}$ also has dimension $N-1$, completing the proof. □

The defining equation of $\mathcal{D} \subset \mathbb{P}^N$ is difficult to write down explicitly. There are determinantal formulas in a few cases: see for example Gelfand et al. [2008] and Eisenbud et al. [2003], but it is a form of high degree (as we shall see) in $\binom{n+d}{d}$ variables, and no good general expression for it is known, though of course it can be computed in principle by elimination theory. Even in very simple cases the discriminant locus has a lot of interesting features, as even a picture of the real points of the discriminant of a cubic in 2 variables suggests:

****insert picture****

In view of Proposition9.1, we can rephrase the first keynote question of this chapter as asking for the number of points of intersection of a general line $L \subset \mathbb{P}^N$ with the hypersurface $\mathcal{D}$; that is, the degree of $\mathcal{D}$. How can we determine this if we can't write down the form?

There is an interpretation of the discriminant hypersurface in $\mathbb{P}^N$ that suggests an interesting generalization. The d-th Veronese map $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ embeds P^{N*} in such a way that the intersection of $\nu_d(\mathbb{P}^n)$ with a hyperplane corresponding to a point $F \in \mathbb{P}^N$ is isomorphic, via ν_d, to the corresponding hypersurface $F = 0$ in $\mathbb{P}^n$. Thus the discriminant is the set of hypersurfaces in $\mathbb{P}^{N*}$ that have singular intersection with $\nu_d(\mathbb{P}^n)$; or, equivalently, those that contain a tangent plane to $\nu_d(\mathbb{P}^n)$. This is the definition of the dual variety to $\mathbb{P}^n$. The proposition shows in particular that the dual of $\nu_d(\mathbb{P}^n)$ is a hypersurface, and that the general tangent hyperplane is tangent at just one point, at which the intersection has an ordinary double point.

It is natural to ask whether these things are true for other smooth varieties $X \subset \mathbb{P}^n$? In general, the answer is no. The conditions for the dual variety to be a hypersurface will be discussed further in Chapter**??**. The question of whether the general singular intersection will have just one ordinary double point is also subtle: it's true in many cases but is often not easy to prove, and can depend on the characteristic of the ground field. See Proposition9.16.

9.2 The bundles of principal parts

We can simplify the problem by linearizing it. We ask not, "Is the hypersurface $C = V(F)$ singular?" but rather asking for each point $p \in \mathbb{P}^n$ in turn the simpler question "Is C singular at p?" As in the context of the problem of counting lines on hypersurfaces, this converts a higher-degree equation into a family of systems of linear equations, which we can then express as the vanishing of a section of a vector bundle.

For each point $p \in \mathbb{P}^n$ we have an $(n+1)$-dimensional vector space

$$E_p = \frac{\{\text{germs of sections of } \mathcal{O}_{\mathbb{P}^n}(d) \text{ at } p\}}{\{\text{germs vanishing to order} \geq 2 \text{ at } p\}}.$$

This space should be thought of as the vector space of first-order Taylor expansions of forms of degree d. We will see that the spaces E_p fit together to form a vector bundle, called the bundle of *first order principal parts*, which we will write as $\mathcal{P}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(d))$. A form F of degree d will give rise to a section τ_F of this vector bundle whose value at the point p is the first-order Taylor expansion of F locally at p, and whose vanishing locus is thus the set of singular points of the hypersurface $F = 0$.

An important feature of the situation is that each vector space E_p has a naturally defined subspace, the space of germs vanishing at p. These subspaces will, as we'll see, glue together into a sub-bundle of $\mathcal{P}^1$. Using the Whitney formula, this will help evaluate the Chern classes of the bundle.

Here is the general definition: for any quasicoherent sheaf $\mathcal{L}$ on a K-scheme X we write $\pi_1, \pi_2 : X \times X$ to X for the projections onto the two factors, and let $\mathcal{I}$ be the ideal of the diagonal in $X \times X$. We set:

$$\mathcal{P}^m_K(\mathcal{L}) = \pi_{2*}\left(\pi_1^*\mathcal{L} \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_{X\times X}/\mathcal{I}^{m+1}\right),$$

which is a quasicoherent sheaf on X. We will parse and explain this rather technical expression below; but first we list its very useful properties:

Theorem 9.2. *The sheaves $\mathcal{P}^m(\mathcal{L}) = \mathcal{P}^m_K(\mathcal{L})$ have the following properties:*

(a) *If $p \in X$ is a K-rational point, then there is a canonical identification of the fiber of $\mathcal{P}^m(\mathcal{L})$ at p with the restriction of $\mathcal{L}$ to the m-th order neighborhood of p; that is,*

$$\mathcal{P}^m(\mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{O}_{X,p}/\mathfrak{m}_{X,p} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,p}/\mathfrak{m}^{m+1}_{X,p}.$$

(b) *If $F \in H^0(\mathcal{L})$ is a global section, then there is a global section $\tau_F \in H^0(\mathcal{P}^m(\mathcal{L}))$ such that the class of τ_F in $\mathcal{P}^m(\mathcal{L})_p$ is the class of F in $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,p}/\mathfrak{m}^{m+1}_{X,p}$*

(c) *$\mathcal{P}^0(\mathcal{L}) = \mathcal{L}$, and for each m there is a natural right exact sequence*

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathrm{Sym}^m(\Omega_{X/K}) \to \mathcal{P}^m(\mathcal{L}) \to \mathcal{P}^{m-1}(\mathcal{L}) \to 0,$$

where $\Omega_{X/K}$ denotes the sheaf of K-linear differential forms on X.

(d) *If X is smooth and of finite type over K and $\mathcal{L}$ is a vector bundle on X, then $\mathcal{P}^m(\mathcal{L})$ is a vector bundle on X, and the right exact sequences of part (c) are left exact as well.*

Example 9.3. In the simplest and most interesting case, where $m = 1$ $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$, this can be made very explicit, though we will not use this below. In fact, in characteristic zero,

$$\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d)) \cong \begin{cases} \Omega_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n} & \text{if } d = 0 \\ \mathcal{O}_{\mathbb{P}^n}(d-1)^{n+1} & \text{if } d \neq 0, \end{cases}$$

while in characteristic p the conditions are replaced by $d \equiv 0$ or $d \not\equiv 0 \bmod p$. This curious dichotomy is explained by the answer to a more refined question: By Part (d) we have a short exact sequence

$$0 \to \Omega_{\mathbb{P}^n}(d) \to \mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d)) \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0.$$

and we can ask for its class in

$$\mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(d), \Omega_{\mathbb{P}^n}(d)) \cong \mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \Omega_{\mathbb{P}^n}) = H^1(\Omega_{\mathbb{P}^n}) = K.$$

More generally, for any line bundle $\mathcal{L}$ on a smooth variety X, the short exact sequence in Part(d) gives us a class in

$$\mathrm{Ext}^1_X(\mathcal{L}, \Omega_X \otimes \mathcal{L}) = H^1(\Omega_X)$$

called the *Atiyah class* $at(\mathcal{L})$ of $\mathcal{L}$ Atiyah [1957] and Illusie [1972]. The formula for $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$ follows at once from the more refined and more uniform result that

$$at(\mathcal{O}_{\mathbb{P}^n}(d)) = d \cdot \eta,$$

where $\eta \in \mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \Omega_{\mathbb{P}^n})$ is the class of the tautological sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \xrightarrow{(x_0,\ldots,x_n)} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

See Perkinson [1996] 2.II, where the theorem is stated just for the characteristic zero case, but the technique works more generally.

Proof of Theorem9.2. Since the constructions all commute with restriction to open sets, we may harmlessly suppose that $X = \operatorname{Spec} R$ is affine. Thus also $X \times X = \operatorname{Spec} S$, where $S := R \otimes_K R$. We may think of $\mathcal{L}$ as coming from an R-module L, and then $\pi_1^* L := L \otimes_K R$. Pushing a (quasicoherent) sheaf $\mathcal{M}$ on $X \times X$ forward by π_{2*} simply means considering the corresponding S-module as an R-module via the ring map $R \to S$ sending r to $1 \otimes_K r$.

In this setting the sheaf of ideals $\mathcal{I}$ defining the diagonal embedding of X in $X \times X$ corresponds to the ideal $I \subset S$ that is the kernel of the multiplication map $S = R \otimes_K R \to R$. If R is generated as a K-algebra by

elements x_i, then I is generated as an ideal of S by the elements $x_i \otimes 1 - 1 \otimes x_i$.

With this notation we see that the R-module corresponding to the sheaf $\mathcal{P}^m(\mathcal{L})$ can be written as

$$P^m(L) = (L \otimes_K R)/I^{m+1}(L \otimes_K R)$$

regarded as an R-module by the action $f \mapsto 1 \otimes r$ as above.

Part (a) now follows: If the K-rational point p corresponds to the maximal ideal

$$\mathfrak{m} = \mathrm{Ker}(R \xrightarrow{\varphi} K); \quad \varphi : x_i \mapsto a_i.$$

then in $R/\mathfrak{m} \otimes_R S \cong R$ the class of $x_i \otimes_K 1 - 1 \otimes_K x_i$ is $x_i \otimes_K 1 - 1 \otimes_K a_i = x_i - a_i$. Thus

$$R/\mathfrak{m} \otimes_R P^m(L) = L/(\{x_i \otimes_K 1 - 1 \otimes_K x_i\})^{m+1}L = L/(\{x_i - a_i\})^{m+1}L,$$

as required.

Part (b) is similarly obvious from this point of view: the section τ_F can be taken to be the image of the element $F \otimes_K 1$ in $(L \otimes_K R)/I^{m+1}(L \otimes_K R)$. As the construction is natural, these elements will glue to a global section when we are no longer in the affine case.

Part (c) requires another important idea: the module of K-linear differentials $\Omega_{R/K}$ is isomorphic, as an R-module, to I/I^2, which has universal derivation $\delta : R \to I/I^2$ given by $\delta(f) = f \otimes_K 1 - 1 \otimes_K f$. This is plausible, since when X is smooth one can see geometrically that the normal bundle of the diagonal, which is $\mathrm{Hom}(I/I^2, R)$, is isomorphic to the tangent bundle of X, which is $\mathrm{Hom}(\Omega_{R/K}, R)$. See Eisenbud [1995], Section 16.8 for further discussion and a general proof. ****Insert picture relating the tangent bundle of X to the normal bundle of the diagonal in $X \times X$.**** Given this fact, the obvious surjection $\mathrm{Sym}^m(\Omega_{R/K}) \cong \mathrm{Sym}^m(I/I^2) \to I^m/I^{m+1}$ yields the desired right exact sequence.

Finally, it is enough to prove part (d) locally at a point $q \in X \times X$. If the point is not on the diagonal, then after localizing I is the unit ideal, and the result is trivial, so we may assume that $q = (p, p)$. Locally at p the module L is free, so it suffices to prove the result when $L = R$.

Write $d : R \to \Omega_{R/K}$ for the universal K-linear derivation of R. Since X is smooth, $\Omega_{R/K}$ is locally free at p, and is generated there by elements $d(x_1), \ldots, d(x_n)$, where $x_1, \ldots, x_n$ is a system of parameters at p, and thus $\mathrm{Sym}^m(\Omega_{R/K})$ is the free module generated by the monomials of degree m in the $d(x_i)$. Also, R is a domain—that is, I is a prime ideal.

Because $\mathrm{Sym}^m(\Omega_{R/K})$ is free, it suffices to show that the map

$$\mathrm{Sym}^m(\Omega_{R/K}) \to S/I^{m+1}$$

is a monomorphism after localizing at the prime ideal I. Since $I/I^2 \cong \Omega_{R/K}$ is free on the classes mod I^2 of the elements $x_i \otimes_K 1 - 1 \otimes_K x_i$ that correspond to the $d(x_i)$, Nakayama's Lemma shows in the local ring S_I, I_I is generated by the images of the $x_i \otimes_K 1 - 1 \otimes_K x_i$ themselves, and it follows that these are a regular sequence. Thus the associated graded ring $S_I/I_I \oplus I_I/I_I^2 \oplus \cdots$ is a polynomial ring on the classes of the elements $x_i \otimes_K 1 - 1 \otimes_K x_i$, and in particular the monomials of degree m in these elements freely generate I_I^m/I_I^{m+1}. Consequently, the map $S_I \otimes_S \mathrm{Sym}^m(\Omega_{R/K}) \to I_I^m/I_I^{m+1}$ is an isomorphism as desired. □

Remark. The name "Bundle of Principle Parts", first used by Grothendieck and Dieudonné, was presumably suggested by the (conflicting) usage that the "principal part" of a meromorphic function at a point is the sum of the terms of negative degree in the Taylor expansion of the function around the point—a finite power series, albeit in the inverse variables. It is not the only terminology in use: $\mathcal{P}^m(\mathcal{L})$ would be called the bundle of *m-jets of sections of* $\mathcal{L}$ by those studying singularities of mappings (see for example Golubitsky and Guillemin [1973]II.2) and some algebraic geometers (for exampePerkinson [1996].)On the other hand, the m-jet terminology is in use in another conflicting sense in algebraic geometry: the "scheme of m-jets" of a scheme X is used to denote the scheme of mappings from Spec $K[x]/x^{m+1}$ into X. So we have thought it best to stick to the Grothendieck-Dieudonné usage.

9.3 Singular Elements of a Pencil

9.3.1 From Pencils to Degeneracy Loci

Using the bundle of principle parts we can tackle Keynote Questiona): how many linear combinations of general polynomials F and G of degree d on $\mathbb{P}^n$ have singular zero loci? By Proposition9.1, none of the hypersurfaces $C_t = V(t_0F + t_1G)$ of the pencil will be singular at more than one point. Furthermore, no two elements of the pencil will be singular at the same point since otherwise every member of the pencil would be singular there. Thus, the Keynote Question is equivalent to the question: For how many points $p \in \mathbb{P}^n$ is some element C_t of the pencil singular at p? This, in turn, amounts to asking, at how many points $p \in \mathbb{P}^n$ are the values $\tau_F(p)$ and $\tau_G(p)$ in the fiber of $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$ at p linearly dependent? Thus we must compute the number of points at which a section of $\wedge^2\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$ vanishes, given that it vanishes at finitely many points. We can do this with Chern classes, provided that the vanishing locus is reduced.

To see that this locus is reduced, consider the behavior of the sections τ_F and τ_G around a point $p \in \mathbb{P}^n$ where they are dependent. At such a

point, some linear combination $t_0F + t_1G$—which we might as well take to be F—vanishes to order 2. If G were also zero at p, then the scheme $V(F,G)$ would have (at least) a double point at p. But Bertini's Theorem shows that a general complete intersection such as $V(F,G)$ is reduced, so this cannot happen.

To show that $V(\tau_F \wedge \tau_G)$ is reduced at p, we restrict our attention to an affine neighborhood of p where all our bundles are trivial. By Proposition9.1 the hypersurface $C = V(F)$ has an ordinary node at p, so if we work on an affine neighborhood where the bundle $\mathcal{O}_{\mathbb{P}^n}(d)$ is trivial, and take p to be the origin with respect to coordinates $x_1, \ldots, x_n$ we may assume that the functions F and G have Taylor expansions at p of the form

$$\begin{aligned} f &= f_2 + (\text{terms of order} > 2) \\ g &= 1 + (\text{terms of order} \geq 1). \end{aligned}$$

The sections τ_F and τ_G are then represented locally by the rows of the matrix

$$\begin{pmatrix} f & \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\ g & \frac{\partial g}{\partial x} & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix}.$$

The vanishing locus of $\tau_F \wedge \tau_G$ near p is by definition defined by the 2×2 minors of this matrix, and to prove that it is a reduced point we need to see that it contains n functions (vanishing at p) with independent linear terms. Suppressing all the terms of the functions in the matrix that could not contribute to the linear terms of the minors, we get the matrix

$$\begin{pmatrix} 0 & \partial f_2/\partial x_1 & \cdots & \partial f_2/\partial x_n \\ 1 & 0 & \cdots & 0. \end{pmatrix}.$$

Thus there are 2×2 minors whose linear terms are $\partial f_2/\partial x_1, \ldots, \partial f_2/\partial x_n$, and these are linearly independent because $f_2 = 0$ is a smooth quadric and the characteristic is not 2.

9.3.2 The Chern Class of a Bundle of Principal Parts

Once again, let F, G be general forms of degree d on $\mathbb{P}^n$. As we saw in the previous section, the linear combinations $t_0F + t_1G$ that are singular correspond exactly points where the two sections τ_F and τ_G are dependent. This number is the n-th Chern class of the rank $n+1$ bundle $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$, so we now turn to the computation of this class. Stated explicitly, if $\zeta \in A^1(\mathbb{P}^n)$ denote the class of a hyperplane in $\mathbb{P}^n$, we want to compute the coefficient of ζ^n in

$$c\Big(\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))\Big) \in A^*(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/(\zeta^n).$$

Parts c) and d) of Theorem9.2 give us a short exact sequence

$$0 \to \Omega_{\mathbb{P}^n}(d) \to \mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d)) \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0,$$

so $c\big(\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))\big) = c\big(\mathcal{O}_{\mathbb{P}^n}(d)\big) \cdot c\big(\Omega_{\mathbb{P}^n}(d)\big)$, a formula that we'll soon generalize. On the other hand, $\Omega_{\mathbb{P}^n}$ fits into a short exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

Tensoring with $\mathcal{O}(d)$ we get an exact sequence

$$0 \to \Omega_{\mathbb{P}^n}(d) \to \mathcal{O}_{\mathbb{P}^n}(d-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0$$

similar to the one involving $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$. This doesn't mean that $\mathcal{P}^1_{\mathbb{P}^n}(d) \cong \mathcal{O}_{\mathbb{P}^n}(d-1)^{n+1}$ are isomorphic (they are not), but by the Whitney sum formula (Proposition7.3) their Chern classes agree,

$$c\left(\mathcal{P}^1_{\mathbb{P}^n}(d)\right) = c\left(\mathcal{O}_{\mathbb{P}^n}(d-1)^{n+1}\right) = (1+(d-1)\zeta)^{n+1}.$$

Putting this formula together with the idea of the previous section we deduce:

Proposition 9.4. *The degree of the discriminant hypersurface in the space of forms of degree d on $\mathbb{P}^n$ is*

$$c_n(\mathcal{P}^1_{\mathbb{P}^n}(d)) = (n+1)(d-1)^n,$$

and this is the number of singular hypersurfaces in a general pencil of hypersurfaces of degree d in $\mathbb{P}^n$. □

In particular this answers Keynote Questiona

It is pleasant to observe that the conclusion agrees with what we get from elementary geometry in the cases where it is easy to check, such as those of plane curves ($n = 2$) with $d = 1$ or $d = 2$. For $d = 1$ the statement $c_2 = 0$ simply means that there are no singular lines in a pencil of lines. The case $d = 2$ corresponds to the number of singular conics in a general pencil $\{C_t\}$ of conics. To see that this is really $3(d-1)^2 = 3$, note that the pencil $\{C_t\}$ consists of all conics passing through the four (distinct) base points, and a singular element of the pencil will thus be the union of a line joining two of the points with the line joining the other two. There are indeed three such pairs of pairs of lines. We could also get the number 3 by viewing the pencil of conics as given by a 3×3 symmetric matrix M of linear forms on $\mathbb{P}^2$ whose entries vary linearly with a parameter t; the determinant of M will then be a cubic polynomial in t.

It is easy to extend the computation above to all the $\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d)$, and this will be useful in the rest of this Chapter:

Proposition 9.5. $c\big(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d)\big) = (1+(d-m)\zeta)^{\binom{n+m}{n}}$.

Proof. We will again use the exact sequences of Theorem9.2. With the Whitney sum formula they immediately give

$$c\Big(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d)\Big) = \prod_{j=0}^{m} c\Big(\mathrm{Sym}_j(\Omega_{\mathbb{P}^n})(d)\Big).$$

To derive the formula we need, we apply Lemma9.6 below to the exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0,$$

and the line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{P}^n}(d)$. To simplify the notation, we set $\mathcal{U} = \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}$. The Lemma yields

$$c\Big(\mathrm{Sym}_j(\Omega_{\mathbb{P}^n})(d)\Big) = c\Big(\mathrm{Sym}_j(\mathcal{U})(d)\Big) \cdot c\Big(\mathrm{Sym}_{j-1}(\mathcal{U})(d)\Big)^{-1},$$

for all $j \geq 1$. Combining this with the obvious $\mathrm{Sym}_0(\Omega_{\mathbb{P}^n})(d) = \mathcal{O}_{\mathbb{P}^n}(d)$, we see that the product in the formula for $c\Big(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d)\Big)$ is

$$c\Big(\mathcal{O}_{\mathbb{P}^n}(d)\Big) \cdot \frac{c\Big(\mathrm{Sym}_1(\mathcal{U})(d)\Big)}{c\Big(\mathcal{O}_{\mathbb{P}^n}(d)\Big)} \cdot \frac{c\Big(\mathrm{Sym}_2(\mathcal{U})(d)\Big)}{c\Big(\mathrm{Sym}_1(\mathcal{U})(d)\Big)} \cdots$$

which collapses to

$$c\Big(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d)\Big) = c\Big(\mathrm{Sym}_m(\mathcal{U})(d)).$$

But

$$\begin{aligned} c\Big(\mathrm{Sym}_m(\mathcal{U})(d)\Big) = & \\ c\Big(\mathrm{Sym}_m(\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1})(d)\Big) &= c\Big(\mathcal{O}_{\mathbb{P}^n}(-m))^{\binom{n+m}{n}}(d)\Big) \\ &= c\Big(\mathcal{O}_{\mathbb{P}^n}(d-m))^{\binom{n+m}{n}}\Big) \\ &= (1+(d-m)\zeta)^{\binom{n+m}{n}} \end{aligned}$$

yielding the formula of the Proposition. □

Lemma 9.6. *If*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is a short exact sequence of vector bundles on a projective variety X with $\mathrm{rank}\,\mathcal{C} = 1$*, and $\mathcal{L}$ is a line bundle on X, then, for every $j \geq 1$,*

$$c\Big(Sym_j(\mathcal{A}) \otimes \mathcal{L}\Big) = c\Big(Sym_j(\mathcal{B}) \otimes \mathcal{L}\Big) \cdot c\Big(Sym_{j-1}(\mathcal{B}) \otimes \mathcal{C} \otimes \mathcal{L}\Big)^{-1}.$$

Proof. For any right exact sequence of sheaves

$$\mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

the universal property of the symmetric powers (see for example Eisenbud [1995], Proposition A.2.2.d) shows that there is for each $j \geq 1$ a right exact sequence

$$\mathcal{E} \otimes \mathrm{Sym}_{j-1}(\mathcal{F}) \to \mathrm{Sym}_j(\mathcal{F}) \to \mathrm{Sym}_j(\mathcal{G}) \to 0,$$

Since $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are vector bundles the dual of the exact sequence in the hypothesis is exact, and we may apply the result on symmetric powers with $\mathcal{G} = \mathcal{A}^*, \mathcal{F} = \mathcal{B}^*$ and $\mathcal{E} = \mathcal{C}^*$.

In this case, since $\operatorname{rank} \mathcal{E} = \operatorname{rank} \mathcal{C} = 1$, the sequence

$$0 \to \mathcal{E} \otimes \mathrm{Sym}_{j-1}(\mathcal{F}) \to \mathrm{Sym}_j(\mathcal{F}) \to \mathrm{Sym}_j(\mathcal{G}) \to 0$$

is left exact as well, as one sees immediately by comparing the ranks of the three terms (this is a special case of an longer exact sequence that one can write down from the Koszul complex, independent of the rank of $\mathcal{E}$.)

Since these are all bundles, dualizing preserves exactness, and we get an exact sequence

$$0 \to \mathrm{Sym}_j(\mathcal{A}^*)^* \to \mathrm{Sym}_j(\mathcal{B}^*)^* \to \mathrm{Sym}_{j-1}(\mathcal{B}^*)^* \otimes \mathcal{C} \to 0.$$

Of course the double dual of a bundle is the bundle itself, and in characteristic zero the dual of the j-th symmetric power is naturally isomorphic to the j-th symmetric power of the dual, so in characteristic zero all the *s cancel, and this, with the Whitney sum formula, proves the Lemma. While there is no natural isomorphism between $\mathrm{Sym}_j(\mathcal{A}^*)^*$ and $\mathrm{Sym}_j(\mathcal{A})$ when the characteristic of the ground field divides j, it *is* true at the level of Chern classes, and remains true if we tensor with a line bundle.

That is, if $\mathcal{H}$ is a vector bundle (on any projective variety, in any characteristic) then

$$c\Big((Sym_m(\mathcal{H}^*))^* \otimes \mathcal{L})\Big) = c\Big(Sym_m(\mathcal{H}) \otimes \mathcal{L})\Big).$$

This follows from the splitting principle, by which it suffices to do the case where $\mathcal{H}$ has a filtration whose successive quotients are line bundles $\mathcal{L}_i$. In this case $\mathrm{Sym}_m(\mathcal{H}^*)$ has a filtration with successive quotients $\mathcal{L}_{i_1}^{-1} \otimes \cdots \otimes \mathcal{L}_{i_m}^{-1}$, so both $\mathrm{Sym}_m(\mathcal{H}^*)^* \otimes \mathcal{L}$ and $\mathrm{Sym}_m(\mathcal{H}^*) \otimes \mathcal{L}$ have filtrations with successive quotients $\mathcal{L}_{i_1} \otimes \cdots \otimes \mathcal{L}_{i_m} \otimes \mathcal{L}$. By the Whitney sum formula, $c\Big(Sym_m(\mathcal{H}^*)^* \otimes \mathcal{L}\Big)$ and $c\Big(Sym_m(\mathcal{H}) \otimes \mathcal{L}\Big)$ are equal. With the Whitney sum formula, and the fact that Chern classes are invertible elements of the Chow ring, this yields the conclusion of the Lemma. □

9.3.3 Triple Points of Plane Curves

We can immediately adapt the preceding ideas to compute the number of points of higher order in linear families of hypersurfaces. By way of example we consider the case of triple points of plane curves.

Suppose that C is a curve of degree d defined by the equation $F = 0$. We say that F has a triple point at p if the Taylor expansion of the restriction f of F to an affine space of which p is the origin has the form

$$f_3 + f_4 + \dots$$

with f_i of degree i.

Let $\mathbb{P}^N$ be the projective space of all plane curves of degree d, and let

$$\Sigma' = \{(C, p) \mid C \text{ has (at least) a triple point at } p\}.$$

The condition that a curve have a triple point at a given point $p \in \mathbb{P}^2$ is six independent linear conditions on the coefficients of the defining equation of C, from which we see that the fibers of the projection map $\Sigma' \to \mathbb{P}^2$ on the second factor are linear spaces $\mathbb{P}^{N-6} \subset \mathbb{P}^N$, and hence that Σ' is irreducible of dimension $N-4$. It follows that the set of curves with a triple point is irreducible as well. An argument similar to that for double points also shows that a general curve $F = 0$ with a triple point has only one. In particular, the projection map $\Sigma' \to \mathbb{P}^N$ on the first factor is birational onto its image. It follows in turn that *the locus $\Phi \subset \mathbb{P}^N$ of curves possessing a point of multiplicity 3 or more is an irreducible variety of dimension $N-4$.* We also see that if C is a general curve with a triple point at p then the cubic term f_3 of the Taylor expansion defines a smooth variety (3 reduced points in $\mathbb{P}^1$)

We ask now for the degree of the variety of curves with a triple point, or, equivalently, the answer to the question: if $F_0, \dots, F_4$ are general polynomials of degree d on $\mathbb{P}^2$, for how many linear combinations $F_a = a_0F_0 + \dots + a_4F_4$ (up to scalars) will the corresponding plane curve $C_a = V(F_a) \subset \mathbb{P}^2$ have a triple point?

If we write τ_F, for the section defined by F in $\mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d))$ then C has a triple point at p if and only if τ_F vanishes at p. The smoothness of the tangent cone at a general triple point shows, ****we need to add an argument here**** that the 5×5 minors of the map $\mathcal{O}^5_{\mathbb{P}^2} \to \mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d)$ generate the maximal ideal locally at a generic point where a linear combination of the F_i defines a curve with a triple point.

Thus the number of triple points in the family is $c_2(\mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d)$. By Proposition9.5,

$$c\Big(\mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d)\Big) = (1 + (d-2)\zeta)^6 = 1 + 6(d-2)\zeta + 15(d^2 - 4d + 4)\zeta^2.$$

In particular, we see that the degree of the locus $\Phi \subset \mathbb{P}^N$ of curves possessing a triple point is $15(d^2 - 4d + 4)$.

In case $d = 1$, the number 15 computed is of course meaningless: there are no lines with triple points! This is because expected dimension $N - 4$ of Φ is negative—any five global sections τ_{F_i} of the bundle $\mathcal{P}^2(\mathcal{O}(1))$ are everywhere dependent. On the other hand, the number 0 computed in the case $d = 2$, which is 0, really does reflect the fact that no conics have a triple point. For $d = 3$ the computation above gives 15, a number we already computed as the degree of the locus of "asterisks" in Section1.2.5.

9.3.4 Higher Dimensions and Other Line Bundles

This whole program of the foregoing sections makes sense if we replace points of order 1 or 2 of hypersurfaces by points of order m of sections of any line bundle $\mathcal{L}$, very ample or not, on any smooth projective variety X. Suppose that the dimension of the variety is n. There are in general three ingredients:

(a) Decide whether singular points of order m occur in the the "expected" codimension, $\binom{d+m}{m}$. For example, if $m = 1$ and $\mathcal{L}$ is very ample, embedding X as a subvariety of $\mathbb{P}^N$, the expected codimension is 1, and this question is the question of whether the dual variety of X is a hypersurface, and the answer is "yes" if X is not covered by linear spaces. See ****need a reference****. In general we expect such points to occur in codimension $\binom{n+m}{m} - n$.

(b) The bundle $\mathcal{P}^m(\mathcal{L})$, has rank $\binom{n+m}{m}$, so that the Chern class $c_n\big(\mathcal{P}^m(\mathcal{L})\big)$ is a cycle corresponding to the locus where a general set of $\binom{n+m}{m} - n+1$ sections of $\mathcal{P}^m(\mathcal{L})$ become linearly dependent, so that some linear combination of them vanishes. Of course if (for example) the dimension of $H^0(\mathcal{L})$ is less than this number, then every set of this many sections will be globally dependent, and the program will not work, and this can happen in other ways as well. Thus we must decide whether a general space of sections of $\mathcal{L}$ of dimension $\binom{n+m}{m} - n + 1$ will have finitely many elements that have an m-th order singularity, and we must decide whether such singularities will correspond to reduced zeros of the wedge product of the $\binom{d+m}{m} - n + 1$ corresponding sections of the bundle $\mathcal{P}^m(\mathcal{L})$. Again, this often happens, but not always.

(c) Compute the number of singular points of order m occuring in a general family of $\binom{d+m}{m}$ sections as the Chern class $c_n\big(\mathcal{P}^m(\mathcal{L})\big)$.

For the last step we need to be able to compute the Chern classes of the bundles $\mathrm{Sym}_i(\Omega_X) \otimes \mathcal{L}$. For $i = 1$ this is equivalent to knowing the Chern classes of the tangent bundle $T_X = \Omega_X^*$, since $c_i(\Omega_X) = (-1)^i c_i(T_X)$; as

we have remarked, these are among the most fundamental invariants of X, and in Section7.4.4 we have seen how to compute them in the case when X is a hypersurface (the same idea can be applied inductively for a complete intersection). We can deduce the Chern classes of the higher symmetric powers from those of Ω by the splitting principle. Rather than work out the general formula, we make the computation in the case of $\mathrm{Sym}_2(\Omega_X)$ when X is a smooth surface.

Suppose that X is a smooth surface and that $c(\Omega_X) = 1 + c_1 + c_2$, so that c_1 is the canonical divisor, and c_2 is a zero-cycle whose degree is the topological Euler class of X by the Hopf Index Theorem (see Section 7.4.4.) If Ω_X had a filtration

$$\Omega_X = \mathcal{E}_2 \supset \mathcal{E}_1 \supset \mathcal{E}_0 = 0$$

such that the successive quotients $\mathcal{E}_{i+1}/\mathcal{E}_i = \mathcal{L}_i$ were line bundles, then $\mathrm{Sym}_2(\Omega_X)$ would have a filtration where the successive quotients were $\mathcal{L}_1^2,\ \mathcal{L}_1 \otimes \mathcal{L}_2,\ \mathcal{L}_2^2$ and $\mathrm{Sym}_2(\Omega_X) \otimes \mathcal{L}$ would have a filtration where the successive quotients were $\mathcal{L}_1^2 \otimes \mathcal{L},\ \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L},\ \mathcal{L}_2^2 \otimes \mathcal{L}$. We now formally factor $1 + c_1 + c_2$ as $(1 + \alpha)(1 + \beta)$, so that (formally) $c_1 = \alpha + \beta$ and $c_2 = \alpha\beta$, and write $c(\mathcal{L}) = 1 + \lambda$. The Chern class of $\mathrm{Sym}_2(\Omega) \otimes \mathcal{L}$ is thus

$$\begin{aligned} c\Big(\mathrm{Sym}_2(\Omega) \otimes \mathcal{L}\Big) &= (1 + 2\alpha + \lambda)(1 + \alpha + \beta + \lambda)(1 + 2\beta + \lambda) \\ &= 1 + 3c_1 + 3\lambda + 2c_1^2 + 4c_2 + 6c_1\lambda + \lambda^2 \\ &\in A^*(X). \end{aligned}$$

Exercise 9.7. Let $\{S_t \subset \mathbb{P}^3\}$ again be a general pencil of surfaces of degree d, and let $X \subset \mathbb{P}^3$ be a fixed general surface of degree d. How many of the surfaces S_t will be tangent to X? (Check your answer in case $d = 1$!)

Exercise 9.8. Suppose now that $\{C_t \subset \mathbb{P}^2\}$ is a pencil of plane curves all of which are singular at a specific point $p \in \mathbb{P}^2$—in other words, let F and G be two general polynomials vanishing to order 2 at p, and take $C_t = V(t_0F + t_1G)$. How many of the curves C_t will be singular somewhere else?

Exercise 9.9. In case $r = 1$, show that Proposition9.4 gives a special case of the *Hurwitz formula*. If C is a smooth projective curve of genus g and $f : C \to \mathbb{P}^1$ a branched cover of degree d, the number b of branch points of f is

$$b = 2d + 2g - 2.$$

(In characteristic $p > 0$ we must as usual assume that f is tamely ramified; that is, the characteristic does not divide any of the ramification exponents.)

Exercise 9.10. Let F and G be two general bihomogeneous polynomials of bidegree (d, e) in two sets of variables $[X_0, X_1]$ and $[Y_0, Y_1, Y_2]$. For how many $t = [t_0, t_1] \in \mathbb{P}^1$ will the hypersurface $Y_t = V(t_0F + t_1G) \subset \mathbb{P}^1 \times \mathbb{P}^2$ be singular?

9.4 Geometry Of Nets of Plane Curves

We now want to consider larger families of curves, and in particular to answer the second keynote question of this chapter. A key step will be to compute the class of the universal singular point $\Sigma = \{(C, p) \mid p \in C_{\text{sing}}\}$ as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^N$ where $\mathbb{P}^N = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(d))$.

9.4.1 Class Of The Universal Singular Point

We can just as easily compute the class of

$$\Sigma_{n,d,m} = \{(H, p) \,|H \subset \mathbb{P}^n \text{ a hypersurface;} \\ p \in H \text{ a singular point of order } \geq m\}$$

as a class in $\mathbb{P}^n \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(d))$. To simplify the notation, we set $V := H^0(\mathcal{O}_{\mathbb{P}^n}(d))$. Let

$$\begin{array}{ccc} \mathbb{P}(V) \times \mathbb{P}^n & \xrightarrow{\tilde{\pi}_1} & \mathbb{P}^n \\ \tilde{\pi}_2 \downarrow & & \\ \mathbb{P}(V) & & \end{array}$$

be the projection maps.

Proposition 9.11. *$\Sigma_{n,d,m}$ is the zero locus of a section of the vector bundle*

$$\mathcal{P}^m := \tilde{\pi}_2^* \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \tilde{\pi}_1^* \mathcal{P}^m\big(\mathcal{O}_{\mathbb{P}^n}(d)\big)$$

which has Chern class

$$c(\mathcal{P}^m) = (1 + (d-m)\zeta_n + \zeta_V)^{\binom{n+m}{n}},$$

where ζ_n and ζ_V are the pullbacks of the hyperplane classes on $\mathbb{P}^n$ and $\mathbb{P}(V)$ respectively. Thus the class of $\Sigma_{n,d,m}$ in $\mathbb{A}^*(\mathbb{P}^n \times \mathbb{P}(V))$ is the sum of the terms of total degree $\binom{n+m}{n}$ in this expression. For example in the case $n = 2, m = 1$ this is

$$[\Sigma] = \zeta_V^3 + 3(d-1)\zeta_2\zeta_V^2 + 3(d-1)^2\zeta_2^2\zeta_V \in A^3(\mathbb{P}^2 \times \mathbb{P}(V)).$$

Proof. The construction is closely analogous to the construction of the universal line in Chapter8. Since every polynomial $F \in V$ defines a section τ_F of $\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))$, we have a map

$$V \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))$$

of vector bundles on $\mathbb{P}^2$. Likewise, we have the tautological inclusion

$$\mathcal{O}_{\mathbb{P}^N}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^(V)}$$

on $\mathbb{P}^(V)$. We pull these maps back to the product $\mathbb{P}^N \times \mathbb{P}^2$ and compose them to arrive at a map

$$\tilde{\pi}_2^* \mathcal{O}_{\mathbb{P}^N}(-1) \to \tilde{\pi}_1^* \mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d)),$$

or equivalently, a section of the bundle $\mathcal{P}$. The zero locus of this map is $\Sigma \subset \mathbb{P}^N \times \mathbb{P}^2$, so the class of $\Sigma_{n,d,m}$ in $\mathbb{A}^*(\mathbb{P}^n \times \mathbb{P}(V))$ is the class of a section of $\mathcal{P}$ as claimed.

To compute the Chern class of $\mathcal{P}^m$ we follow the argument of Proposition9.5, pulling back the sequences

$$0 \to \mathrm{Sym}_i(\Omega_{\mathbb{P}^n})(d) \to \mathcal{P}^i(\mathcal{O}_{\mathbb{P}^n}(d)) \to \mathcal{P}^{i-1}(\mathcal{O}_{\mathbb{P}^n}(d)) \to 0$$

and tensoring with the line bundle $\tilde{\pi}_2^* \mathcal{O}_{\mathbb{P}(V)}(1)$ to get

$$c\Big(\mathcal{P}^m\Big) = \prod_{j=0}^{m} c\Big(\mathrm{Sym}_j(\tilde{\pi}_1^* \Omega_{\mathbb{P}^n}) \otimes \mathcal{O}(d\zeta_n + \zeta_V)\Big).$$

where we have written $\mathcal{O}(d\zeta_n + \zeta_V)$ for the line bundle associated to the pullback of d times the hyperplane section of $\mathbb{P}^n$ tensored with the pullback of the hyperplane section of $\mathbb{P}(V)$. Using the exact sequences

$$0 \to \mathrm{Sym}_i(\Omega_{\mathbb{P}^n}) \to \mathrm{Sym}_i(\mathcal{O}_{\mathbb{P}^n}(-1)) \to \mathrm{Sym}_{i-1}(\mathcal{O}_{\mathbb{P}^n}(-1)) \to 0$$

we get a collapsing product as before, yielding the desired formula for the Chern class of $\mathcal{P}^m$. To deduce the special case at the end of the Proposition it suffices to remember that since ζ_2 is the pullback from a 2-dimensional variety we have $\zeta_2^2 = 0$. □

9.4.2 Singular Points and Discriminant in a Net of Plane Curves

We return to the case of a net of Plane curves of degree d, with notation introduced a the beginning of Section9.4. If $\mathcal{B} \subset \mathbb{P}^N$ is a linear subspace of dimension 2, then the set of curves $\{V(F) \mid F \in \mathcal{B}\}$, or simply the 2-plane $\mathcal{B}$, is called a *net* of curves.

Throughout this section we fix a general net of curves $\mathcal{B}$. Let $\Gamma \subset \mathbb{P}^2$ be the plane curve traced out by the singular points of members of the

net, and let $\mathcal{D} \subset \mathcal{B}$ be the set of singular curves, called the *discriminant curve* of the net $\mathcal{B}$. Since $\mathcal{D}$ is the intersection of $\mathcal{B}$ with the discriminant hypersurface in $\mathbb{P}^N$, its degree is $\deg \mathcal{D} = 3(d-1)^2$ by Proposition9.4.

If we set

$$\Sigma_{\mathcal{B}} := \Sigma \cap (\mathbb{P}^2 \times \mathcal{B}),$$

then the projection maps π_i on Σ restrict to surjections

$$\begin{array}{ccc} \Sigma_{\mathcal{B}} & \xrightarrow{\pi_1|_{\mathcal{B}}} & \Gamma \\ {\scriptstyle \pi_2|_{\mathcal{B}}}\downarrow & & \\ \mathcal{D}. & & \end{array}$$

Since Σ is smooth of codimension 3, Bertini's Theorem shows that $\Sigma_{\mathcal{B}}$ is a smooth curve in $\mathbb{P}^2 \times \mathcal{B}$. Since the generic singular plane curve is singular at only one point, the map $\Sigma_B \to \Gamma$ is birational. Since the fiber of Σ over a given point $p \in \mathbb{P}^2$ is a linear space of dimension $N-3$, the general 2-plane $\mathcal{B}$ containing a curve singular at p will contain a unique such curve. Thus the map $\Sigma_B \to \mathcal{D}$ is also birational, and $\Sigma_{\mathcal{B}}$ is the normalization (resolution of singularities) of each of Γ and $\mathcal{D}$. In particular the geometric genus of $\mathcal{D}$ and of Γ are the same as the genus of $\Sigma_{\mathcal{B}}$.

From the previous section we know that $\Sigma_{\mathcal{B}}$ is the zero locus of a section of the rank 3 bundle $\mathcal{P}^1|_{\mathbb{P}^2 \times \mathcal{B}}$ on $\mathbb{P}^2 \times \mathcal{B}$. This makes it easy to compute the degree and genus of $\Sigma_{\mathcal{B}}$, and we will derive the degree and genus of Γ, answering Keynote Questionb.

Proposition 9.12. *With notation as above, the map $\Sigma_{\mathcal{B}} \to \Gamma$ is an isomorphism, so both curves are smooth. The curve Γ has degree $3d-3$ and thus has genus $\binom{3d-4}{2}$. When $d \geq 2$ the curve $\mathcal{D}$ is singular.*

We will see how the singularities of $\mathcal{D}$ arise, what they look like and how many there are, in Chapter13.

Proof. We begin with the degree of Γ, the number of points of intersection of Γ with a line $L \subset \mathbb{P}^2$. Since $\Sigma_{\mathcal{B}} \to \Gamma$ is birational, this is the same as the degree of the product $[\Sigma_{\mathcal{B}}]\zeta_2 \in A^*(\mathbb{P}^2 \times \mathcal{B})$. (More formally, $\pi_{1*}[\Sigma_{\mathcal{B}}] = \Gamma$, and $\pi_{1*}[\Sigma_{\mathcal{B}}][L] = [\Sigma_{\mathcal{B}}]\zeta_2$.) Write $\zeta_{\mathcal{B}}$ for the restriction of ζ_V, the pullback of the hyperplane section from $\mathbb{P}^N$, to $\mathbb{P}^2 \times \mathcal{B}$. The degree of a class in $\mathbb{P}^2 \times \mathcal{B}$ is the coefficient of $\zeta_2^2\zeta_{\mathcal{B}}^2$ in its expression in

$$A^*(\mathbb{P}^2 \times \mathcal{B}) = \mathbb{Z}[\zeta_2, \zeta_{\mathcal{B}}]/(\zeta_2^3, \zeta_{\mathcal{B}}^3).$$

Since $\zeta_{\mathcal{B}}^3 = 0$ the last formula in Proposition9.11 gives

$$\begin{aligned} \deg(\Gamma) &= \deg \zeta_2\big(3(d-1)\zeta_2\zeta_{\mathcal{B}}^2 + 3(d-1)^2\zeta_2^2\zeta_{\mathcal{B}}\big) \\ &= \deg 3(d-1)\zeta_2^2\zeta_{\mathcal{B}}^2 \\ &= 3d-3. \end{aligned}$$

Since Γ is a plane curve, the arithmetic genus of the curve Γ is $\binom{3d-4}{2}$.

Next we compute the genus g_{Σ_B} of the smooth curve $\Sigma_{\mathcal{B}}$. The normal bundle of $\Sigma_{\mathcal{B}}$ in $\mathbb{P}^2 \times \mathcal{B}$ is the restriction of the rank 3 bundle $\mathcal{P}^1$, and the canonical divisor on $\mathbb{P}^2 \times \mathcal{B}$ has class $-3\zeta_2 - 3\zeta_{\mathcal{B}}$ so by the Adjunction Formula (Hartshorne [1977] Proposition 8.20) the degree of the canonical class of $\Sigma_{\mathcal{B}}$ is is the degree of the line bundle obtained by tensoring the canonical bundle of $\mathcal{P}^2 \times \mathcal{B}$ with $\wedge^3\mathcal{P}^1$ and restricting the result to $\Sigma_{\mathcal{B}}$. This is the degree of the class

$$\begin{aligned}(-3\zeta_2 &- 3\zeta_{\mathcal{B}} + c_1(\mathcal{P}^1))[\Sigma_B] \\ &= (-3\zeta_2 - 3\zeta_{\mathcal{B}} + c_1(\mathcal{P}^1)) \cdot (3(d-1)\zeta_2\zeta_{\mathcal{B}}^2 + 3(d-1)^2\zeta_2^2\zeta_{\mathcal{B}}).\end{aligned}$$

Substituting the value $c_1(\mathcal{P}^1) = 3((d-1)\zeta_2 + \zeta_V)$ from Proposition9.11 and taking account of the fact that $\zeta_V\zeta_{\mathcal{B}} = \zeta_{\mathcal{B}}^2$, this becomes

$$(3d-6)\zeta_2 \cdot (3(d-1)\zeta_2\zeta_{\mathcal{B}}^2 + 3(d-1)^2\zeta_2^2\zeta_{\mathcal{B}}) = (3d-6)(3d-3)\zeta_2^2\zeta_{\mathcal{B}}^2$$

with degree $2g_{\Sigma_B} - 2 = (3d-3)(3d-6)$, and we see that

$$g(\Sigma_{\mathcal{B}}) = \frac{(3d-4)(3d-5)}{2} = \binom{3d-4}{2}.$$

Since this coincides with the arithmetic genus of Γ computed above, we see that Γ is smooth and the map $\Sigma_{\mathcal{B}} \to \Gamma$ is an isomorphism. On the other hand the degree $3(d-1)^2$ of $\mathcal{D}$ is different from that of Γ for all $d \geq 2$, so in these cases the arithmetic and geometric genuses of $\mathcal{D}$ differ, and $\mathcal{D}$ must be singular, completing the proof. □

Here is a different method for computing the degree of Γ: the net $\mathcal{B}$ of curves, having no base points, defines a regular map

$$\varphi_{\mathcal{B}} : \mathbb{P}^2 \to \Lambda$$

where $\Lambda \cong \mathbb{P}^2$ is the projective plane dual to the plane parametrizing the curves in the net $\mathcal{B}$. This map expresses $\mathbb{P}^2$ as a d^2-sheeted branched cover of Λ, and the curve $\Gamma \subset \mathbb{P}^2$ is just the ramification divisor of this map!

By definition,

$$\varphi^*\mathcal{O}_\Lambda(1) = \mathcal{O}_{\mathbb{P}^2}(d);$$

so that if we denote by ζ_Λ the hyperplane class on Λ, we have $\varphi^*\zeta_\Lambda = d\zeta$. Now, we can apply the Hurwitz formula to the cover $\varphi : \mathbb{P}^2 \to \Lambda$; it says that

$$K_{\mathbb{P}^2} = \varphi^*K_\Lambda + \Gamma,$$

and since $K_\Lambda = -3\zeta_\Lambda$, this yields

$$-3\zeta = -3d\zeta + [\Gamma]$$

or $[\Gamma] = (3d-3)\zeta$.

Exercise 9.13. How much of this extends to the case of an n-dimensional family $\mathbb{P}^n \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(d))$ of hypersurfaces of degree d in $\mathbb{P}^n$?

Exercise 9.14. Let $S \subset \mathbb{P}^3$ be a general surface of degree d, and $\mathcal{B}$ a general net of plane sections of S (that is, intersections of X with planes containing a general point $p \in \mathbb{P}^3$. What are the degree and genus of the curve $\Gamma \subset S$ traced out by singular points of this net? What are the degree and genus of the discriminant curve?

9.5 Flexes, and inflection points in general

We'll take up now the third of the keynote questions of this chapter: how to extend the notion of flexes to curves in $\mathbb{P}^n$, and how to count them.

9.5.1 Inflection points

Although we generally avoid multiplicities in this book, we will use them here in a very well-behaved special case: If $C \subset X$ is a reduced curve and $D \subset X$ a Cartier divisor on a scheme X, and no component of C is contained in D then for any closed point $p \in \cap C$ we define $m_p(D \cdot C)$ to be the length (or the dimension over the ground field K, which are the same since we are supposing that K is algebraically closed) of $\mathcal{O}_{C,p}/\mathcal{I}(D) \cdot \mathcal{O}_{C,p}$. Thus for example when $p \notin C \cap D$ we have $m_p(D \cdot C) = 0$, and $m_p(D \cdot C) = 1$ if and only if C and D are both smooth at p and meet transversely there.

For the purpose of this chapter it is convenient to expand this notion beyond what is normally useful in the theory of multiplicities. Suppose that C is a smooth curve over K, $f : C \to X$ is a morphism and D is any subscheme of X such that $f^{-1}(D \cap C)$ is a finite scheme. We then define $m_p(D \cdot C)$ as before to be

$$m_p(D \cdot C) := \dim_{\kappa(p)} \mathcal{O}_{C,p}/f^{-1}(\mathcal{I}(D)) \cdot \mathcal{O}_{C,p}.$$

For example, if $p \in C \subset \mathbb{P}^2$ is a smooth point of a plane curve $L \supset p$ is any line through p then $m_p(L \cdot C) \geq 1$, while L is tangent to C at p if $m_p(L \cdot C) \geq 2$. We say that p is a *flex* of C and L is a *flex tangent* if $m_p(L \cdot C) \geq 3$, and a hyperflex if $m_p(L \cdot C) \geq 4$. We adopt the same definitions in the situation where $f : C \to \mathbb{P}^2$ is a nonconstant morphism from a smooth curve.

To extend the definition beyond plane curves, consider a map $f : C \to \mathbb{P}^n$ from a smooth curve to $\mathbb{P}^n$ whose image is not contained in a hyperplane. We will say that $p \in C$ is an *inflection point* of C is there's a hyperplane

$H \subset \mathbb{P}^r$ with intersection multiplicity $m_p(H \cdot C) \geq r+1$. In this section, we'll derive a formula for the number of such points.

A map f as above corresponds to a line bundle $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^r}(1)$ on C and an $(r+1)$-dimensional vector space $V \subset H^0(\mathcal{L})$ of sections of $\mathcal{L}$ without common zeroes. However, the analysis we're about to carry out applies more generally to arbitrary pairs $(\mathcal{L}, V \subset H^0(\mathcal{L}))$, called a *linear system* on C, so we will work in this more general setting. We set $d = \deg \mathcal{L}$ and continue to write $r+1$ for the vector space dimension of V; such a pair is said to be a *linear system of dimension r and degree d* on C.

Let $p \in C$ be any point, and consider the set $\{\mathrm{ord}_p\sigma\}_{\sigma \neq 0 \in V}$ of orders of vanishing at p of all the nonzero sections $\sigma \in V$. We can find a basis for V consisting of sections vanishing to distinct orders at p (start with any basis; if two sections vanish to the same order replace one with a linear combination of the two vanishing to higher order, and repeat); it follows that the cardinality of this set is exactly $r+1$. We can write

$$\{\mathrm{ord}_p\sigma\}_{\sigma \neq 0 \in V} = \{a_0, a_1, \ldots, a_r\} \quad \text{with} \quad a_0 < a_1 < \cdots < a_r;$$

we call the sequence $(a_0, a_1, \ldots, a_r)$ the *vanishing sequence* of V at p, and the associated sequence $(\alpha_0, \ldots, \alpha_r)$ with $\alpha_i = a_i - i$ the *ramification sequence* of V at p. (If there's any ambiguity about which point, or what linear system, we're referring to, we'll write $a_i(V,p)$ and $\alpha_i(V,p)$ for a_i and α_i.) For example, if $(\mathcal{L}, V)$ arises from a morphism $f : C \to \mathbb{P}^r$ then since p is not a base point of V we must have $a_0(V,p) = \alpha_0(V,p) = 0$.

We say that p is an *inflection point* for a linear system $(\mathcal{L}, V)$ of dimension r if the ramification sequence $(\alpha_0, \ldots, \alpha_r) \neq (0, \ldots, 0)$; equivalently, if $\alpha_r \neq 0$. In case $(\mathcal{L}, V)$ arises from a morphism $f : C \to \mathbb{P}^r$ that is an embedding near p, the number $a_i(V,p)$ can be computed from the multiplicities with which planes meet $f(C)$ at $q = f(p)$: The equations of an i-plane L pull back to a vector space of $n-i$ independent sections of V, and the multiplicity $m_p(L \cdot C)$ is the minimum of the orders of vanishing of these sections at p. Thus for $i \geq 1$

$$a_i \max_{L \subset \mathbb{P}^r \text{ a plane of dimension } i} m_p(L \cdot C).$$

In particular, p is an inflection point of C if and only if some hyperplane meets C with multiplicity $\geq r+1$ at p.

We will define the *weight* of inflection of a point $p \in C$ with respect to $(\mathcal{L}, V)$ to be

$$w(V,p) := \sum_{i=0}^{r} \alpha_i(V,p).$$

9.5.2 The Plücker formula

Our main result gives the sum of the weights of the inflection points of a linear system in characteristic zero:

Theorem 9.15 (Plücker Formula)**.** *Let C be a smooth projective curve of genus g over an algebraically closed field of characteristic zero. If $(\mathcal{L}, V)$ is a linear system of degree d and dimension r on C then*

$$\sum_{p \in C} w(V,p) = (r+1)d + (r+1)r(g-1).$$

The result is false in positive characteristic because of the possible presence of wild ramification and inseparable maps. See for example Laksov [1984] and Osserman [2006] for more information.

Proof. The key observation is that both sides of the desired formula are equal to the degree of the first Chern class of the bundle $\mathcal{P}^r(\mathcal{L})$. We can compute the class of this bundle from Theorem9.2 as

$$c\Big(\mathcal{P}^r(\mathcal{L})\Big) = \prod_{j=0}^{r} c\Big(\mathrm{Sym}_j(\Omega_C) \otimes \mathcal{L}\Big).$$

Since Ω_C is a line bundle we have $\mathrm{Sym}_j(\Omega_C) = \Omega_C^j$, and thus $c((\mathrm{Sym}_j(\Omega_C) \otimes \mathcal{L}) = 1 + jc_1(\Omega_C) + c_1(\mathcal{L})$. It follows that

$$c_1\Big(\mathcal{P}^r(\mathcal{L})\Big) = (r+1)c_1(\mathcal{L}) + \binom{r+1}{2} c_1(\Omega_C).$$

Since the degree of Ω_C is $2g-2$, the degree of this class is $(r+1)d + (r+1)r(g-1)$, the right hand side of the Plücker formula.

We may define a map $\varphi : \mathcal{O}_C^{r+1} \to \mathbb{P}^r(CL)$ by choosing any basis $\sigma_0, \cdots, \sigma_r$ of V and sending the i-th basis element of $\mathcal{O}_C^{r+1}$ to τ_{σ_i}. We will complete the proof by showing that for any point $p \in C$ the determinant of the map φ vanishes at p to order exactly $w(V,p)$, and that there are only finitely many points $w(V,p)$ where the determinant is nonzero.

To this end fix a point $p \in C$. Since the determinant of φ depends on the choice of basis $\sigma_0, \ldots, \sigma_r$ only up to scalar, we may choose the basis σ_i so that the order of vanishing $\mathrm{ord}_p(\sigma_i) = a_i$ at p is $a_i(V,p)$. Trivializing $\mathcal{L}$ in a neighborhood of p, we may think of the section σ_i locally as a function, and φ is represented by the matrix

$$\begin{pmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_r \\ \sigma_0' & \sigma_1' & \ldots & \sigma_r' \\ \vdots & \vdots & & \vdots \\ \sigma_0^{(r)} & \sigma_1^{(r)} & \ldots & \sigma_r^{(r)} \end{pmatrix}.$$

where σ_i' denotes the derivative, and $\sigma_i^{(r)}$ the r-th derivative. Because σ_i vanishes to order $\geq i$ p, the matrix evaluated at p is lower triangular, and the entries on the diagonal are all nonzero if and only if $a_i = i$ for each i; that is, if and only if p is not an inflection point for V.

We can compute the exact order of vanishing of $\det\varphi$ at an inflection point as follows. Denote by $v(z)$ the $(r+1)$-vector $(\sigma_0(z), \ldots, \sigma_r(z))$, so that the determinant of φ is the wedge product

$$\det(\varphi) = v \wedge v' \wedge \cdots \wedge v^{(r)}.$$

Applying the product rule, the n^{th} derivative of $\det(\varphi)$ is then a linear combination of terms of the form

$$v^{(\beta_0)} \wedge v^{(\beta_1+1)} \wedge \cdots \wedge v^{(\beta_r+r)}$$

with $\sum \beta_i = n$. Now, $v^{(\beta_0)}(p) = 0$ unless $\beta_0 \geq \alpha_0$; similarly, $v^{(\beta_0)}(p) \wedge v^{(\beta_1)}(p) = 0$ unless $\beta_0 + \beta_1 \geq \alpha_0 + \alpha_1$, and so on. We conclude that *any derivative of* $\det(\varphi)$ *of order less than* $w = \sum \alpha_i$ *vanishes at* p; and the expression for the w^{th} derivative of $\det(\varphi)$ has exactly one term nonzero at p, namely

$$v^{(\alpha_0)} \wedge v^{(\alpha_1+1)} \wedge \cdots \wedge v^{(\alpha_r+r)}.$$

Since this term appears with nonzero coefficient (we have used characteristic zero here), we conclude that $\det(\varphi)$ vanishes to order exactly w at p.

It remains to show that not every point of C can be an inflection point for V—that is, that $\det\varphi$ is not identically zero. To prove this, suppose that $\det(\varphi)$ does vanish identically, that is, that

$$v \wedge v' \wedge \cdots \wedge v^{(k)} \equiv 0 \tag{9.1}$$

for some $k \leq r$. Suppose in addition that k is the smallest such integer, so that at a general point $p \in C$ we have

$$v(p) \wedge v'(p) \wedge \cdots \wedge v^{(k-1)}(p) \neq 0;$$

in other words, $v(p), \ldots, v^{(k-1)}(p)$ are linearly independent, but $v^{(k)}(p)$ lies in their span Λ. Again using the product rule to differentiate the expression9.1, we see that

$$\frac{d}{dz}\big(v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k)}\big) = v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k+1)} \equiv 0;$$

so that $v^{(k+1)}(p)$ also lies in the span of $v(p), \ldots, v^{(k-1)}(p)$. Similarly, taking the second derivative of9.1, we see that

$$\frac{d^2}{dz^2}\big(v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k)}\big) = v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k+2)} \equiv 0,$$

where are all the other terms in the derivative are zero because they are $(k+1)$-fold wedge products of vectors lying in a k-dimensional space. Continuing

in this way, we see that $v^{(m)}(p) \in \Lambda$ for all m. Since we are in characteristic zero it follows by integration that $v(z) \in \Lambda$ for all z. This implies that the linear system V has dimension $k < r+1$, contradicting our assumptions. □

The situation in higher dimensions is equally interesting: ****does this require char 0?****

Proposition 9.16. *Let $S \subset \mathbb{P}^3$ be a surface of degree $d \geq 2$, $p \in S$ a general point and $H = \mathbb{T}_p S \subset \mathbb{P}^3$ the tangent plane to S at p.*

(a) *Assuming that S is general, show that the intersection $H \cap S$ has an ordinary double point at p.*

(b) *Now prove this for arbitrary S.*

The proof, which uses the second fundamental form of S, is described in Griffiths and Harris [1979]. ****give a precise reference.****

9.5.3 Consequences of the Plücker formula

The Plücker formula gives the answer to Keynote Questionc. We don't even need to assume C is smooth; if C is singular, as long as it's reduced and irreducible we view it as the image of the map $\nu : \tilde{C} \to \mathbb{P}^2$ from its normalization. If we apply the Plücker formula to this linear system corresponding to this map, we see that C has

$$(r+1)d + r(r+1)(g-1) = 3d + 6g - 6$$

flexes, where g is the genus of $\tilde{C}$, that is, geometric genus of C. If the curve C is indeed smooth, then $2g - 2 = d(d-3)$, and so this yields

$$3d + 6g - 6 = 3d + 3d(d-3) = 3d(d-2).$$

To be explicit, this formula counts points $p \in \tilde{C}$ such that for some line $L \subset \mathbb{P}^2$, the multiplicity of the pullback divisor $\nu^* L$ at p is at least 3. In particular:

(a) It does not necessarily count nodes of C, even though at a node p of C, there will be lines having intersection multiplicity 3 or more with C at p.

(b) It does count singularities where the differential $d\nu$ vanishes, for example cusps.

There is an alternative notion of a flex point of a (possibly singular) curve $C \subset \mathbb{P}^2$: a point $p \in C$ such that for some line $L \subset \mathbb{P}^2$ through p, we have

$$m_p(C \cdot L) \geq 3;$$

that is, the restriction to L of the defining equation of C vanishes to order at least 3 at p. In this sense, a node p of a plane curve C is in fact a flex point, since the tangent lines to either of the branches of the curve at the node will have intersection multiplicity at least 3 with C at p. (In talking about formulas for the number of flexes of a plane curve, we will always specify which notion is being invoked.) This is the number that can be counted by the intersections of the curve with its Hessian (see Exercise 9.19. In Chapter 10 we'll see a different approach to counting flexes of a plane curve that will count them in the latter sense, including nodes.

Exercise 9.17. First, verify that for a general curve $C \subset \mathbb{P}^2$ of degree d the number $3d(d-2)$ is the actual number of flexes of C; that is, all inflection points of C will have weight 1.

Exercise 9.18. Now, let $\{C_t = V(t_0F + t_1G)\}$ be a general pencil of plane curves of degree $d \geq 3$; suppose C_0 is a singular element of C (so that in particular by Proposition9.1, C_0 will have just one node as singularity). By our formula, C_0 will have 6 fewer flexes than the general member C_t of the pencil. Where do the other 6 flexes go? If we consider the incidence correspondence

$$\Phi = \{(t, p) : C_t \text{ is smooth and } p \text{ is a flex of } C_t\} \subset \mathbb{P}^1 \times \mathbb{P}^2,$$

what is the geometry of the closure of Φ near $t = 0$?

There is another way to calculate the number of flexes of a plane curve that may shed some light on the last problem (this is not the approach we're going to take in Chapter 10; it's a third approach). Briefly, if C is the zero locus of a homogeneous polynomial $F(X, Y, Z)$, we define the *Hessian* of C be be the zero locus of the polynomial

$$H = \begin{vmatrix} \frac{\partial^2 F}{\partial X^2} & \frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial X \partial Z} \\ \frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial Y^2} & \frac{\partial^2 F}{\partial Y \partial Z} \\ \frac{\partial^2 F}{\partial X \partial Z} & \frac{\partial^2 F}{\partial Y \partial Z} & \frac{\partial^2 F}{\partial Z^2} \end{vmatrix}$$

Exercise 9.19. Let C be a plane curve and D its Hessian.

(a) Let $p \in C$ be a smooth point. Show that p is a flex of C if and only if it's a point of intersection of C with D.

(b) With $p \in C$ as above, show moreover that the weight of p as an inflection point is equal to the intersection multiplicity $m_p(C \cdot D)$.

(c) Deduce that a smooth plane curve has exactly $3d(d-2)$ flexes, counting multiplicity.

(d) Now suppose C is a general singular curve, $p \in C$ its node. What is the intersection multiplicity of C with D at p? If C is special, under what circumstances might $m_p(C \cdot D)$ be larger?

Here are some special cases of the Plücker formula in general:

Exercise 9.20. Show that the only smooth, irreducible and nondegenerate curve $C \subset \mathbb{P}^r$ with no inflection points is the rational normal curve.

Exercise 9.21. Observe that in case $g = 1$ and $d = r + 1$—that is, the curve is an elliptic normal curve E—the Plücker formula yields the number $(r + 1)^2$ of inflection points. Show that these are exactly the translates of any one by the points of order $r + 1$ on E, each having weight 1.

Exercise 9.22. Let C be a smooth curve of genus $g \geq 2$. A point $p \in C$ is called a *Weierstrass point* if there exists a nonconstant rational function on C with a pole of order g or less at p and regular on $C \setminus \{p\}$.

(a) Show that the Weierstrass points of C are exactly the inflection points of the canonical map $\varphi : C \to \mathbb{P}^{g-1}$; and

(b) Use this to count the number of Weierstrass points on C.

9.5.4 Curves with Hyperflexes

As we've asked you to verify in Exercise9.17, a general plane curve has only inflection points of weight 1. What about the curves that *do* have a flex of order ≥ 2?

Exercise 9.23. Let $\mathbb{P}^N$ be the space of all plane curves of degree $d \geq 4$, and let $H \subset \mathbb{P}^N$ be the closure of the locus of smooth curves with a hyperflex. Show that H is a hypersurface.

What is the degree of the hypersurface of Exercise9.23? We will be able to answer this question once we have developed the techniques of Chapter13.

If we try to generalize thes ideas, there are many open questions. For example: *Let $\mathcal{H}$ be any component of the Hilbert scheme parametrizing curves in $\mathbb{P}^r$ whose general point corresponds to a smooth, irreducible nondegenerate curve $C \subset \mathbb{P}^r$. Do all inflection points of C have weight 1?*

Here is one case in which the answer is yes:

Exercise 9.24. Let $C \subset \mathbb{P}^r$ be a general complete intersection of hypersurfaces of degrees $d_1, \ldots, d_{r-1} \geq 2$. Show that C has no inflection points of weight 2 or more.

9.6 The Topological Hurwitz Formula

Over the complex numbers we can use the topological Euler characteristic to give a different approach to the whole question of ramification. It sheds

additional light on the formula of Proposition9.4, and is applicable in many circumstances in which Proposition9.4 cannot be used.

Suppose that X is a smooth projective variety over $\mathbb{C}$, and $Y \subset X$ a divisor. Denoting the topological Euler characteristic (in the classical, or analytic topology) by χ_{top} we have

$$\chi_{\text{top}}(X) = \chi_{\text{top}}(Y) + \chi_{\text{top}}(X \setminus Y)$$

This relation is not hard to prove: we apply the Mayer-Vietoris sequence to the covering of X by $U = X \setminus Y$ and a small open neighborhood V of Y, and argue that, since Y is the zero locus of a section of a line bundle in X, the Euler characteristic of the intersection $U \cap V$ is 0. In fact, the formula $\chi_{\text{top}}(X) = \chi_{\text{top}}(Y) + \chi_{\text{top}}(X \setminus Y)$ applies much more generally, to an arbitrary subvariety Y of an arbitrary X. But this requires a much harder result, that we can triangulate X in such a way that Y forms a subcomplex; even the statement that an arbitrary variety admits a triangulation is difficult. See **?**]. ****insert something about tubular neighborhoods. This is very standard for smooth subvarieties, less so for hypersurfaces or locally complete intersections, still less for arbitrary singular subvarieties. Is there a good reference in Milnor?****

Let X be a smooth projective variety and let $f : X \to B$ be a map to a smooth curve B of genus g. This being characteristic 0, there are only a finite number of points $p_1, \ldots, p_\delta$ over which the fiber X_{p_i} is singular. We can apply the relation on Euler characteristics to the divisor

$$Y = \bigcup_{i=1}^{\delta} X_{p_i} \subset X.$$

Naturally, $\chi_{\text{top}}(Y) = \sum \chi_{\text{top}}(X_{p_i})$; and on the other hand the open set $X \setminus Y$ is a fiber bundle over the complement $B \setminus \{p_1, \ldots, p_\delta\}$, so that

$$\begin{aligned} \chi_{\text{top}}(X \setminus Y) &= \chi_{\text{top}}(X_\eta)\chi_{\text{top}}(B \setminus \{p_1, \ldots, p_\delta\}) \\ &= (2 - 2g - \delta)\chi_{\text{top}}(X_\eta) \end{aligned}$$

where again η is a general point of B. Combining these, we have

$$\begin{aligned} \chi_{\text{top}}(X) &= (2 - 2g - \delta)\chi_{\text{top}}(X_\eta) + \sum_{i=1}^{\delta} \chi_{\text{top}}(X_{p_i}) \\ &= \chi_{\text{top}}(B)\chi_{\text{top}}(X_\eta) + \sum_{i=1}^{\delta} \big(\chi_{\text{top}}(X_{p_i}) - \chi_{\text{top}}(X_\eta)\big). \end{aligned}$$

In this form, we can extend the last summation over all points $q \in B$. We have proven:

Theorem 9.25 (Topological Hurwitz Formula)**.** *Let $f : X \to B$ be a morphism from a smooth projective variety to a smooth projective curve;*

let $\eta \in B$ be a general point. Then

$$\chi_{\text{top}}(X) = \chi_{\text{top}}(B)\chi_{\text{top}}(X_\eta) + \sum_{q\in B}\big(\chi_{\text{top}}(X_q) - \chi_{\text{top}}(X_\eta)\big).$$

In English: the Euler characteristic of X is what it would be if it were a fiber bundle over B—that is, the product of the Euler characteristics of B and the general fiber X_η—with a "correction term" coming from each singular fiber, equal to the difference between its Euler characteristic and the Euler characteristic of the general fiber.

To see why Theorem9.25 is a generalization of the classical Hurwitz formula (see for example Hartshorne [1977] Theorem *****), consider case where X is a smooth curve of genus h, and $f : X \to C$ a branched cover of degree d. For each point $p \in C$, we write the fiber X_p as a divisor:

$$f^*(p) = \sum_{q\in f^{-1}(p)} m_q \cdot q.$$

We call the integer $m_q - 1$ the *ramification index* of f at q; we define the *ramification divisor* R of f to be the sum

$$R = \sum_{q\in X}(m_q - 1)\cdot q$$

and we define the *branch divisor* B of f to be the image of R (as a divisor, not as a scheme!)—that is,

$$B = \sum_{p\in C} b_p \cdot p \quad \text{where} \quad b_p = \sum_{q\in f^{-1}(p)} m_q - 1.$$

Now, since the degree of any fiber $X_p = f^*(p)$ of f is equal to d, for each $p \in C$ the cardinality of $f^{-1}(p)$ will be $d - b_p$, so its contribution to the topological Hurwitz formula is $-b_p$. The formula then yields

$$2 - 2h = d(2 - 2g) - \deg(B),$$

the classical Hurwitz formula.

9.6.1 Application to pencils

To apply the topological Hurwitz formula to Keynote Questionc, suppose that $\{C_t = V(t_0F + t_1G) \subset \mathbb{P}^2\}$ is a general pencil of plane curves of degree d. The polynomials F and G being general, the base locus $\Gamma = V(F, G)$ of the pencil will consist of d^2 reduced points, and the graph of the rational map $[F, G] : \mathbb{P}^2 \to \mathbb{P}^1$, which is

$$X = \{(p, t) : t_0F + t_1G = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^1,$$

is be the blow-up of $\mathbb{P}^2$ along Γ. In particular, X is be smooth, so Theorem9.25 can be applied to the map $f : X \to \mathbb{P}^1$ that is the projection on the second factor.

Since X is the blow-up of $\mathbb{P}^2$ at d^2 points, we have

$$\chi_{\text{top}}(X) = \chi_{\text{top}}(\mathbb{P}^2) + d^2 = d^2 + 3.$$

Next, we know that a general fiber C_η of the map f is a smooth plane curve of degree d; as we've seen in Section ** of Chapter 3 ****add genus formula for smooth plane curves to discussion of adjunction in Chapter 3**** its genus is $\binom{d-1}{2}$ and hence

$$\chi_{\text{top}}(C_\eta) = -d^2 + 3d.$$

We know from Proposition9.1 that each singular fiber C appearing in a general pencil of plane curves has a single node as singularity. Thus its normalization $\tilde{C}$ will be a curve of genus $\binom{d-1}{2} - 1$ and hence Euler characteristic $-d^2 + 3d + 2$. Since C is obtained from $\tilde{C}$ by identifying two points, we have

$$\chi_{\text{top}}(C) = -d^2 + 3d + 1,$$

so the contribution of each singular fiber of f to the topological Hurwitz formula is exactly 1. It follows that the number of singular fibers is

$$\begin{aligned}\delta &= \chi_{\text{top}}(X) - \chi_{\text{top}}(\mathbb{P}^1)\chi_{\text{top}}(C_\eta) \\ &= d^2 + 3 - 2(-d^2 + 3d) \\ &= 3d^2 - 6d + 3,\end{aligned}$$

as we saw before.

This same analysis can be applied to a pencil of curves on any smooth surface S. Let L be a line bundle on S with first Chern class $\lambda = c_1(L) \in A^1(S)$, and let $V \subset H^0(L)$ be a two-dimensional vector space of sections with

$$\{C_{[\sigma]} = V(\sigma) \subset S\}_{[\sigma]\in\mathbb{P}V}$$

the corresponding pencil of curves. We make—for the time being—two assumptions:

(a) The base locus $\Gamma = V(\{\sigma\}_{\sigma\in V})$ of the pencil is reduced, that is, consists of λ^2 points; and

(b) Each of the finitely many singular elements of the pencil has just one node as singularity.

We also denote by $c_i = c_i(T_S)$ the Chern classes of the tangent bundle to S.

Given this, the calculation proceeds as before: we let X be the blow-up of S along Γ, and apply the topological Hurwitz formula to the natural

map $f : X \to \mathbb{P}V^* \cong \mathbb{P}^1$. By the Hopf formula (Section7.4.4), the Euler characteristic $\chi_{\text{top}}S$ is its second Chern class c_2, so that

$$\chi_{\text{top}}(X) = \chi_{\text{top}}(S) + \deg(\Gamma) = c_2 + \lambda^2.$$

Next, by the adjunction formula, the Euler characteristic of a smooth member C_η of the pencil is given by

$$\chi_{\text{top}}(C_\eta) = -\deg(\omega_{C_\eta}) = -(\omega_S + \lambda)\cdot\lambda = -\lambda^2 - c_1\lambda.$$

and we know that the Euler characteristic of each singular element of the pencil is one greater than the Euler characteristic of the general one. In sum, then, the number of singular fibers is

$$\begin{aligned}\delta &= \chi_{\text{top}}(X) - \chi_{\text{top}}(\mathbb{P}^1)\chi_{\text{top}}(C_\eta)\\ &= \lambda^2 + c_2 - 2(-\lambda^2 - c_1\lambda)\\ &= 3\lambda^2 + 2\lambda c_1 + c_2,\end{aligned}$$

agreeing with our previous calculation.

Exercise 9.26. Let $X \subset \mathbb{P}^r$ be a general hypersurface of degree d, and $X_0 \subset \mathbb{P}^r$ a hypersurface of the same degree having one ordinary double point as singularity. Show that

$$\chi_{\text{top}}(X_0) = \chi_{\text{top}}(X) + (-1)^{r-1}.$$

Exercise 9.27. Use the topological Hurwitz formula, and the preceding exercise, to derive the formula of Proposition9.4.

9.6.2 Multiplicities of the discriminant hypersurface

One striking thing about this derivation of the formula for the number of singular elements in a pencil is that it gives a description of the multiplicities with which a given singular element counts that allows us to determine these multiplicities at a glance. In the derivation of the formula, we assumed that the singular elements of the pencil had only nodes as singularities. But what if an element C of the pencil has a cusp? In that case the geometric genus of the curve—the genus of its normalization $\tilde{C}$—is again 1 less than the genus of the smooth fiber, but this time instead of identifying two points we're just "crimping" the curve at one point. (In the analytic topology, C and $\tilde{C}$ are homeomorphic.) Thus

$$\chi_{\text{top}}(C) = \chi_{\text{top}}(\tilde{C}) = \chi_{\text{top}}(C_\eta) + 2,$$

and the fiber C counts with multiplicity 2. Similarly, if C has a tacnode, we have $g(\tilde{C}) = g(C_\eta) - 2$, so that $\chi_{\text{top}}(\tilde{C}) = \chi_{\text{top}}(C_\eta) + 4$; but we identify two points of $\tilde{C}$ to form C so in all

$$\chi_{\text{top}}(C) = \chi_{\text{top}}(C_\eta) + 3,$$

and the fiber C counts with multiplicity 3. Again, if C has a triple point, then $g(\tilde{C}) = g(C_\eta) - 3$, but we identify 3 points of $\tilde{C}$ to form C, so

$$\chi_{\text{top}}(C) = \chi_{\text{top}}(C_\eta) + 4,$$

and the fiber C counts with multiplicity 4. Moreover, if a fiber has more than one isolated singularity, the same analysis shows that the multiplicity with which it appears in the formula above is just the sum of the contributions coming from the individual singularities.

In addition to giving us a way of determining the contribution of a given singular fiber to the expected number, this approach tells us something about the geometry of the discriminant locus $\mathcal{D} \subset \mathbb{P}^N$. To see this, suppose that $C \subset \mathbb{P}^2$ is any plane curve of degree d with isolated singularities. Let D be a general plane curve of the same degree, and consider the pencil $\mathcal{B}$ of plane curves they span—in other words, take $\mathcal{B} \subset \mathbb{P}^N$ a general line through the point $C \in \mathbb{P}^N$. By what we've said, the number of singular elements of the pencil $\mathcal{B}$ other than C will be $3(d-1)^2 - (\chi_{\text{top}}(C) - \chi_{\text{top}}(C_\eta)$, where C_η is a smooth plane curve of degree d; it follows that the multiplicity of the intersection $\mathcal{B} \cap \mathcal{D}$ at C is $\chi_{\text{top}}(C) - \chi_{\text{top}}(C_\eta)$. We deduce the

Proposition 9.28. *Let $C \subset \mathbb{P}^2$ be any plane curve of degree d with isolated singularities. Then*

$$\text{mult}_C(\mathcal{D}) = \chi_{\text{top}}(C) - \chi_{\text{top}}(C_\eta).$$

Thus, a plane curve with a cusp (and no other singularities) is a double point of $\mathcal{D}$; a plane curve with a tacnode is a triple point, and so on. Note also that a curve C with one node and no other singularities is necessarily a smooth point of $\mathcal{D}$.

Exercise 9.29. Prove the last statement directly by analyzing the defining equations of the universal singular point $\Sigma \subset \mathbb{P}^N \times \mathbb{P}^2$, as in the first section of this chapter. Can you deduce other cases of Proposition9.28 as well in this way?

9.6.3 Tangent cones

Let $\mathbb{P}^N$ be the projective space of plane curves of degree d. We can use the ideas above to describe the tangent spaces and tangent cones to the discriminant hypersurface $\mathcal{D} \subset \mathbb{P}^N$. To do this, we have to remove the first assumption in our application of the topological Hurwitz formula to pencils of curves, and deal with pencils whose base locus is not reduced.

For example, suppose the base locus $\Gamma = V(F, G)$ of the pencil $C_t = V(t_0F + t_1G)$ has $d^2 - 2$ reduced points and one point p with multiplicity 2, but that the pencil is otherwise general (that is, F, G are general forms of degree d in the ideal of a subscheme of the plane of length 2.) In this

case, exactly one member C_0 of the pencil is singular at p and all the others are smooth and have a common tangent line at p.

Ler X be the minimal blow-up resolving the indeterminacy of the map $S \to \mathbb{P}^1$ associated to the pencil—that is, X is obtained by blowing up $\mathbb{P}^2$ at p, then blowing up the resulting surface at the point p' on the exceptional divisor corresponding to the common tangent line to the smooth members of the pencil at p. (This is not the blow up of S along the scheme Γ, which is singular! See for example Eisenbud and Harris [2000]****give precise ref****) The fiber of the map $X \to \mathbb{P}^1$ corresponding to C_0 is the union of the proper transform of C_0 and the proper transform E of the first exceptional divisor, so that, for example, if C_0 had a node at p the fiber is the union of its normalization $\tilde{C}_0$ and a copy of $\mathbb{P}^1$, meeting at the two points of $\tilde{C}_0$ lying over the node p. In sum, the Euler characteristic of the fiber is

$$\chi_{\text{top}}(\tilde{C} \cup E) = \chi_{\text{top}}(\tilde{C}) + \chi_{\text{top}}(E) - 2 = \chi_{\text{top}}(\tilde{C}) = \chi_{\text{top}}(C_\eta) + 2$$

and the fiber counts with multiplicity 2. We can use this to analyze the tangent planes to $\mathcal{D}$ at its simplest points:

Proposition 9.30. *Let C be a plane curve with a node at p and no other singularities. The tangent plane $\mathbb{T}_C\mathcal{D} \subset \mathbb{P}^N$ is the hyperplane $H_p \subset \mathcal{D}$ of curves containing the point p.*

Proof. If $C \subset \mathbb{P}^2$ is a plane curve with one node p and no other singularities, then by Proposition9.28 C is a smooth point of $\mathcal{D}$, so it suffices to show that H_p is contained in the tangent space to $\mathcal{D}$. But if $\mathcal{B} \subset \mathbb{P}^N$ is a general pencil including C and having p as a base point, $\mathcal{B}$ will meet $\mathcal{D}$ in exactly $3(d-1)^2-1$ points. By Bertini's Theorem it can't be tangent to $\mathcal{D}$ anywhere except at C, and so it must be tangent at C. □

Exercise 9.31. Again, verify this statement by analyzing the equations of $\Sigma \subset \mathbb{P}^N \times \mathbb{P}^2$.

****Give a reference for the following.**** More generally, if C is a curve with a unique singular point p, the tangent cone to $\mathcal{D}$ at C will be a multiple of the hyperplane H_p; and more generally still, if C has isolated singularities $p_1, \ldots, p_\delta$ the tangent cone $\mathbb{T}_C\mathcal{D}$ is supported on the union of the planes H_{p_i}.

There is also a sort of converse to Proposition9.28:

Proposition 9.32. *The smooth locus of $\mathcal{D}$ consists exactly of those curves with a single node and no other singularity.*

Proof. Proposition9.28 gives one inclusion: if C has a node and no other singularity, it is a smooth point of $\mathcal{D}$. Moreover, if C has more than one

(isolated) singular point, then the projection map $\Sigma \to \mathcal{D}$ is finite but not one-to-one over C; it is intuitively clear (and follows from Zariski's Main Theorem) that $\mathcal{D}$ is analytically reducible and hence singular at C. Moreover, we observe that if $d \geq 3$ any curve with multiple components is a limit of curves with isolated singularities and at least 3 nodes—just deform each multiple component $mC_0 \subset C$ to a union of m general translates of C_0—so these must also lie in the singular locus of $\mathcal{D}$.

It remains to see that if C is a singular curve having a singularity p other than a node, then $\mathcal{D}$ is singular at C. This follows from an analysis of plane curve singularities: if C has isolated singularities including a point p of multiplicity $k \geq 3$, then as we saw in Section ** of Chapter 3 the genus of the normalization $\tilde{C}$ is at most

$$g(\tilde{D}) \leq \binom{d-1}{2} - \frac{k(k-1)}{2}$$

and since at most k points of the normalization lying over p are identified in C, the Euler characteristic

$$\chi_{\text{top}}(C) \geq 2 - 2g(\tilde{C}) + k - 1 \geq -d(d-3) + (k-1)^2.$$

As for double points p other than a node, we've already done the case of a cusp; other double points will drop the genus of the normalization by 2 or more, and since we have at most two points of the normalization lying over p, we must have $\chi_{\text{top}}(C) \geq -d(d-3) + 3$. □

We should add that that there are many, many problems having to do with the local geometry of $\mathcal{D}$ and its stratification by singularity type, only a small fraction of which we know how to answer. The statements above barely scratch the surface; for more, see ****insert a reference****.

Exercise 9.33. (a) Let $\mathbb{P}^5$ be the space of conic plane curves, and $\mathcal{D} \subset \mathbb{P}^5$ the discriminant hypersurface. Let $C \in \mathcal{D}$ be a point corresponding to a double line. What is the multiplicity of $\mathcal{D}$ at C, and what is the tangent cone?

(b) Let $\mathbb{P}^{14}$ now be the space of quartic plane curves, and $\mathcal{D} \subset \mathbb{P}^{14}$ the discriminant hypersurface. Let $C \in \mathcal{D}$ be a point corresponding to a double conic. What is the multiplicity of $\mathcal{D}$ at C, and what is the tangent cone?

10

Compactifying Parameter Spaces

Keynote Questions

(a) (Five Conic Problem): Given five general plane conics $C_1, \ldots, C_5 \subset \mathbb{P}^2$, how many smooth conics are tangent to all 5?

(b) Given 11 general points $p_1, \ldots, p_{11} \in \mathbb{P}^2$ in the plane, how many rational quartic curves $C \subset \mathbb{P}^2$ contain them all?

All the applications of enumerative geometry exploit the fact that interesting "families" of algebraic varieties form the points of an algebraic variety themselves, the *parameter space*, and our efforts have all been toward counting intersections on these spaces. But to use intersection theory to count something, the parameter space must be projective (or at least proper) so that we have a degree map, as defined in Chapter 1. In the first case we treated in this book, that of the family of planes of a certain dimension in projective space, the natural parameter space was the Grassmannian, and the fact that it is projective is what makes the Schubert calculus so useful for enumeration ****did we use the words Schubert calculus? – we should have.**** When we studied the questions about hypersurfaces containing lines, we were similarly concerned with parameter spaces that were projective—either the projective space of hypersurfaces itself, or the universal hypersurface. These spaces have an additional feature of importance: a universal family of the geometric objects we are studying, or what comes to the same thing, the property of representing a functor we under-

stand. This property is useful in many ways, first of all for understanding tangent spaces, and thus transversality questions.

In many interesting cases, however, the "natural" parameter space for a problem is *not* projective. To use the tools of intersection theory to count something, we must add points to the parameter space to complete it to a projective (or at least proper) variety. It is customary to call these new points the *boundary*, although this is not a topological boundary in any ordinary sense—the boundary points may look like any other point of the space—and (more reasonably) to call the enlarged space a *compactification* of the original space. If we are lucky, the boundary points of the compactification still parametrize some sort of geometric object we understand. In such cases we can use this structure to solve geometric problems. But as we shall see, the boundary can also get in the way, even when it seems quite natural. In such cases, we might look for a "better" compactification... but just how to do this is a matter of art rather than of science.

Perhaps the first problem in enumerative geometry where this tension became clear is the Five Conic Problem, which was solved in a naive way, not taking the difficulty into account (and therefore getting the wrong answer) bySteiner [1848], and again, with the necessary subtlety and correct answers, by Chasles [1864]. In this case there is a very beautiful and classical construction of a good parameter space, the space of *complete conics*. In this chapter we will explore the construction, and briefly discuss two more general constructions: Hilbert schemes and Kontsevich spaces.

10.1 Approaches To The Five Conic Problem

Conics in characteristic 2 are interesting, but rather different than conics in characteristic not 2. For simplicity: *We will assume throughout this Chapter that the characteristic of the ground field is not 2.*

To reiterate the problem: Given five general plane conics $C_1, \ldots, C_5$, how many smooth conics are tangent to all five? Here is a naive approach:

(a) The set of plane conics is parametrized by $\mathbb{P}^5$, and for each of the given conics C_i the locus of conics tangent to C_i is an irreducible hypersurface $Z_i \subset \mathbb{P}^5$. One sees this by considering the incidence correspondence:

$$\begin{array}{ccc} & \{(C', p) \in \mathbb{P}^5 \times C_i \mid C' \text{ a conic tangent to } C_i \text{ at } p\} & \\ {\scriptstyle}\swarrow & & \searrow{\scriptstyle \pi_2} \\ Z_i & & C_i \end{array}$$

and noting that the fibers of π_2 are linear subspaces of $\mathbb{P}^5$ of codimension 3. (Here, "tangent to C_i at p" means $m_p(C' \cdot C_i) \geq 2$, that is,

the restriction to C_i of the defining equation of C' vanishes to order at least 2 at p.)

(b) The degree of Z_i is 6. To see this, we intersect W_i with a general pencil of conics. The conic C_i may be thought of as the embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ by the complete linear system of degree 2. Thus a general pencil of conics cut out a general linear series on C_i of degree 4, and the degree of the Z_i is the number of divisors in this family with less than 4 distinct points. The linear series defines a general map $C_i \to \mathbb{P}^1$ of degree 4 with distinct branch points, and by Hurwitz' theorem the number of branch points of this map is 6.

(c) Thus the number of points of intersection of $Z_1, \ldots Z_5$, *assuming they intersect transversely,* will be $6^5 = 7,776$.

Alas, this is *not* the answer to the question we posed: the hypersurfaces Z_i don't even meet in a finite set. The part of the intersection within the set of smooth conics is what we wanted. The trouble is with the compactification: we used the space of all (possibly singular) conics, and "excess" intersection of the Z_i takes place along the boundary.

In detail: the hypersurface Z_j is the *closure* in $\mathbb{P}^5$ of the locus of smooth conics C tangent to C_j. A smooth conic C' is tangent to C_j exactly when the defining equation F of C', restricted to $C_j \cong \mathbb{P}^1$ and viewed as a quartic polynomial on $\mathbb{P}^1$, has a multiple root. When we extend this characterization to arbitrary conics C' we see that a double line is tangent to every conic! Thus the five hypersurfaces $Z_1, \ldots, Z_5 \subset \mathbb{P}^5$ will all contain the locus $S \subset \mathbb{P}^5$ of double lines, which is the Veronese surface in the $\mathbb{P}^5$ of conics. As we shall see, the intersection $\bigcap Z_i$ is the union of S and the finite set of smooth conics tangent to the five C_i. The presence of this extra component S means that the number we seek has little to do with the intersection product $\prod[Z_i] \in A^5(\mathbb{P}^5)$.

Before we explain the solution of this problem by complete conics, we pause to discuss some different approaches to dealing with this issue.

Blowing Up The Excess Locus. Suppose we are interested in intersections inside some quasi-projective variety U, and we have a compactification V of U; in the example above U is the space of smooth conics, and V the space of all conics. We could blow up some locus in the boundary $V \setminus U$ to obtain a new compactification. This is the classical way of separating subvarieties of a given variety that we don't want to meet. In the Five Conic Problem we would blow up the surface S in $\mathbb{P}^5$ and consider the proper transforms $\tilde{Z}_i$ of the hypersurfaces Z_i in the blow-up $X = Bl_S\mathbb{P}^5$. If we're lucky (and in this case we would be) we will have eliminated the excess intersection—that is, the $\tilde{Z}_i$ will not intersect anywhere in the exceptional divisor of the blow up, and will intersect transversely inside U. To carry

this out, we would need to determine the Chow ring $A^*(X)$ of the blow-up, find the class $\zeta \in A^1(X)$ of the $\tilde{Z}_i$, and evaluate the product $\zeta^5 \in A^5(X)$.

This approach has the virtue of being universally applicable, at least in theory: any component of any intersection of cycles can be eliminated by blowing up. It is not too hard to carry out this attack in the case of the the Five Conic Problem—see Griffiths and Harris [1979]. But often we cannot recognize the blowup as the parameter space of any nice geometric objects, and this makes the computations less intuitive and sometimes unwieldy. For example, if we look at cubics rather than conics, the problem analogous to the Five Conic Problem, solved heuristically by Schubert in the 19th century and rigorously in Aluffi [1990], requires multiple blow-ups and complex calculations. (For a different approach to the analogous problem that is capable of handling curves of any degree, seeFulton [1984] Section 10.4.)

Excess Intersection Formulas. As we mentioned in the initial Chapter, there is a general formula that assigns to every connected component of an intersection $\bigcap Z_i \subset X$ a class in the appropriate dimension, in such a way that the sum of these classes (viewed as classes on the ambient variety X via the inclusion) equals the product of the classes of the intersecting cycles. In the present case, the formula gives us a class of dimension zero in the Chow ring of the Veronese surface. Subtracting the degree of this class from $7,776$ yields the number we want.

Excess intersection problems were already considered by Salmon in 1847, and were much generalized by Cayley around 1868 (seeFulton [1984] for references.) They will have a chapter of their own later in this book. The reader can find a solution to the Five Conic Problem done this way (following ideas of Fulton and MacPherson) in Fulton [1984] Example 9.1.9. As a general method, excess intersection formulas are often an improvement on blowing up. They are universally applicable in principle, but apt to be difficult to apply in practice.

Changing the Parameter Space. To understand what sort of compactification is "right" for a given problem is, as we have said, an art. In the case of the Five Conic Problem, we can take a hint from the fact that the problem is about tangencies. The set of lines tangent to a nonsingular conic is again a conic in the dual space (we will identify it explicitly below.) But when a conic degenerates to the union of two lines or a double line, the dual conic seems to disappear! – the dual of a line is only a point. This leads us to ask for a compactification of the space of smooth conics that preserves information about limiting positions of tangents.

There are at least two ways to make a compactification that encodes the necessary information. One is to use the *Kontsevich space*. It parametrizes,

not subschemes of $\mathbb{P}^2$, but rather maps $f : C \to \mathbb{P}^2$, with C a nodal curve of arithmetic genus 0. This is an important construction, which generalizes to a parametrization of curves of any degree and genus in any variety. We will discuss it informally in the second half of this Chapter. But proving even the existence of Kontsevich spaces requires a considerable development, and we will not take this route; the reader will find an exposition inFulton and Pandharipande [1997].

The other way to describe a compactification of the space of smooth conics that preserves the tangency information is through the idea of *complete conics*. The space of complete conics is very well-behaved, and we will spend the first half of this Chapter on this beautiful construction. It turns out that the space we will construct is isomorphic to the Kontsevich space for conics (and, for that matter, to the blow-up $Bl_S\mathbb{P}^5$ of $\mathbb{P}^5$ along the surface of double lines), but generalizes in a different direction: there are analogues for quadric hypersurfaces of any dimension, for linear transformations ("complete colineations") and more generally for symmetric spaces (seeDe Concini and Procesi [1983],De Concini and Procesi [1985],De Concini et al. [1988],Bifet et al. [1990]), but not for curves of higher degree or genus.

10.2 Complete conics

We begin with an informal discussion. Later in this section we will provide a rigorous foundation for what we describe. Recall that the *dual* of a smooth conic $C \subset \mathbb{P}^2$ is the set of lines tangent to C, regarded as a curve $C^* \subset \mathbb{P}^{2*}$. As we shall see, C^* is also a smooth conic (this would not be true in characteristic 2.)

10.2.1 Informal Description

Degenerating the dual. Consider what happens to the dual conic as a smooth conic degenerates to a singular conic—either two distinct lines or a double line: That is, let $\{C_t\}$ be a one-parameter family of conics parametrized by a disc $\mathcal{D}$ with coordinate t, with C_t smooth for $t \neq 0$. Associating to each curve C_t the dual conic $C_t^* \subset \mathbb{P}^{2*}$ we get a regular map from the punctured disc $\mathcal{D} \setminus \{0\}$ to the space $\mathbb{P}^{5*}$ of conics in $\mathbb{P}^{2*}$. Since the space of all conics is proper, this extends to a regular map on all of $\mathcal{D}$—in other words, there is a well-defined conic $C_0^* = \lim_{t\to 0} C_t^*$. However, this limit depends, in general, on the family $\{C_t\}$ and not just on the curve C_0: the limiting tangent behavior is *not* determined by the limit curve.

To provide a compactification of the space U_1 of smooth conics that captures this phenomenon, we write $\mathbb{P}^5$ for the space of conics in $\mathbb{P}^2$ and $\mathbb{P}^{5*}$ for the space of conics in $\mathbb{P}^{2*}$. Consider the graph of the map $C \mapsto C^*$ from the set of smooth conics to $\mathbb{P}^{5*}$, that is, the set

$$U = \{(C, C^*) \in \mathbb{P}^5 \times \mathbb{P}^{5*} \mid C \text{ a smooth conic in } \mathbb{P}^2 \text{ and } C^* \subset \mathbb{P}^{2*} \text{ its dual.}\}$$

The desired compactification, the *variety of complete conics* is the closure

$$X = \overline{U} \subset \mathbb{P}^5 \times \mathbb{P}^{5*}.$$

The dual of the dual of a smooth conic is the original conic, as we shall soon see (in fact the same statement holds for varieties much more generally, and will be proven in Chapter ****) so the set U is symmetric under exchanging $\mathbb{P}^5$ and $\mathbb{P}^{5*}$. It follows that X is symmetric too. The set $U_1 \subset \mathbb{P}^5$ of smooth conics is an open subset of $\mathbb{P}^5$, and it follows that $U \cong U_1$ and hence X are irreducible and of dimension 5 as well.

As we shall soon see, the map $C \mapsto C^*$ is regular on smooth conics, so its graph U is closed in $U_1 \times \mathbb{P}^{5*}$ and if $(C, C') \in X$ with C smooth, then C' must be smooth too. By symmetry, if either C or C' is smooth, then the other is too.

What happens when C is singular? Let's first consider the case of a family $\{C_t\}$ of smooth conics approaching a conic C_0 of rank 2, that is, $C_0 = L \cup M$ is the union of a pair of distinct lines.

****insert picture****

The picture makes it easy to guess what happens: any collection $\{L_t\}$ of lines with L_t tangent to C_t for $t \neq 0$ approaches a line L_0 through the point $p = L \cap M$, and conversely any line L_0 through p is a limit of lines L_t tangent to C_t. (Actually, the second half follows from the first, given that the limit $C_0' = \lim_{t \to 0} C_t^*$ is one-dimensional.) Since C_0' is by definition a conic, it must be the double of the line in $\mathbb{P}^{2*}$ dual to the point p, irrespective of the family $\{C_t\}$ used to construct it.

Next consider what happens when the family $\{C_t\}$ of smooth conics approaches a double line $C_0 = 2L$. Let $D_t = p_t + q_t$ be the divisor cut on the line L by the conic C_t for $t \neq 0$, and let $D = p + q$ be the limit as $t \to 0$ of these divisors. Again, let C_0' be the limit of the family of dual conics C_t^*, so that $(C_0, C_0') \in X$.

****insert picture****

We see from the picture that the limit L_0 of any family $\{L_t\}$ of lines tangent to C_t must pass through one of the points p, q, and conversely any line through one of these points is a limit of tangent lines to nearby smooth conics in the family. Thus if $p \neq q$ then C_0' is the union of the lines in $\mathbb{P}^{2*}$ dual to p and q in case p and q. On the other hand, if $p = q$ then C_0' the double of the line dual to p.

****insert picture****

Types of Complete Conics. In conclusion, there are four types of complete conics, that is, points $(C, C') \in X$:

(a) C and C' are both smooth, and $C' = C^*$. We will call these *smooth* complete conics.

(b) $C = L \cup M$ is of rank 2, and $C' = 2p^*$, where $p^* \subset \mathbb{P}^{2*}$ is the line dual to $p = L \cap M$;

(c) $C = 2L$ is of rank 1, and $C' = p^* \cup q^*$ is the union of the lines in $\mathbb{P}^{2*}$ dual to two points $p, q \in L$; or

(d) $C = 2L$ is of rank 1, and $C' = 2p^*$ is the double of the line in $\mathbb{P}^{2*}$ dual to a point $p \in L$.

Note that the description is exactly the same if we reverse the roles of C and C', except that the second and third types are exchanged. Note also that the points of each type form a locally closed subset of X, with the first open and the last closed; and all four are orbits of the action of PGL_3 on $\mathbb{P}^5 \times \mathbb{P}^{5*}$.

As we have already explained, the complete conics of type (a) form a set of dimension 5. It is easy to see that those of type (b) are determined by the pair of lines L, M, and thus form a set of dimension 4. By symmetry (or inspection) the same is true for type (c). Finally, those of type (d) are determined by the line L and the point $p \in L$; thus these form a set of dimension 3, which is in fact the intersection of the closures of the sets of points described in (b) and (c).

10.2.2 Rigorous Description

Let's now verify all these statements, using the equations defining the locus $X \subset \mathbb{P}^5 \times \mathbb{P}^{5*}$. We could do this explicitly in coordinates, but it will save a great deal of ink if we use a little multilinear algebra. The reader to whom this is new will find more than enough background in Appendix 2 ofEisenbud [1995]. The multilinear algebra allows us to treat some basic properties in all dimensions with no extra effort, so we begin with some general results about duality for quadrics.

Duals of Quadrics. Let V be a vector space. Recall that, since we are assuming the characteristic of the ground field K is not 2, the following three notions are equivalent:

- A symmetric linear map $\varphi : V \to V^*$;
- A quadratic map $q : V \to K$;

- An element $q' \in \mathrm{Sym}^2(V^*)$.

Explicitly, if we start with a symmetric map $\varphi : V \to V^*$ then we take $Q(x) = \langle \varphi(x), x \rangle$, and the element $q \in \mathrm{Sym}^2(V^*)$ comes about from the identification of $\mathrm{Sym}(V^*)$ with the ring of polynomial functions on V. The quadric $Q = \{x \mid q(x) = 0\} \subset \mathbb{P}(V)$ is thus the locus

$$\{\overline{v} \in \mathbb{P}(V) : \langle \varphi(v), v \rangle = 0\},$$

where we denote by $\overline{v}$ the point in $\mathbb{P}(V)$ associated to a nonzero vector $v \in V$. The quadric will be smooth if and only if φ is an isomorphism; more generally, the singular locus of C will be the (projectivization of the) kernel of φ. What, in these terms, is the dual to $Q = 0$? That is, what is the set of hyperplanes tangent to this locus in $\mathbb{P}(V)$?

To state the result, recall that if $\varphi : V \to W$ is any map of vector spaces of rank $n+1$ then there is a *cofactor map* $\varphi^c : W \to V$ represented by a matrix whose entries are signed $n \times n$ minors of φ, satisfying $\varphi \circ \varphi^c = det(\varphi) Id_W$ and $\varphi^c \circ \varphi = det(\varphi) Id_V$. In invariant terms, φ^c is the composite

$$W \cong \wedge^n W^* \xrightarrow{\wedge^n \varphi^*} \wedge^n V^* \cong V$$

where the identifications $W \cong \wedge^n W^*$ and $\wedge^n V^* \cong V$ are defined by choices of bases in the 1-dimensional spaces $\wedge^{n+1} W$ and $\wedge^{n+1} V^*$ respectively. Note that when the rank of φ is $< n$ the map φ^c is zero.

Proposition 10.1. *Let $\overline{v} \in Q \subset \mathbb{P}(V) = \mathbb{P}^n$ be a point on a quadric over a field of characteristic not 2, corresponding to the symmetric map $\varphi : V \to V^*$ and the vector $v \in V$ such that $\langle \varphi(v), v \rangle = 0$. The tangent hyperplane to Q at $\overline{v}$ is $\{w \in V \mid \langle \varphi(v), w \rangle = 0$. The dual of Q is the variety*

$$Q^* = \{\overline{\varphi(v)} \mid \overline{v} \in Q\}.$$

If Q is nonsingular, that is, has rank $n+1$, then Q^* is the quadric corresponding to the cofactor map φ^c. If the rank of Q is n, then the quadric Q^c corresponding to the cofactor map φ^c is the unique double plane containing Q^*.

Proof. For any $w \in V$, the line $\overline{vw} \subset \mathbb{P}^2$ spanned by $\overline{v}$ and $\overline{w}$ is tangent to Q at v if and only if

$$\langle \varphi(v + \epsilon w), v + \epsilon w \rangle = 0 \mod(\epsilon^2)$$

Expanding this out we get

$$\langle \varphi(w), v \rangle + \langle \varphi(v), w \rangle = 0$$

and by the symmetry of φ and the assumption that we are not in characteristic 2 this is the case if and only if

$$\langle \varphi(v), w \rangle = 0,$$

proving the first statement and identifying the dual variety as $Q^* = \varphi(Q)$.

Suppose that $v \in Q$, and that the rank of Q is at least n. Let φ^c be the matrix of cofactors of φ, so that $\varphi^c\varphi = \det\varphi \circ I$, where I is the identity map. Since $\operatorname{rank} Q = \operatorname{rank}\varphi \geq n$, the map φ^c is nonzero. Setting $w = \varphi(v)$ we see that $\langle w, \varphi^c(w)\rangle = (\det\varphi)\langle\varphi(v), v\rangle = 0$, so the dual variety $\varphi(Q)$ is contained in the quadric hypersurface Q^c corresponding to φ^c.

If $\operatorname{rank}\varphi = n+1$, so that φ is an isomorphism, then $Q^* = \varphi(Q)$ is again a quadric hypersurface, and we must have $Q^* = Q^c$. If $\operatorname{rank}\varphi = n$ then since $\varphi^c\varphi = 0$ the rank of φ^c is 1, and the associated quadric is a double plane. On the other hand, Q is the cone over a nonsingular quadric in $\mathbb{P}^{n-1}$, and Q^* is the dual of that quadric inside a hyperplane (corresponding to the vertex of Q) in $\mathbb{P}^{n*}$. Thus Q^* spans the plane contained in Q^c. □

The following easy consequence will be useful for the Five Conic Problem.

Corollary 10.2. *If Q and Q' are smooth quadrics, then Q and Q' have the same tangent hyperplane $w = 0$ at some point of intersection $\overline{v}$ if and only if Q^* and Q'^* have the common tangent hyperplane $v = 0$ at the point of intersection $\overline{w}$. In particular, if D is a smooth plane conic then the divisor $Z_D \subset X$, which is the closure of the set of complete conics (C, C') such that C is smooth and tangent to D is equal to the divisor defined similarly starting from the dual conic D^*.*

Proof. The first statement is no more than the statement that duality interchanges points and tangent hyperplanes. In detail: Suppose that Q, Q' corresponding to symmetric maps φ and φ'. Since the tangent planes at $\overline{v}$ are the same, Proposition10.1 shows that $\overline{\varphi(v)} = \overline{\varphi}'(v) \in Q^* \cap Q'^*$. Since $v = \varphi^{-1}(\varphi(v)) = \underline{\psi^{-1}(\psi(v))} \sim \psi^{-1}(\varphi(v))$, we see that Q^* and Q'^* are in fact tangent at $\overline{\varphi(v)}$.

The second statement follows at once from the first. □

Equations For the Variety of Complete Conics. We now return to the case of conics in $\mathbb{P}^2$, and suppose that V is 3-dimensional. We wish to give equations for X:

Proposition 10.3. *The variety*

$$X \subset \mathbb{P}(\operatorname{Sym}^2(V^*)) \times \mathbb{P}(\operatorname{Sym}^2(V)) = \mathbb{P}^5 \times \mathbb{P}^{5*}$$

of complete conics is smooth and irreducible. Thinking of $(\overline{\varphi}, \overline{\psi}) \in \mathbb{P}^5 \times \mathbb{P}^{5*}$ as coming from a pair of symmetric matrices $\varphi : V \to V^*$ and $\psi : V^* \to V$, the scheme X is defined by the ideal I generated by the 8 bilinear equations specifying that the product $\psi \circ \varphi$ has all diagonal entries equal (two equations) and off-diagonal entries zero (6 equations).

(For the experts: It follows from the Proposition that the ideal I has codimension 5, and that its saturation, in the bihomogeneous sense, is prime. Computation shows that (in characteristic zero, anyhow) the polynomial ring modulo I is Cohen-Macaulay. With the Proposition, this implies that I is prime. In particular, I is preserved under the interchange of factors φ and ψ, which does not seem evident from the form given.)

Proof. Let Y be the subscheme defined by the given equations. We first show that Y agrees set-theoretically with X on at least the locus of those points (φ, ψ) where $\operatorname{rank}\varphi \geq 2$ or $\operatorname{rank}\psi \geq 2$, that is, where φ or ψ corresponds to a smooth conic or the union of two distinct lines. On the locus of smooth conics, φ has rank 2 and $(\varphi, \psi) \in Y$ if and only if $\psi = \varphi^{-1}$ up to scalar, so Proposition10.1 shows that the dual conic is defined by ψ. Moreover, if the rank of φ is 2, and $(\varphi, \psi) \in Y$ then we see from the equations that $\psi \circ \varphi = 0$. Up to scalar, $\psi = \varphi^c$ is the unique possibility, and again Proposition10.1 shows that the corresponding conic C' is the dual of C. To see the uniqueness (up to scalars) in terms of matrices, note that in suitable bases

$$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the symmetric matrices annihilating the image have the form

$$\psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} = a\varphi^c.$$

The same arguments show that when $\operatorname{rank}\psi \geq 2$ then $\varphi = \psi^c$ and again they correspond to dual conics. (Note that, since $\operatorname{rank}\psi^c = 1$ on this locus we do *not* have $\psi = \varphi^c$ there.)

Since X is defined as the closure of U, we see now in particular that $X \subset Y$. We will show next that Y is smooth of dimension 5 locally at any point $(\varphi, \psi) \in Y$ where both φ and ψ have rank 1. We will use this to show that Y is everywhere smooth of dimension 5, which implies $X = Y$.

To this end, suppose that $(\varphi, \psi) \in Y$ and that both φ and ψ have rank 1. The tangent space to Y at the point (φ, ψ) may be described as the locus of pairs of symmetric matrices $\alpha : V \to V^*,\ \beta : V^* \to V$ such that

$$(\psi + \epsilon\beta) \circ (\varphi + \epsilon\alpha) \mod(\epsilon^2)$$

has its diagonal entries equal and its off-diagonal entries zero. Since both φ and ψ have rank 1, this is equivalent to saying that

$$\psi \circ \alpha + \beta \circ \varphi = 0.$$

We must show that this condition is equivalent to 5 independent linear conditions on the pair (α, β). In suitable coordinates the maps α, β will be represented by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Multiplying out, we see that $\psi\alpha$ and $\beta\varphi$ have the forms

$$\begin{pmatrix} \psi_{1,1} & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ \psi_{3,1} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{2,1} & \varphi_{2,2} & \varphi_{2,3} \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the equation $\psi\alpha+\beta\varphi = 0$ is equivalent to the equations $\psi_{2,1}+\varphi_{2,1} = 0$ and $\psi_{1,1} = \psi_{3,1} = \varphi_{2,2} = \varphi_{3,2} = 0$; five independent linear conditions, as required.

To complete the proof, note that Y is preserved scheme-theoretically by the action of the orthogonal group G; if $(\varphi, \psi) \in Y$ and α is orthogonal, then $(\alpha\varphi\alpha^*, \alpha\psi\alpha^*) \in Y$ since $\alpha^*\alpha = 1$. Any closed point on Y where $\operatorname{rank}\varphi \geq 2$ degenerates under the action of G to a point where $\operatorname{rank}\varphi = 1$; working in a basis for which φ is diagonal, we stretching one of the coordinates will make the correponding entry of φ approach zero, and $\psi = \varphi^c$ moves at the same time. The same is true when $\operatorname{rank}\psi \geq 2$. Thus if the singular locus of Y were not empty it would have to intersect the locus of pairs of matrices of rank 1, and we have seen that this is not the case. □

The classification of the points of X into the four types above follows from Proposition10.3: if $(\varphi, \psi) \in Y$, then:

(a) (Smooth Complete Conics) If φ is of rank 3, then ψ must be its inverse;

(b) If φ is of rank 2, then (since X is symmetric) the products $\psi \circ \varphi$ and $\varphi \circ \psi$ must both be zero; it follows that ψ is the unique (up to scalars) symmetric map $V^* \to V$ whose kernel is the image of φ and whose image is the kernel of φ;

(c) If φ is of rank 1, ψ may have rank 1 or 2; if the latter, it may be any symmetric map $V^* \to V$ whose kernel is the image of φ and whose image is the kernel of φ; and finally

(d) If φ and ψ both have rank 1, they simply have to satisfy the condition that the kernel of ψ contains the image of φ and vice versa.

10.2.3 Solution to the Five Conic Problem

Now that we have established that the space X of complete conics is smooth and projective, we will show how to solve the Five Conics Problem. To any

smooth conic $D \subset \mathbb{P}^2$ we associate a divisor $Z = Z_D \subset X$, which we define to be the closure in X of the locus of pairs $(C, C^*) \in X$ with C smooth and tangent to D, and let $\zeta = [Z_D] \in \mathbb{A}^1(X)$ be its class. We will address each of the following issues:

Outline.

(a) We have to show that in passing from the "naive" compactification $\mathbb{P}^5$ of the space U_1 of smooth conics to the more sensitive compactification X, we have in fact eliminated the problem of extraneous intersection: in other words, that for five general conics C_i the corresponding divisors $Z_{C_i} \subset X$ intersect only in points $(C, C') \in X$ with C and $C' = C^*$ smooth.

(b) We have to show that the five divisors Z_{C_i} are transverse at each point where they intersect.

(c) We have to determine the Chow ring of the space X, or at least the structure of a subring of $A^*(X)$ containing the class ζ of the hypersurfaces Z_{C_i} we wish to intersect;

(d) We have to identify the class ζ in this ring and find the degree of the fifth power $\zeta^5 \in A^5(X)$.

Complete Conics Tangent To Five General Conics are Smooth. We begin by remarking that Z is symmetric under the operation of interchanging the factors $\mathbb{P}^5$ and $\mathbb{P}^{5*}$. It is of course enough to prove this for $U \cap Z$, and in this case it follows at once from Corollary10.2.

We will now describe the points (C, C') of types (b), (c), and (d) lying in Z. For type (b) this is immediate: if $C = L \cup M$ is a conic of rank 2 which is a limit of smooth conics tangent to D, then C also must have a point of intersection multiplicity 2 or more with D; thus either L or M is a tangent line to D, or the point $p = L \cap M$ lies on D. By symmetry, a similar description holds for the points of type (c).

We now show that no complete conic (C, C') of type 2 lies in the intersection of the divisors $Z_i = Z_{C_i}$ associated to five general conics C_i. Write $C = L \cup M$, and set $p = L \cap M$. Suppose first that p lies on one of the C_i, say C_1. Because the C_i are general, the tangent lines to one of the C_i through p are tangent at only one C_i; and no other C_i passes through p. It follows that $L \cup M$ cannot have a point of multiplicity 2 in common with more than three of the C_i.

Next, suppose that p is not on any C_i there are only finitely many lines tangent to two distinct smooth conics (in fact, at most 4, the intersection points of the dual conics) so that L and M can each have contact of order ≥ 2 with at most 2 of the C_i, and $L \cup M$ is thus "tangent" to at most 4 of

the C_i, proving our claim. Since Z is symmetric, the same argument works for complete conics of type (c).

Finally, to verify that no complete conic (C, C') of type (d) can lie in the intersection $\bigcap Z_i$, we have to characterize the intersection of a cycle $Z = Z_D$ with the locus of complete conics of type (d). If (C, C') is of type (d), then we can write $C = 2M$ and $C' = 2q^*$, with $q \in M$. If (C, C') lies in the divisor $Z = Z_D$ then there is a one-parameter family $(C_t, C_t') \in Z_D$ with C_t smooth and $C_t' = C_t^*$ for $t \neq 0$, and with $(C, C') = (C_0, C_0')$.

Let $p_t \in C_t \cap D$ be the point of tangency of C_t with D and set $p = \lim_{t\to 0} p_t \in M$. The tangent line L_t to C_t at p_t will have as limit the tangent line L to D at p so $L \in q^*$. Thus both p and q are in both L and M.

If $p = q$ then in particular $q \in D$. This can be true when $D = D_i$ for at most two values of i, since at most two of the D_i meet in a point. Otherwise, $p \neq q$, in which case $M = L$ is tangent to D. Since no line is tangent to more than 2 of the C_i, we see that no complete conic $(2M, 2q^*)$ can lie more than 4 of the cycles Z_i. ****these cases should have pictures with them!****

****the language here is very complex-analytic sounding. Of course we know what we mean, but... . Also, we use the words like specialization and limit interchangeably. Will the reader be comfortable?****

Transversality. In order to prove that the cycles $Z_{C_i} \subset X$ intersect transversely when the conics $C_1, \ldots, C_5$ are general we need a description of the tangent spaces to the Z_{C_i} at points of intersection. We already know that such points are represented by smooth conics, and the open subscheme parametrizing smooth conics is the same whether we are working in $\mathbb{P}^5$ or in the space of complete conics, so we may express the answer in terms of the geometry of $\mathbb{P}^5$.

Lemma 10.4. *Let $D \subset \mathbb{P}^2$ be a smooth conic curve, and $Z_D^\circ \subset \mathbb{P}^5$ the variety of smooth plane conics $C \neq D$ tangent to D. Write $[C] \in Z_D$ for the point corresponding to C.*

(a) *If C has a point p of simple tangency with D and is otherwise transverse to D, then Z_D° is smooth at $[C]$; and*

(b) *In this case, the projective tangent plane $\mathbb{T}_{[C]} Z_D^\circ$ to Z_D° at $[C]$ is the hyperplane $H_p \subset \mathbb{P}^5$ of conics containing the point p.*

Proof. Identify D with $\mathbb{P}^1$ and consider the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \to H^0(\mathcal{O}_D(2)) = H^0(\mathcal{O}_{\mathbb{P}^1}(4)).$$

The kernel of this map is the subspace spanned by the section representing D itself. In terms of projective spaces, the restriction induces the linear

projection map

$$\pi_{[D]} : \mathbb{P}^5 = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \to \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(4)) = \mathbb{P}^4$$

from the point $[D] \in \mathbb{P}^5$. The set Z_D° is an open subset (not containing $[D]$) of the preimage, under this projection, of the hypersurface $\mathcal{D} \subset \mathbb{P}^4$ of singular divisors in the linear system $|\mathcal{O}_{\mathbb{P}^1}(4)|$—that is, the cone over $\mathcal{D}$ with vertex $[D]$. By **??******though we do have the methods to do this in Ch 7 (Singular Elements), and we do compute the tangent space to singular curves, I don't think we have this particular statement. It should be added, early on, to that chapter.******** the hypersurface $\mathcal{D}$ is smooth at a point corresponding to a divisor having one double point p and otherwise smooth and that its tangent space at such a point is the hyperplane of divisors containing p; the statement of the Lemma follows. □

In order to apply this, we need to establish some more facts about a conic tangent to five general conics:

Lemma 10.5. *Let $C_1, \ldots, C_5 \subset \mathbb{P}^2$ be general conics, and $C \subset \mathbb{P}^2$ any smooth conic tangent to all five. Each conic C_i is simply tangent to C at a point p_i and otherwise transverse to C; and the points $p_i \in C$ are distinct.*

Proof. Let U be the set of smooth conics and consider incidence correspondences

$$\Phi = \left\{(C_1, \ldots, C_5; C) \in (U^5 \times U) \mid \text{each } C_i \text{ is tangent to } C\right\}$$
$$\subset \Phi' = \left\{(C_1, \ldots, C_5; C) \in ((\mathbb{P}^5)^5 \times U) \mid \text{each } C_i \text{ is tangent to } C\right\}$$

The set Φ is an open subset of the set Φ'. Since U is irreducible of dimension 5 and the projection map $\Phi' \to U$ on the last factor has irreducible fibers $(Z_C)^5$ of dimension 20, we see that Φ', and thus also Φ, is irreducible of dimension 25.

There are certainly points in Φ were the conditions of the Lemma are satisfied: just choose a conic C and five general conics C_i tangent to it. Thus the set of $(C_1, \ldots, C_5; C) \in \Phi$ where the conditions of the Lemma are not satisfied is a proper closed subset, and as such it can have dimension at most 24, and cannot dominate U^5 under the projection to the first factor. This proves the Lemma. □

To complete the argument for transversality, let $[C] \in \cap Z_{C_i}$ be a point corresponding to the conic $C \subset \mathbb{P}^2$. By Lemma10.5 the points p_i of tangency of C with the C_i are distinct points on C. Since C is the unique conic through these five points, the intersection of the tangent spaces to Z_{C_i} at $[C]$

$$\bigcap \mathbb{T}_{[C]} Z_{C_i} = \bigcap H_{p_i} = \{[C]\}$$

is zero-dimensional, proving transversality.

10.2.4 Chow Ring Of The Space Of Complete Conics

Having confirmed that the intersection $\bigcap Z_i$ indeed behaves well, let us turn now to computing the intersection number. We start with a calulcation in the Chow ring of X:

********as I understand it, we agreed that we're going to add a discussion of Chow rings of blow-ups in general later on, but at this point we might as well confine ourselves to intersections of divisors in X; we should say something to that effect********

It is easy to list some classes in $A^*(X)$. First, let α and $\beta \in A^1(X)$ be the pullbacks to $X \subset \mathbb{P}^5 \times \mathbb{P}^{5*}$ of the hyperplane classes on $\mathbb{P}^5$ and $\mathbb{P}^{5*}$. These are respectively represented by the divisors

$$A_p = \{(C, C^*) : p \in C\}$$

(for any point $p \in \mathbb{P}^2$,) and

$$B_L = \{(C, C^*) : L \in C^*\}$$

(for any point $L \in \mathbb{P}^{2*}$.)

Also, let $\gamma, \varphi \in A^4(X)$ be the classes of the curves Γ and Φ that are the pullbacks to X of general lines in $\mathbb{P}^5$ and $\mathbb{P}^{5*}$.

Lemma 10.6. *The group $A^1(X)$ of divisor classes on X has rank 2, and is generated over the rationals by α and β. The intersection number of these divisors with Γ and Φ are given by the table*

$$\begin{array}{c} \\ \gamma \\ \varphi \end{array} \begin{array}{c} \begin{array}{cc} \alpha & \beta \end{array} \\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array}$$

Proof. We first show that the rank of $A^1(X)$ is at most two. The open subset $U \subset X$ of pairs (C, C^*) with C and C^* smooth is isomorphic to the complement of a hypersurface in $\mathbb{P}^5$, and hence has torsion Picard group: any line bundle on U extends to a line bundle on $\mathbb{P}^5$, a power of which is represented by a divisor supported on the complement $\mathbb{P}^5 \setminus U$. Thus, if L is any line bundle on X, a power of L is trivial on U and hence represented by a divisor supported on the complement $X \setminus U$. But the complement of U in X has just two irreducible components, the closures D_2 and D_3 of the loci of complete conics of type (b) and (c). Any divisor class on X is thus a rational linear combination of the classes of D_2 and D_3, from which we see that the rank of the Picard group of X is at most 2.

Since passing through a point is one linear condition on a quadric we have $\deg \alpha\gamma = 1$ and dually $\deg \beta\varphi = 1$. Arguing as for the degree of the divisor Z_{C_i} above, we get $\deg \alpha\varphi = 2$ and again by duality $\deg \beta\gamma = 2$.

Since the matrix of intersections between α, β and γ, φ is nonsingular, we conclude that α and β generate $\mathrm{Pic}(X) \otimes \mathbb{Q}$. □

The Class of the Divisor of Complete Conics Tangent to C. It follows that we can write $\zeta = p\alpha + q\beta \in A^1(X) \otimes \mathbb{Q}$ for some $p, q \in \mathbb{Q}$. To compute p and q we use the fact that, restricted to the open set $U \subset X$, the divisor Z is a sextic hypersurface; it follows that $\deg \zeta\gamma = p + 2q = 6$. By ****, ζ is symmetric, so we get $\deg \zeta\gamma = q + 2p = 6$ as well. Thus

$$\zeta = 2\alpha + 2\beta \in A^1(X) \otimes \mathbb{Q}.$$

From this we see that $\deg(\zeta^5) = 32 \deg(\alpha+\beta)^5$, and it suffices to evaluate $\deg \alpha^i\beta^{5-i} \in A^5(X)$ for $i = 0 \ldots 5$. By symmetry, it's enough to do this for $i = 3$, 4 and 5:

(a) For $p_1, \ldots, p_5 \in \mathbb{P}^2$ general points, there is a unique conic C containing them, and C will be smooth; thus the corresponding cycles A_{p_i} intersect in the one point $[C] \in X$, and (modulo checking that they intersect transversely), we conclude that $\deg(\alpha^5) = 1$.

(b) For $p_1, \ldots, p_4 \in \mathbb{P}^2$ general points, and $L \subset \mathbb{P}^2$ a general line, there will be a pencil of conics passing through $p_1, \ldots, p_4$; these will cut out a pencil of degree 2 without base points on L, which will then have exactly two ramification points. There will thus be exactly two conics passing through the points p_i and tangent to L; again modulo checking transversality, we conclude that $\deg(\alpha^4\beta) = 2$.

(c) Finally, let $p_1, p_2, p_3 \in \mathbb{P}^2$ be general points and $L_1, L_2 \subset \mathbb{P}^2$ general lines. There will be a net of conics passing through the 3 points p_i, and in the $\mathbb{P}^2$ parametrizing this net the locus of conics tangent to each L_i will (by the last calculation) be a conic curve B_i. Moreover, the conics B_1 and B_2 intersect transversely: if C is a conic through p_1, p_2 and p_3 and tangent to L_i at a point q_i, the intersection of the tangent lines to B_1 and B_2 at $[C]$ will be the linear system of conics passing through p_1, p_2, p_3, q_1 and q_2. This will be positive-dimensional only if four of these five points are collinear, but this can't be the case: there will be a unique conic in the net tangent to L_1 at its point of intersection with each line $\overline{p_ip_j}$ joining two of the points, and for L_2 general none of these conics will be tangent to L_2. Thus B_1 and B_2 intersect in 4 points, from which (modulo transversality) we deduce that $\deg(\alpha^3\beta^2) = 4$.

Thus

$$\begin{aligned}\deg((\alpha+\beta)^5) &= \binom{5}{0} + 2\binom{5}{1} + 4\binom{5}{2} + 4\binom{5}{3} + 2\binom{5}{4} + \binom{5}{5} \\ &= 1 + 10 + 40 + 40 + 10 + 1 \\ &= 102\end{aligned}$$

and correspondingly,

$$\zeta^5 = 2^5 \cdot 102 = 3264.$$

This proves:

Theorem 10.7. *There are 3264 plane conics tangent to five general plane conics.* □

Of course, the fact that we are imposing the condition of being tangent to a conic is arbitrary; we can use the space of complete conics to count conics tangent to five general plane curves of any degree, as the following two exercises show; and indeed we can extend this to the condition of tangency with singular curves, as the exercises after that indicate.

Exercise 10.8. Let $D \subset \mathbb{P}^2$ be a smooth curve of degree d, and let $Z \subset X$ be the closure, in the space X of complete conics, of the locus of smooth conics tangent to D. Find the class $[Z_D] \in A^1(X)$ of the cycle Z.

Exercise 10.9. Now let $D_1, \ldots, D_5 \subset \mathbb{P}^2$ be general curves of degrees $d_1, \ldots, d_5$. Show that the corresponding cycles $Z_{D_i} \subset X$ intersect transversely, and using the result of the preceding exercise, find the number of conics tangent to all five .

Exercise 10.10. Now let $D \subset \mathbb{P}^2$ be a curve of degree d with a node at a point $p \in D$ (and smooth otherwise), and let $Z \subset X$ be the closure, in the space X of complete conics, of the locus of smooth conics tangent to D at a smooth point of D. Find the class $[Z_D] \in A^1(X)$ of the cycle Z.

Exercise 10.11. Finally, let $\{D_t\}$ be a family of plane curves of degree d, with D_t smooth for $t \neq 0$ and D_0 having a node at a point p. What is the limit of the cycles Z_{D_t} as $t \to 0$?

Exercise 10.12. Finally, here's a very 19^{th} century way of deriving the result of Exercise 10.8 above. Let $\{D_t\}$ be a pencil of plane curves of degree d, with D_t smooth for general t and D_0 consisting of the union of d general lines in the plane. Using the description of the limit of the cycles Z_{D_t} as $t \to 0$ in the preceding exercise, find the class of the cycle Z_{D_t} for t general.

********Put in a general description of complete quadrics, perhaps as an exercise series. (We have to say that the locus of k-planes tangent to Q is a divisor in the linear system $|\mathcal{O}_{\mathbb{G}(k,n)}(2)|$, and consider correspondingly the incidence correspondence in $\prod_k |\mathcal{O}_{\mathbb{G}(k,n)}(2)|$)********

10.3 Parameter spaces of curves

So far in this chapter we have been studying parameter spaces of smooth conics. The most obvious is perhaps $\mathbb{P}^5$, which we can identify as the space of all subschemes of $\mathbb{P}^2$ having pure dimension 1 and degree 2 (and thus arithmetic genus zero), and we have shown how the compactification by complete conics was more useful for dealing with tangencies. Here we have used the fact that the dual of a smooth conic is again a smooth conic. It would have been a different story if the problem had involved twisted cubics in $\mathbb{P}^3$ rather than conics in $\mathbb{P}^2$—if we had asked, for example, for the number of twisted cubic curves meeting each of 12 lines, or tangent to each of 12 planes. In that case it is not so clear how to make any parameter space and compactification at all!

In this section, we'll discuss two general approaches to constructing parameter spaces for curves: the Hilbert scheme of curves and the Kontsevich space of stable maps. Each of these has advantages, as we'll see.

10.3.1 Hilbert schemes

The Hilbert scheme $\mathcal{H} = \mathcal{H}_{H(m)}(\mathbb{P}^n)$. is a parameter space for subschemes of $\mathbb{P}^n$ with Hilbert polynomial $H(m)$; in the case of curves (one-dimensional subschemes) this means all subschemes with fixed degree and arithmetic genus. The Hilbert scheme has many good properties. For example, there is a useful cohomological description of its tangent spaces, and, beyond that, a deformation theory that in some cases can describe its local structure. And, of course, associated to a point on the Hilbert scheme is all the rich structure of a homogenous ideal in the ring $K[x_0, \ldots, x_n]$ and its resolution.

10.3.2 Examples of Hilbert Schemes

Hypersurfaces. The Hilbert schemes containing smooth plane curves of degree d (Hilbert polynomial $H(m) = dm + 1 - \binom{d-1}{2}$) is the projective space $\mathbb{P}^{\binom{2+d}{d}-1}$ of homogeneous forms of degree d, and similarly the Hilbert scheme containing smooth hypersurfaces of degree d in $\mathbb{P}^n$ is the projective space of homogeneous forms of degree d on $\mathbb{P}^n$. This is appealingly simple but, as we have seen above, not always the best parameter space for a given enumerative problem.

Twisted Cubics. Beyond hypersurfaces, the simplest case is the Hilbert scheme containing twisted cubic curves in $\mathbb{P}^3$ (Hilbert polynomial $H(m) = 3m + 1$.) It has one component of dimension 12 whose general point corresponds to a twisted cubic curve, but it has a second component, whose

general point corresponds to the union of a plane cubic $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ and a point $p \in \mathbb{P}^3$. Moreover, this second component has dimension 15 (the choice of plane has 3 degrees of freedom; the cubic inside the plane 9 more; and the point gives an additional 3.) These two components meet along the 11-dimensional subscheme of singular plane cubics C with an embedded point at the singularity, not contained in the plane spanned by C. See Piene and Schlessinger [1985]

Report Card for the Hilbert Scheme. The Hilbert scheme, as a compactification of the space of smooth curves, has drawbacks that sometimes make it difficult to use:

(a) **It has extraneous components, often of differing dimensions.** We see this phenomenon already in the case of twisted cubics, above. Of course we could take just the closure $\mathcal{H}^\circ$ in the Hilbert scheme of the locus of smooth curves, but we would lose some of the nice properties, like the description of the tangent space. Thus while it is relatively easy to describe the singular locus of $\mathcal{H}$, we don't know how to describe singular locus of $\mathcal{H}^\circ$ along the locus where it intersects other components.

****picture here****

In fact, we don't know for curves of higher degree how many such extraneous components there are, or their dimensions: for $r \geq 3$ and large d the Hilbert scheme of zero-dimensional subschemes of degree d in $\mathbb{P}^r$ will have an unknown number of extraneous components of unknown dimensions, and this creates even more extraneous components in the Hilbert schemes of curves.

(b) **No one knows what's in the closure of the locus of smooth curves**. If we do choose to deal with the closure of the locus of smooth curves rather than the whole Hilbert scheme—as it seems we must—we face another problem: Except in a few special cases, we can't tell if a given point in the Hilbert scheme is in this closure. That is, we don't know how to tell whether a given singular 1-dimensional scheme $C \subset \mathbb{P}^r$ is smoothable.

(c) **It has many singularities**. A famous result of Hartshorne says that the Hilbert scheme is connected, and thus singular at least where two components meet. But in fact the singularities are, in a precise sense, arbitrarily bad: Vakil [2006b] has shown that the completion of every affine local K-algebra appears (up to adding variables) as the completion of a local ring on a Hilbert scheme of curves.

10.3.3 The Kontsevich space

These drawbacks often make it difficult to use the Hilbert scheme for enumerative purposes. An alternative is the *Kontsevich space*. Unfortunately, its construction—the proof that it exists—requires far more time, and far more techniques, than we have available. We refer the reader to Fulton and Pandharipande [1997]for a systematic treatment.

The Kontsevich space $\overline{M}_{g,0}(\mathbb{P}^r, d)$ parametrizes what are called *stable maps* of degree d and genus g to $\mathbb{P}^r$. These are morphisms

$$f : C \to \mathbb{P}^r$$

with C a connected curve of arithmetic genus g having only nodes as singularities, such that the image $f_*[C]$ of the fundamental class of C is equal to d times the class of a line in $A_1(\mathbb{P}^r)$, and satisfying the one additional condition that the automorphism group of the map f—that it, automorphisms φ of C such that $f \circ \varphi = f$—is finite. (This last condition is automatically satisfied if the map f is finite; it's relevant only for maps that are constant on an irreducible component of C, and amounts to saying that any smooth, rational component C_0 of C on which f is constant must intersect the rest of the curve C in at least three points.) Two such maps $f : C \to \mathbb{P}^r$ and $f' : C' \to \mathbb{P}^r$ are said to be the same if there exists an isomorphism $\alpha : C \to C'$ with $f' \circ \alpha = f$. There's an analogous notion of a family of stable maps, and the Kontsevich space $\overline{M}_{g,0}(\mathbb{P}^r, d)$ is a coarse moduli space for the functor of families of stable maps. Note that we're taking the quotient by automorphisms of the domain, but not of the image, so that $\overline{M}_{g,0}(\mathbb{P}^r, d)$ shares with the Hilbert scheme $\mathcal{H}_{dm-g+1}(\mathbb{P}^r)$ a common subset parametrizing smooth curves $C \subset \mathbb{P}^r$ of degree d and genus g.

There are naturally variants of this: the space $\overline{M}_{g,n}(\mathbb{P}^r, d)$ parametrizes maps $f : C \to \mathbb{P}^r$ with C a nodal curve having n marked distinct smooth points $p_1, \ldots, p_n \in C$. (Here an automorphism of f is an automorphism of C fixing the points p_i and commuting with f; the condition of stability is thus that any smooth, rational component C_0 of C on which f is constant must have at least three distinguished points, counting both marked points and points of intersection with the rest of the curve C.) More generally, for any projective variety X and numerical equivalence class $\beta \in \mathrm{Num}_1(X)$, we have a space $\overline{M}_{g,n}(X, \beta)$ parametrizing maps $f : C \to X$ with fundamental class $f_*[C] = \beta$, again with C nodal and f having finite automorphism group.

The remarkable aspect of the Kontsevich space is simply that it is indeed proper: in other words, if $\mathcal{C} \subset \mathcal{D} \times \mathbb{P}^r$ is a flat family of subschemes of $\mathbb{P}^r$ parametrized by a smooth, one-dimensional base $\mathcal{D}$, and the fiber C_t is a smooth curve for $t \neq 0$, then no matter what the singularities of C_0 there is a unique stable map $f : \tilde{C}_0 \to \mathbb{P}^r$ which is the limit of the inclusions $\iota_t : C_t \hookrightarrow \mathbb{P}^r$. Note that this limiting stable map $f : \tilde{C}_0 \to \mathbb{P}^r$ depends

on the family, not just on the scheme C_0; the import of this in practice is that the Kontsevich space is often locally a blow-up of the Hilbert scheme along locii of curves with singularities worse than nodes. (This is not to say we have in general a regular map from the Kontsevich space to the Hilbert scheme; as we'll see in the examples below, the limiting stable map $f : \tilde{C}_0 \to \mathbb{P}^r$ doesn't determine the flat limit C_0 either.) We'll see how this plays out in four relatively simple cases below.

10.3.4 Examples of Kontsevich spaces

Plane conics. One indication of how useful the Kontsevich space can be is that, in the case of $\overline{M}_0(\mathbb{P}^2, 2)$ (that is, plane conics), the Kontsevich space is actually equal to the space of complete conics:

To begin with, if $C \subset \mathbb{P}^2$ is a conic of rank 2 or 3—that is, anything but a double line—then the inclusion map $\iota : C \hookrightarrow \mathbb{P}^2$ is a stable map; thus the open set $W \subset \mathbb{P}^5$ of such conics is likewise an open subset of the Kontsevich space $\overline{M}_0(\mathbb{P}^2, 2)$.

But when a family $\mathcal{C} \subset \mathcal{D} \times \mathbb{P}^2$ of conics specializes to a double line $C_0 = 2L$, the limiting stable map is a finite, degree 2 map $f : C \to L$, with C either isomorphic to $\mathbb{P}^1$, or two copies of $\mathbb{P}^1$ meeting at a point. Such a map is characterized, up to automorphisms of the domain curve, by its branch divisor $B \subset L$, a divisor of degree 2. (If B consists of two distinct points, $C \cong \mathbb{P}^1$, while if $B = 2p$ for some $p \in L$, the curve C will be reducible.)

****picture here****

Thus we have a birational morphism

$$\pi : \overline{M}_0(\mathbb{P}^2, 2) \to \mathcal{H}_{2m+1}(\mathbb{P}^2) = \mathbb{P}^5$$

from the Kontsevich space to the Hilbert scheme, with 2-dimensional fibers over the locus in $\mathbb{P}^5$ corresponding to double lines.

Conics in space. By contrast, there is not a regular map in either direction between the Hilbert scheme of conics in space and the Kontsevich space $\overline{M}_0(\mathbb{P}^3, 2)$. Of course there is a common open set: its points correspond to reduced conics—that is, embedded nodal curves of degree 2. To see that this does not extend to a regular map in either direction, note first that, as before, if $\mathcal{C} \subset \mathcal{D} \times \mathbb{P}^3$ is a family of conics specializing to a double line C_0, the limiting stable map is a finite, degree 2 cover $f : C \to L$, and this cover is not determined by the flat limit C_0 of the schemes $C_t \subset \mathbb{P}^3$.

But on the other hand the scheme C_0 is again a complete intersection of a plane and a quadric surface, which is to say it lies in a unique plane H; and this plane is not determined by the data of the map f.

The relationship in this case between the Hilbert scheme and the Kontsevich space is what's called in higher-dimensional birational geometry a *flip* ****maybe this is actually a flop?****: the Kontsevich space is obtained from the Hilbert scheme $\mathcal{H}$ by blowing up the locus of double lines, and then blowing down the exceptional divisor along another ruling. (The blow-up of $\mathcal{H}$ along the double line locus could be described as the space of pairs $(H; (C, C^*))$, where $H \subset \mathbb{P}^3$ is a plane and (C, C^*) a complete conic in $H \cong \mathbb{P}^2$.)

Plane cubics. Here, we do have a regular map from the Kontsevich space $\overline{M}_1(\mathbb{P}^2, 3)$ to the Hilbert scheme $\mathcal{H}_{3m}(\mathbb{P}^2) \cong \mathbb{P}^9$, and it does some interesting things: it blows up the locus of triple lines, much as in the example of plane conics, and the locus of cubics consisting of a double line and a line as well. But it also blows up the locus of cubics with a cusp, and cubics consisting of a conic and a tangent line, and these are trickier: the blow-up along the locus of cuspidal cubics, for example, can be obtained either by three blow-ups with smooth centers, or one blow-up along a nonreduced scheme supported on this locus.

But what we really want to illustrate here is that the Kontsevich space $\overline{M}_1(\mathbb{P}^2, 3)$ is not irreducible—in fact, it's not even 9-dimensional! For example, maps of the form $f : C \to \mathbb{P}^2$ with C consisting of the union of an elliptic curve E and a copy of $\mathbb{P}^1$, with f mapping $\mathbb{P}^1$ to a nodal plane cubic C_0 and mapping E to a smooth point of C_0 form a 10-dimensional family of stable maps; in fact, these form an open subset of a second irreducible component of $\overline{M}_1(\mathbb{P}^2, 3)$.

****picture here****

And there's also a third component, whose general member looks like this:

****picture here****

Twisted cubics. Here the shoe is on the other foot. The Hilbert scheme $\mathcal{H} = \mathcal{H}_{3m+1}$ has, as we saw, a second irreducible component besides the closure $\mathcal{H}_0$ of the locus of actual twisted cubics, and the presence of this component makes it difficult to work with. For example, it takes quite a

bit of analysis to see that $\mathcal{H}_0$ is smooth, since we have no simple way of describing its tangent space; see **Piene-Schlessinger** for details. By contrast, the Kontsevich space is irreducible, and has only relatively mild (finite quotient) singularities.

Exercise 10.13. True or False: There are only finitely many PGL_4 orbits in the Kontsevich space $\overline{M}_0(\mathbb{P}^3, 3)$.

Report Card for the Kontsevich Space. As with the Hilbert scheme, there are difficulties in using the Kontsevich space:

(a) **It has extraneous components**. These arise in a completely different way from the extraneous components of the Hilbert scheme, but they're there. A typical example of an extraneous component of the Kontsevich space $\overline{M}_g(\mathbb{P}^r, d)$ would consist of maps $f : C \to \mathbb{P}^r$ in which C was the union of a rational curve $C_0 \cong \mathbb{P}^1$, mapping to a rational curve of degree d in $\mathbb{P}^r$, and C_1 an arbitrary curve of genus g meeting C_0 in one point and on which f was constant; if the curve C_1 does not itself admit a nondegenerate map of degree d to $\mathbb{P}^r$, this map can't be smoothed.

****picture here****

So, using the Konsevitch space rather than the Hilbert scheme doesn't solve this problem, but it does provide a frequently useful alternative: there are situations where the Kontsevich space has extraneous components and the Hilbert scheme not—like the case of plane cubics described above—and also situations where the reverse is true, such as the case of twisted cubics.

(b) **No one knows what's in the closure of the locus of smooth curves**. This, unfortunately, remains an issue with the Kontsevich space. Even in the case of the space $\overline{M}_g(\mathbb{P}^2, d)$ parametrizing plane curves, where it might be hoped that the Kontsevich space would provide a better compactification of the Severi variety than simply taking its closure in the space $\mathbb{P}^N$ of all plane curves of degree d, the fact that we don't know which stable maps are smoothable represents a real obstacle to its use.

(c) **It has points corresponding to highly singular schemes, and these tend to be in turn highly singular points of** $\overline{M}_g(\mathbb{P}^r, d)$. Still true; but in this respect, at least, it might be said that the Kontsevich space represents an improvement over the Hilbert scheme: even when the image $f(C)$ of a stable map $f : C \to \mathbb{P}^r$ is highly singular, the fact that the domain of the map is at worst nodal makes the deformation theory of the map relatively tractable.

10.4 Rational plane curves

One case in which the Kontsevich space is truly well-behaved is the case $g = 0$. Here the space $\overline{M}_0(\mathbb{P}^r, d)$ is irreducible—it has no extraneous components—and moreover its singularities are at worst finite quotient singularities (in fact, it's the coarse moduli space of a smooth, Deligne-Mumford stack). Indeed, the use of the Kontsevich space has been phenomenally successful in answering enumerative questions about rational curves in projective space. We'll close out this chapter with an example of this; specifically, we'll answer the second keynote question, and more generally: how many rational curves $C \subset \mathbb{P}^2$ of degree d are there passing through $3d-1$ general points in the plane?

We should be clear that, since we have not defined the Kontsevich space precisely, or proved that it exists, or given a description of its local structure, this analysis will be far from a complete and logically rigorous proof of the formula we'll derive. On the other hand, a reader with knowledge of the basic facts about Kontsevich spaces—such as in **ref Fulton-Pandharipande** should be able to supply the missing details and verifications.

Before launching in to the calculation, let's check that we do in fact expect a finite number. Maps of degree d from $\mathbb{P}^1$ to $\mathbb{P}^2$ are given by triples $[F, G, H]$ of homogeneous polynomials of degree d on $\mathbb{P}^1$ with no common zeroes; since the vector space of polynomials of degree d on $\mathbb{P}^1$ has dimension $d+1$, the space U of all such triples has dimension $3d+3$. But now the map $U \to \mathbb{P}^N$ sending such a triple to the image (as divisor) of the corresponding map $\mathbb{P}^1 \to \mathbb{P}^2$ has four-dimensional fibers (we can multiply F, G and H by a common scalar, or compose the map with an automorphism of $\mathbb{P}^1$), so we conclude that the image has dimension $3d-1$. In particular, we see that *there are no rational curves of degree d passing through $3d$ general points of* $\mathbb{P}^2$; and we expect a finite number (possibly 0) through $3d-1$. We'll denote by $N(d)$ the number.

We will work on the space $M_d := \overline{M}_{0,4}(\mathbb{P}^2, d)$ of stable maps from curves with 4 marked points. Specifically, fix a point $p \in \mathbb{P}^2$ and two lines $L, M \subset \mathbb{P}^2$ passing through p; fix two more general points $q, r \in \mathbb{P}^2$ and a collection $\Gamma \subset \mathbb{P}^2$ of $3d-4$ general points. We consider the locus

$$B = \left\{ f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2 \;\middle|\; \begin{array}{c} f(p_1) = q;\ f(p_2) = r; \\ f(p_3) \in L;\ f(p_4) \in M \text{ and} \\ \Gamma \subset f(C) \end{array} \right\} \subset M_d.$$

Since as we said the space of rational curves of degree d in $\mathbb{P}^2$ has dimension $3d-1$, and we're requiring the curves in our family to pass through $3d-2$ points (the points q and r, and the $3d-4$ points of Γ), our locus B will be a curve.

Two further observations before we proceed with the calculation. There may be points in B for which the domain C of the corresponding map $f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ is reducible. But in these cases C *will have no more than two components*. To see this, note that if the image of C has components $D_1, \ldots, D_k$ of degrees $d_1, \ldots, d_k$, by the above the curve D_i can contain at most $3d_i - 1$ of the $3d - 2$ points $\Gamma \cup \{q, r\}$. Thus

$$3d - 2 \geq \sum_{i=1}^{k} (3d_i - 1) = 3d - k,$$

whence $k \leq 2$. And if the map f is nonconstant on at most two components, by the stability condition it can't be constant on any. This argument also shows that there are only finitely many points in B for which the domain C is reducible: if $D = D_1 \cup D_2 \subset \mathbb{P}^2$, with D_i a rational curve of degree d_i, and $\Gamma \cup \{q, r\} \subset D$, then by the above D_i must contain exactly $3d_i - 1$ of the $3d - 2$ points $\Gamma \cup \{q, r\}$. The number of such plane curves D is thus

$$\binom{3d-2}{3d_1-1} N(d_1) N(d_2).$$

Moreover, for each such plane curve D there are $d_1 d_2$ stable maps $f : C \to \mathbb{P}^2$ with image D: we can take C the normalization of D at all but any one of the points of intersection $D_1 \cap D_2$.

Exercise 10.14. Let Γ_1 and Γ_2 be collections of $3d_1 - 1$ and $3d_2 - 1$ general points in $\mathbb{P}^2$, and $D_i \subset \mathbb{P}^2$ any of the finitely many rational curves of degree d passing through Γ_i. Show that D_1 and D_2 intersect transversely.

On with the calculation! On B we have a rational function φ, given by the *cross-ratio*: at a point of B corresponding to a map $f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ with $C \cong \mathbb{P}^1$ irreducible, it's just the cross-ratio of the points $p_1, p_2, p_3, p_4 \in \mathbb{P}^1$; that is, in terms of an affine coordinate z on $\mathbb{P}^1$,

$$\varphi = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)},$$

where $z_i = z(p_i)$. Note that φ takes on the values $0, 1$ and ∞ only when two of the points p_i coincide, or when the curve C is reducible: for example, if C has two components C_1 and C_2, with $p_1, p_2 \in C_1$ and $p_3, p_4 \in C_2$,

****picture here****

we can realize $(C, p_1, \ldots, p_4)$ as a limit of pointed curves $(C_t, p_1(t), \ldots, p_4(t))$ with C_t irreducible and $\lim_{t\to 0} p_1(t) = \lim_{t\to 0} p_2(t)$; thus φ has a zero at such a point.

****picture here****

Note that if three of the points p_i, or all four, lie on one component of C, then φ will be the cross-ratio of four distinct points on $\mathbb{P}^1$, and so will not be 0, 1 or ∞.

The calculation now is very straightforward: we simply equate the number of zeroes and the number of poles of φ on B. To begin with, we consider points $f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ of B with C irreducible. Since $f(p_1) = q$ and $f(p_2) = r$ are fixed and lie off the lines L and M, the only way any two of the points p_i can coincide on such a curve is if

$$f(p_3) = f(p_4) = p \quad \text{where} \quad L \cap M = \{p\}.$$

Such points are zeroes of φ; the number of these zeroes is the number of rational plane curves of degree d through the $3d-1$ points p, q, r and Γ, that is to say, $N(d)$.

What about zeroes and poles of φ coming from points $f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ in B with $C = C_1 \cup C_2$ reducible? As we've observed, we get a zero of φ at such a point iff the points p_1 and p_2 lie on one component of C and p_3 and p_4 on the other. How many such points are there? Well, letting d_1 be the degree of the component C_1 of C containing p_1 and p_2, and $d_2 = d - d_1$ the degree of the other component C_2, we see that $f(C_1)$ must contain q, r and $3d_1 - 3$ of the points of Γ, while C_2 contains the remaining $3d - 4 - (3d_1 - 3) = 3d_2 - 1$ points of Γ. For any subset of $3d_1 - 3$ points of Γ, the number of such plane curves is $N(d_1)N(d_2)$, and for each such plane curve there are d_2 choices of the point $p_3 \in C_2 \cap f^{-1}(L)$ and d_2 choices of the point $p_4 \in C_2 \cap f^{-1}(M)$, as well as $d_1 d_2$ choices of the point $f(C_1 \cap C_2) \in f(C_1) \cap f(C_2)$. We thus have a total of

$$\sum_{d_1=1}^{d-1} d_1 d_2^3 \binom{3d-4}{3d_1-3} N(d_1)N(d_2)$$

zeroes of φ arising in this way.

The poles of φ are counted similarly. These can occur only at points $f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ in B with $C = C_1 \cup C_2$ reducible, specifically with the points p_1 and p_3 lying on one component of C and p_2 and p_4 on the other. Again letting d_1 be the degree of the component C_1 of C containing p_1 and p_3, and $d_2 = d - d_1$ the degree of the other component C_2, we see that $f(C_1)$ must contain q and $3d_1 - 2$ points of Γ, and $f(C_2)$ the remaining $3d - 4 - (3d_1 - 2) = 3d_2 - 2$ points of Γ, plus r. For any subset of $3d_1 - 2$ points of Γ, the number of such plane curves is $N(d_1)N(d_2)$, and for each such plane curve there are d_1 choices of the point $p_3 \in C_2 \cap f^{-1}(L)$ and d_2 choices of the point $p_4 \in C_2 \cap f^{-1}(M)$, as well as $d_1 d_2$ choices of the point

$f(C_1 \cap C_2) \in f(C_1) \cap f(C_2)$. We thus have a total of

$$\sum_{d_1=1}^{d-1} d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N(d_1)N(d_2)$$

poles of φ arising in this way. Now adding up the poles and zeroes, we conclude that

$$N(d) = \sum_{d_1=1}^{d-1} d_1 d_2 \left[d_1 d_2 \binom{3d-4}{3d_1-2} - d_2^2 \binom{3d-4}{3d_1-3} \right] N(d_1)N(d_2),$$

a recursive formula that allows us to determine $N(d)$ if we know $N(d')$ for $d' < d$.

For example, we know that $N(d_1) = N(d_2) = 1$—there's a unique line through two points, and a unique conic through 5—so that if we take $d = 3$ we have

$$N(3) = 2\left[2\binom{5}{1} - 4\binom{5}{0}\right] + 2\left[2\binom{5}{4} - \binom{5}{3}\right] = 12$$

as we've already seen. In case $d = 4$, we have

$$\begin{aligned} &N(4) \\ &\quad = 3 \cdot 12\left[3\binom{8}{1} - 9\binom{8}{0}\right] + 4\left[4\binom{8}{4} - 4\binom{8}{3}\right] + 3 \cdot 12\left[3\binom{8}{7} - \binom{8}{6}\right] \\ &\quad = 620. \end{aligned}$$

Modulo checking transversality, which we will not do in this case, this gives the answer to Keynote Question (b): through 11 general points of $\mathbb{P}^2$, there are 620 rational quartic curves.

Here are some additional problems that can be solved in a similar fashion:

Exercise 10.15. Let $p_1, \ldots, p_7 \in \mathbb{P}^2$ be general points and $L \subset \mathbb{P}^2$ a general line. How many rational cubics pass through $p_1, \ldots, p_7$ and are tangent to L?

Exercise 10.16. (Harder): More generally, let $p_1, \ldots, p_k \in \mathbb{P}^2$ be general points and $L_1, \ldots, L_{8-k} \subset \mathbb{P}^2$ general lines. Given the result of Exercise10.17 below, how many rational cubics pass through all the points and are tangent to all the lines?

Finally, we should mention one other virtue of the Kontsevich space: it allows us to work with tangency conditions, without modifying the space and without excess intersection. The reason is simple: if $X \subset \mathbb{P}^r$ is a smooth hypersurface, the closure Z_X in $\overline{M}_g(\mathbb{P}^r, d)$ of the locus of embedded curves tangent to X is contained in the locus of maps $f : C \to \mathbb{P}^r$ such that the preimage $f^{-1}(X)$ is nonreduced or positive-dimensional. Thus, for example, a point in $\overline{M}_g(\mathbb{P}^2, d)$ corresponding to a multiple curve—that is, a map

$f : C \to \mathbb{P}^2$ that is multiple-to-one onto its image—is not necessarily in Z_X.

Exercise 10.17. (a) Let $M = \overline{M}_0(\mathbb{P}^2, d)$ be the Kontsevich space of rational plane curves of degree d, and let $U \subset M$ be the open set of immersions $f : \mathbb{P}^1 \to \mathbb{P}^2$ that are birational onto their images. For $D \subset \mathbb{P}^2$ a smooth curve, let $Z_D^\circ \subset U$ be the locus of maps $f : \mathbb{P}^1 \to \mathbb{P}^2$ such that $f(\mathbb{P}^1)$ is tangent to D at a smooth point of $f(\mathbb{P}^1)$, and $Z_D \subset M$ its closure. Verify the statement above: that is, show that Z_D is contained in the locus of maps $f : C \to \mathbb{P}^r$ such that the preimage $f^{-1}(D)$ is nonreduced or positive-dimensional.

(b) Given this, show that for $D_1, \ldots, D_{3d-1}$ general curves the intersection $\cap Z_{D_i}$ is contained in U.

11

Chow Rings of Projective Bundles

Keynote Questions

(a) Given eight general lines $L_1, \ldots, L_8 \subset \mathbb{P}^3$, how many conic curves in planes in $\mathbb{P}^3$ meet all eight?

In this chapter we'll derive a basic formula describing the Chow rings of projective bundles. This will enlarge the range of parameter spaces whose Chow rings we can compute and allow us to answer the question above. It will also help us to describe the Chow ring of a blow-up, which we'll do in Chapter 16.

11.1 Projective bundles and the tautological divisor class

Definition 11.1. Let X be a scheme and E a vector bundle of rank $r+1$ on X. As usual, we'll use the same notation for the corresponding locally free sheaf. By the *projectivization* of E we will mean the X-scheme

$$\mathrm{Proj}(\mathrm{Sym}^\bullet E^*) \to X;$$

informally, this is the scheme whose points correspond to pairs (x, ξ) with $x \in X$ and ξ a one-dimensional subspace $\xi \subset E_x$ of the fiber E_x of E. By a *projective bundle over* X we mean a morphism $\pi : Y \to X$ that can be realized as the projectivization $\mathbb{P}E \to X$ of a vector bundle E over X.

Note that in this definition, we do not specify the vector bundle E of which Y is the projectivization. We will explain the ambiguity in Proposition 11.3 below.

The reader should be aware that there is a conflicting definition that is also in use. Some sources, following Grothendieck, define the projectivization of E to be what we would call the projectivization of E^*, that is, $\mathrm{Proj}(\mathrm{Sym}^\bullet E) \to X$. Its points correspond to 1-quotients of fibers of E. We are following the classical tradition, which is also that adopted in Fulton [1984]. (Grothendieck's definition would allow us incorporate $\mathrm{Proj}(\mathrm{Sym}(\mathcal{F}))$ for a non-reflexive quasicoherent sheaf $\mathcal{F}$ in our discussion; but we will be working exclusively with vector bundles.)

If $\pi : \mathbb{P}E \to X$ is a projective bundle, then there is a *tautological line bundle* S that is a sub bundle of π^*E. Its fiber at a point $(x, \xi) \in \mathbb{P}E$ is the the subspace $\xi \subset E_x$ of the fiber E_x corresponding to the point $\xi \in \mathbb{P}E_x$. It is defined as the locally free rank 1 sheaf on $\mathbb{P}E$ associated to the graded $(\mathrm{Sym}^\bullet E^*)$-module $(\mathrm{Sym}^\bullet E^*)(-1)$, which we denote by $\mathcal{O}_{\mathbb{P}E}(-1)$, and similarly we write $\mathcal{O}_{\mathbb{P}E}(1)$ for the dual, S^*. The n-th tensor power of S is denoted $\mathcal{O}_{\mathbb{P}E}(-n)$; its dual is the n-th tensor power of S^*, denoted $\mathcal{O}_{\mathbb{P}E}(n)$. For any quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}E$ we write $\mathcal{F}(n)$ to denote $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}E}(n)$.

The dual S^* of S is the sheaf associated to the sheaf of $\mathrm{Sym}^\bullet E^*$-modules $(\mathrm{Sym}^\bullet E^*)(1)$, which we denote by $\mathcal{O}_{\mathbb{P}E}(1)$. Dualizing the inclusion $S \subset E$ we get a surjection $E^* \to \mathcal{O}_{\mathbb{P}E}(1)$. Thus any global section σ of E^* gives rise to a global section $\tilde{\sigma}$ of $\mathcal{O}_{\mathbb{P}E}(1)$. By the Theorem on Cohomology and Base Change (Theorem 6.4) the corresponding map

$$H^0(E^*) \to H^0(\mathcal{O}_{\mathbb{P}E}(1)),$$

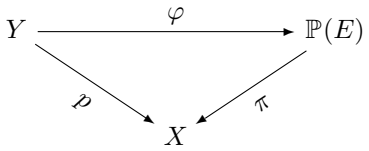
this map is an isomorphism.

The zero locus of the section $\tilde{\sigma}$ in $\mathbb{P}E$ is the locus of pairs (x, ξ) such that σ_x vanishes on ξ; thus the divisor $(\tilde{\sigma})$ meets a general fiber of $\pi : \mathbb{P}E \to X$ in a hyperplane. It will not in general meet *every* fiber of $\mathbb{P}E \to X$ in a hyperplane, however; the section σ of E^* may have zeroes $x \in X$; and the divisor $(\tilde{\sigma}) \subset \mathbb{P}E$ will contain the corresponding fibers $(\mathbb{P}E)_x = \pi^{-1}(x)$.

If L is any line bundle on X, there is a natural correspondence between one-dimensional subspaces of E_x and one-dimensional subspaces of $E_x \otimes L_x$. This defines a set-theoretic isomorphism between $\mathbb{P}(E) \to X$ and $\mathbb{P}(L \otimes E) \to X$. To see that it is a map of schemes, we use the universal property of Proj. This gives us a way to define mappings from schemes over X to projective bundles over X, generalizing the familar way of defining maps to projective space by means of vector spaces of sections:

Proposition 11.2 (Universal Property of Proj)**.** *Maps of schemes over X from $p : Y \to X$ to $\pi : \mathbb{P}(E) \to X$, that is, commutative diagrams of maps*

of schemes



are in natural one-to-one correspondence with line sub-bundles (that is, locally split inclusions of locally free rank one sheaves) $L \to p^*E$.

Proof. Given φ we pull back $S \to \pi^*E$ via φ and get $\varphi^*S \to \varphi^*\pi^*E = p^*E$. Conversely, given $L \to p^*E$, we may cover X by open sets on which E and L are trivial, and get a unique map over each of these using the universal property of ordinary projective space. By uniqueness, these maps glue together to give a map over all of X. □

We can use Proposition 11.2 to see when two vector bundles give the same projective bundle:

Proposition 11.3. *Let X be a variety, and let $\mathbb{P}(E) \to X$ be a projective bundle. The tautological line bundle $S = \mathcal{O}_{\mathbb{P}(E)}$ determines E by the formula $E^* = \pi_* S^* = \pi_* \mathcal{O}_{\mathbb{P}(E)}(1)$. Two projective bundles $\mathbb{P}(E) \to X$ and $\mathbb{P}(E') \to X$ are isomorphic (as schemes over X) if and only if there is a line bundle L on X such that $L \otimes E = E'$.*

Proof. To prove that $\mathbb{P}(E) \to X$ is isomorphic to $\mathbb{P}(L \otimes E) \to X$ we use the description of maps above. Writing $p : \mathbb{P}(L \otimes E) \to X$ for the projection, we have a tautological sub-bundle $S' \subset p^*L \otimes E$ on $\mathbb{P}(L \otimes E)$. Tensoring with the pullback of L^{-1} we get a sub-bundle $p^*(L^{-1}) \otimes S' \to p^*(L^{-1}) \otimes p^*E = p^*(L^{-1} \otimes E)$ that defines a morphism of X-schemes $\mathbb{P}(L \otimes E) \to \mathbb{P}(E)$. The inverse map is defined similarly. The proof that they are inverse to each other is that the composite $\mathbb{P}(E) \to \mathbb{P}(L \otimes E) \to \mathbb{P}(E)$ corresponds to the original sub bundle $S \subset E$.

Conversely, given an isomorphism $\varphi : \mathbb{P}(E') \to \mathbb{P}(E)$, we may pull back the bundle S to get a line bundle S'' on $\mathbb{P}(E')$ that restricts on each fiber $\mathbb{P}(E_x) \cong \mathbb{P}^n$ to the bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$. By Corollary6.11 any such line bundle is of the form $p^*(L) \otimes S'$, where L is a line bundle on X and $p_*(S''^{-1}) = L^{-1} \otimes p_*(S') = L^{-1} \otimes E'$. □

11.2 Chow ring of a projective bundle

We now turn to the central problem of this Chapter: to describe the Chow ring of a projective bundle $Y = \mathbb{P}E \to X$. We'll see that the Chow groups

of Y depend only on the rank of E, but the ring structure reflects the Chern classes of E.

To see what to expect, suppose that $Y \cong X \times \mathbb{P}^r$ is a product, the projectivization of the trivial bundle, and that we are working over the complex numbers and consider the analogous case of the integral cohomology ring. We can compute it by the Künneth formula:

$$H^*(Y) = H^*(X) \otimes_{\mathbb{Z}} H^*(X) = H^*(X)[\zeta]/(\zeta^{r+1}),$$

where $\zeta \in H^2(X \times \mathbb{P}^r)$ is the pullback of the generator of $H^2(\mathbb{P}^r)$. In particular,

$$H^*(Y) = \bigoplus_{i=0}^{r} \zeta^i H^*(X)$$

as groups.

Here is the corresponding statement for the Chow ring of any projective bundle. Much of the statement remains true without the assumption that X is smooth; see Fulton [1984] Section ****.

Theorem 11.4. *Let E be a vector bundle of rank $r+1$ on a smooth projective scheme X, and let $\zeta = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \in A^1(\mathbb{P}(E))$. Let $\pi : \mathbb{P}(E) \to X$ be the projection. The map $\pi^* : A^*(X) \to A^*(\mathbb{P}(E))$ is an injection of rings, and via this map*

$$A^*(\mathbb{P}(E)) \cong A^*(X)[\zeta]/\big(\zeta^{r+1} + c_1(E)\zeta^r + \cdots + c_{r+1}(E)\big).$$

In particular, the group homomorphism $A^*(X)^{\oplus r+1} \to A^*(\mathbb{P}(E))$ given by $(\alpha_0, \ldots, \alpha_r) \mapsto \sum \zeta^i \pi^*(\alpha_i)$ is an isomorphism, so that

$$A^*(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{r} \zeta^i A^*(X)$$

as groups.

The result of Theorem 11.4 can be inverted and used to define the Chern classes of E, as the coefficients of the unique monic polynomial relation $\zeta^{r+1} + c_1(E)\zeta^r + \cdots + c_{r+1}(E) = 0$ on the Chow ring of $\mathbb{P}E$. It has the advantage of not depending on global sections, and the disadvantage that it does not suggest such an immediate intuition about the properties and uses of the Chern classes. This works when X is any scheme, so it yields a good definition of Chern classes in a very general case. ****put a forward reference to this where we define Chern classes.****

We isolate part of the proof that will be useful elsewhere:

Lemma 11.5. *Let the hypotheses be as in Theorem 11.4. If $\alpha \in A(X)$ then*

$$\pi_*(\zeta^i \alpha) = \begin{cases} \alpha & \text{if } i = r \\ 0 & \text{if } i < r. \end{cases}$$

Proof. Let $A \subset X$ be a subvariety, and write $\alpha = [A]$. Because π is flat we have $\pi^*\alpha = [\pi^{-1}A]$, and $\pi^{-1}A$ has each fiber of dimension r. Since the restriction of ζ to any fiber is the hyperplane class, we see that for $i < r$ the class $\zeta^i\pi^*[A]$ is represented by a cycle whose fibers under π have positive dimension. Thus $\pi_*(\zeta^i\alpha) = \pi_*(\zeta^i\pi^*[A]) = 0$ for $i < r$.

Choosing a cycle $\sum_i n_i[B_i] \equiv \zeta^r$ such that the B_i are generically transverse to $\pi^{-1}A$, we can compute $\zeta^r\alpha$ as $\sum_i n_i[B_i \cap \pi^{-1}A]$. Since $\pi(\pi^{-1}A) = A$ we must have $\pi_*(\zeta^r[A]) = m[A]$ for some $m \in \mathbb{Z}$, and m is the sum, over those $B_i \cap \pi^{-1}A$ that dominate A, of the numbers $n_i \deg(\pi : B_i \cap \pi^{-1}A \to A)$. Equivalently, $m_{[X]}$ is the degree of the intersection of ζ^r with the generic fiber of π over a point of A.

Thus we may restrict our question to a fiber of π, and assume that A is a point. In this case $\mathbb{P}E = \mathbb{P}^r$, and ζ is the class of a hyperplane. Thus $m = 1$. □

Proof of Theorem 11.4. Let $\psi : A(\mathbb{P}E) \to \oplus_{i=0}^r A(X)\zeta^i$ be the map

$$\beta \mapsto \sum_i \pi_*(\zeta^{r-i}\beta)\zeta^i.$$

and let $\varphi : \oplus_{i=0}^r A(X) \to A(\mathbb{P}E)$ be the sum of the multiplications by powers of ζ,

$$\varphi : (\alpha_0, \ldots, \alpha_r) \mapsto \sum_i \zeta^i\alpha_i$$

By Lemma 11.5 the composite $\psi\varphi$ is upper triangular with ones on the diagonal; in particular, φ is a monomorphism.

Thus, to prove the Theorem, it suffices to show that the subgroups $\zeta^i A(X)$ generate $A(\mathbb{P}(E))$ additively. We will do this by induction on the rank $r+1$ of E. If $r = 0$ then $\mathbb{P}(E) \cong X$ and the assertion is obvious. ****The following argument for generation doesn't work in our context because the induction leaves the category of smooth projective varieties. See below for an outline of an alternate.**** First, suppose that $E = \mathcal{O}_X^{r+1}$ is the trivial bundle, so that $\mathbb{P}(E) \cong \mathbb{P}^r \times X$. Let H be a hyperplane in $\mathbb{P}^r$, and consider the right exact sequence of Chow groups

$$A_*(H \times X) \to A_*(\mathbb{P}(E)) \to A_*(\mathbb{A}^n \times X) \to 0$$

coming from Proposition4.11. The composite of the pullback map from $A_*(X)$ to $A_*(\mathbb{P}(E))$ with the restriction to $A_*(\mathbb{A}^n \times X)$, is the pullback map from $A_*(X)$ to $A_*(\mathbb{A}^n \times X)$, which is surjective by Theorem4.21.

Let $\pi_2 : H \times X \to X$ be the projection to the second factor, and let ζ_1 be the hyperplane class in $A_*(H) = A_*(\mathbb{P}^{r-1})$. The classes $\zeta^i\pi_2^*(A_*(X))$ generate $A_*(H \times X)$ by induction on r. Since H meets the preimage of any subvariety $Z \subset X$ generically transversely, the class $\zeta_1^i\pi_2^*[Z]$ is the

restriction to $H \times X$ of the class $\zeta^i \pi^*[Z] \in A_*(\mathbb{P}^r \times X)$. It follows that the classes $\zeta^i A_*(X)$ generate $A_*(\mathbb{P}^r \times X)$.

To reduce to the case of the trivial bundle, we use induction on the dimension again—this time on the dimension of X. Let $U \subset X$ be an open dense set over which E is trivial, and set $X' = X \setminus U$, so that $\dim X' < \dim X$. ****For the induction to work as stated, we need X' to be smooth, as well as needing E to trivialize on $X \setminus X'$. A red herring – and I'm not sure it's even possible!**** Consider the exact sequence

$$A_*(\mathbb{P}(E|_{X'})) \to A_*(\mathbb{P}(E)) \to A_*(\mathbb{P}(E|_U)) \to 0.$$

If z is a class in $A_*(\mathbb{P}(E))$ then by the case of a trivial bundle the image of z in $A_*(\mathbb{P}(E|_U))$ is a sum of classes of the form $\zeta^i z_i$, where the z_i are classes in $A_*(U)$ ****this is a slight abuse of notation, since the class ζ here is the tautological class over U. I think it doesn't matter.**** Using the exact sequence

$$A_*(X') \to A_*(X) \to A_*(U) \to 0.$$

we may lift these classes back to X, so we see z minus a suitable class in $\sum_i \zeta^i \mathbb{A}_*(X)$will go to zero in $A_*(\mathbb{P}(E|_U))$, and thus come from a class in $A_*(\mathbb{P}(E|_{X'}))$. By induction on the dimension this class is in $\sum_i \zeta^i A_*(X')$. Since π^* commutes with the maps induced by the inclusions $X' \subset X$ and $\mathbb{P}(E|_{X'}) \subset \mathbb{P}(E)$, we are done. ****end of bad argument****

****alternate argument****

(a) Lemma: If we tensor E with a sufficiently positive bundle from the base, a general section will have smooth zero locus.
To prove this, twist until $\mathcal{O}_{PE}(1)$ is very ample. If we look locally, so that a section of E is $r = rankE$ functions, then the condition for all these functions to vanish to order 2 at a point of X is the condition for the divisor in $\mathbb{P}(H^0 E)^*$ to contain the first neighborhood of the fiber. Containing the fiber $\mathbb{P}^{r-1}$ is r conditions and containing the neighborhood, even at one point of the fiber, is $\dim X$ more conditions, total of at least $r + \dim X$ conditions to be satisfied at $\dim X$ points, thus at most a codim $\geq r$ family of sections will be singular somewhere on X. (Question: is ever bundle trivial off a smooth subvariety?)

(b) Twist up our bundle til a section has a smooth zero locus, and then do an induction on the dimension of the base to reduce to the case where $E = E' \oplus \mathcal{O}_{\mathbb{P}E}$. Then we have an open set of $\mathbb{P}E$ consisting of an affine bundle, and another consisting of a lower dimensional projective space bundle.

(c) Affine space bundles are treated by Theorem 4.21.

****end of alternate generation argument.****

From this description of the Chow groups we see that we can write ζ^{r+1} as a linear combination of products of (pullbacks of) classes in $A^*(X)$ with lower powers of ζ—that is, ζ satisfies a monic polynomial f of degree $r+1$ over $A^*(X)$. Thus the ring homomorphism $A^*(X)[\zeta] \to A^*(\mathbb{P}(E))$ factors through the quotient $A^*(X)[\zeta]/(f)$. Since $A^*(X)[\zeta]/(f) \cong \oplus \zeta^i \mathbb{A}_*(X)$ as groups, it follows that the map $A^*(X)[\zeta]/(f) \to A^*(\mathbb{P}(E))$ is an isomorphism of rings.

It remains to identify the polynomial f. Let Q be the cokernel of the natural inclusion $S \to \pi^*E$, a bundle of rank r. We have an exact sequence

$$0 \to S \to \pi^*E \to Q \to 0.$$

Identifying $A^*(X)$ with $\pi^*(A^*(X))$ as before, we have

$$c(S) \cdot c(Q) = c(E)$$

by the Whitney sum formula, Proposition7.3.

We defined the class ζ to be the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$, which is the dual of S; thus $c(S) = 1 - \zeta$ and we can write this as

$$c(Q) = c(E) \cdot c(S)^{-1} \tag{11.1}$$

$$= c(E)(1 + \zeta + \zeta^2 + \dots). \tag{11.2}$$

Since Q is a vector bundle of rank r, we conclude that

$$0 = c_{r+1}(Q) = \zeta^{r+1} + c_1(E)\zeta^r + c_2(E)\zeta^{r-1} + \cdots + c_r(E)\zeta + c_{r+1}(E),$$

so the polynomial f is given by the formula in the Theorem. □

Exercise 11.6. Let X, E and $\mathbb{P}(E)$ be as above, and let L be any line bundle on X.

(a) How does the class $c_1(\mathcal{O}_{\mathbb{P}(E\otimes L)}(1))$ relate to $c_1(\mathcal{O}_{\mathbb{P}E}(1))$?

(b) Using the results of Section7.2.5, show that the two descriptions of the Chow ring of $\mathbb{P}(E) = \mathbb{P}E = \mathbb{P}(E \otimes L)$ agree.

Exercise 11.7. Let $\pi : Y \to X$ be a projective bundle.

(a) Show that the direct sum decomposition of the group $A^*(X)$ given in Theorem 11.4 depends on the choice of vector bundle E with $Y \cong \mathbb{P}E$; but

(b) Show that if we define group homomorphisms $\psi_i : A^*(Y) \to A^*(X)^{\oplus i+1}$ by

$$\psi_i : \alpha \mapsto \big(\pi_*(\alpha), \pi_*(\zeta\alpha), \dots, \pi_*(\zeta^i\alpha)\big)$$

then the filtration of $A^*(Y)$ given by

$$A^*(Y) \supset \mathrm{Ker}(\psi_0) \supset \mathrm{Ker}(\psi_1) \supset \cdots \supset \mathrm{Ker}(\psi_{r-1}) \supset \mathrm{Ker}(\psi_r) = 0$$

is independent of the choice of E. (Hint: give a geometric characterization of the cycles in each subspace of $A^*(Y)$.)

Exercise 11.8. Consider the flag variety of pairs consisting of a point $p \in \mathbb{P}^4$ and a line $\Lambda \subset \mathbb{P}^3$ containing p; that is,

$$\mathbb{F} = \{(p, L) : p \in L \subset \mathbb{P}^3\} \subset \mathbb{P}^3 \times \mathbb{G}(1, 3).$$

$\mathbb{F}$ may be viewed as a $\mathbb{P}^1$-bundle over $\mathbb{G}(1, 3)$, or as a $\mathbb{P}^2$-bundle over $\mathbb{P}^3$. Calculate the Chow ring $A^*(\mathbb{F})$ via each map, and show that the two descriptions agree.

Exercise 11.9. Let $\mathbb{P}^5$ be the space of conic curves in $\mathbb{P}^2$ and let $Y \to \mathbb{P}^5$ be the universal conic, that is,

$$Y = \{(C, p) : p \in C\} \subset \mathbb{P}^5 \times \mathbb{P}^2.$$

(a) Observing that projection on the second factor expresses Y as a $\mathbb{P}^4$-bundle over $\mathbb{P}^2$, calculate its Chow ring.

(b) Use this to prove that $Y \to \mathbb{P}^5$ has no rational sections, as stated in Exercise **??**.

11.2.1 Example: the blow-up of $\mathbb{P}^n$ along a linear space

Let $X = Bl_{\mathbb{P}^k}\mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ along a linear subspace $\mathbb{P}^k$. We can realize X as the graph of the projection $\mathbb{P}^n \to \mathbb{P}^{n-k-1}$ from $\mathbb{P}^k$; that is, if we realize $\mathbb{P}^{n-k-1}$ as the set of $(k+1)$-planes $\Lambda \subset \mathbb{P}^n$ containing $\mathbb{P}^k$, we have

$$X = \{(p, \Lambda) : p \in \Lambda\} \subset \mathbb{P}^n \times \mathbb{P}^{n-k-1}$$

Exercise 11.10. Show that the projection $\pi_2 : X \to \mathbb{P}^{n-k-1}$ expresses X as a $\mathbb{P}^{k+1}$-bundle over $\mathbb{P}^{n-k-1}$; specifically, show that

$$X \cong \mathbb{P}\big(\mathcal{O}^{k+1}_{\mathbb{P}^{n-k-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-k-1}}(-1)\big)$$

Exercise 11.11. Use the preceding Exercise to give a description of the Chow ring of X. In particular, find in these terms the classes

(a) of the exceptional divisor E of the blow-up;

(b) of a linear space $\mathbb{P}^l$ disjoint from $\mathbb{P}^k$; and

(c) of the proper transform on a linear space $\mathbb{P}^m$ containing $\mathbb{P}^k$.

11.3 Chow ring of a Grassmannian bundle

Suppose X is any variety, and E a vector bundle of rank n on X. Generalizing the projective bundle associated to E we can form the *Grassmann bundle* $G(k, E)$ of k-planes in the fibers of E; that is,

$$G(k, E) = \{(x, V) : x \in X;\ V \subset E_x\} \xrightarrow{\pi} X.$$

There is a description of the Chow ring of $G(k, E)$ that extends both the description of the Chow ring of a projective bundle and the Chow ring of $G(k, n)$; we'll explain it here without proof.

As in the projective bundle case, there is a *tautological subbundle* $S \subset \pi^* E$ defined on $G(k, E)$; this is a rank k bundle whose fiber over a point (x, V) is the vector space $V \subset E_x$. Let $Q = \pi^*(E)/S$ the *tautological quotient bundle*. As in the case of projective bundles, the Chow ring $A^*(G(k, E))$ is generated as an $A^*(X)$-algebra by the Chern classes $c_i(S)$, and also by the classes $c_i(Q)$. To understand the relations they satisfy, consider the exact sequence

$$0 \to S \to \pi^* E \to Q \to 0.$$

By the Whitney sum formula

$$c(Q) = \frac{c(E)}{c(S)}.$$

Since Q has rank $n-k$, the Chern classes $c_l(Q)$ vanish for $l > n-k$, and as in the projective bundle case (above) or the case of $\mathbb{G}(k, n)$ (Theorem7.14) this gives all the relations:

Theorem 11.12. *Let X be a variety and let E be a vector bundle of rank n on X. If $G = G(k, E) \to X$ the bundle of k-planes in the fibers of E then*

$$A^*(G) = A^*(X)[\zeta_1, \dots \zeta_k] / \left(\left\{ \frac{c(E)}{1 + \zeta + \zeta^2 + \dots} \right\}_l, l < \dim G - n + k \right)$$

where $\zeta = \zeta_1 + \dots + \zeta_k$ and $\{-\}_l$ denotes the component of $-$ of dimension *l, an element of $A_l(G)$.*

One can go further, and, fixing a sequence of ranks $0 < r_1 < \cdots r_m < \operatorname{rank} E$, consider the *flag bundle* $Flag(r_1, \dots, r_m, E)$ whose fiber over a point of X is the set of all flags of subspaces of the given ranks in E. There is again an analogous description of the Chow ring of this space. See Grayson and Stillman [$\geq$ 2011] for this result and an interesting proof that is in some ways more explicit than the one we have given, even in the case of $A^*(G(k, n))$. ****is there a published proof?****

Exercise 11.13. Consider the flag variety of pairs consisting of a point $p \in \mathbb{P}^4$ and a 2-plane $\Lambda \subset \mathbb{P}^4$ containing p; that is,

$$\mathbb{F} = \{(p, L) : p \in \Lambda \subset \mathbb{P}^4\} \subset \mathbb{P}^4 \times \mathbb{G}(2, 4).$$

$\mathbb{F}$ may be viewed as a $\mathbb{P}^2$-bundle over $\mathbb{G}(2, 4)$, or as a $\mathbb{G}(1, 3)$-bundle over $\mathbb{P}^4$. Calculate the Chow ring $A^*(\mathbb{F})$ via each map, and show that the two descriptions agree.

11.4 Conics in $\mathbb{P}^3$ Meeting Eight Lines

We come now to a classical problem in enumerative geometry: to determine the number of conic plane curves in $\mathbb{P}^3$ intersecting each of 8 general lines $L_1, \ldots, L_8 \subset \mathbb{P}^3$. A key step in solving the problem is the choice of parameter space for plane conics. We will choose the Hilbert scheme, and we will use the description of its local structure to characterize transversality of intersections. It turns out that the Hilbert scheme of conics is a natural example of a projective bundle, so we can use the theory developed earlier in this chapter for the computation.

We'll see how this description allows us to solve other enumerative problems as well—and, on the other hand, exhibit some related enumerative problems whose solution requires a different choice of parameter space.

The steps we'll carry out to solve this problem are:

(a) Describe a parameter space $\mathcal{H}$ for plane conics in $\mathbb{P}^3$.

(b) Let $D_L \subset \mathcal{H}$ be the cycle of plane conics in meeting a line L. Verify that if the lines L_i are generic then $\cap_{i=1}^8 D_{L_i}$ is a transverse intersection, so that the number of intersections coincides with the degree of the element $[D_L]^8 \in A^*(\mathcal{H})$.

(c) Determine the Chow ring $A^*(\mathcal{H})$.

(d) Find the class $[D_L] \in A^*\mathcal{H}$.

(e) Combine the results of parts (b) and (c)—that is, take the product in $A^*(\mathcal{H})$ of the classes of the cycles—to calculate the expected number of objects satisfying the conditions.

The subtlety in the choice in part (a) is to find a parameter space that makes the statement of part (b) true while making the computations in parts (c) and (d) possible!

11.4.1 The Parameter Space

By a smooth conic in $\mathbb{P}^3$, we'll mean just a smooth plane curve $C \subset \mathbb{P}^2$ of degree 2, embedded in $\mathbb{P}^3$ via a linear embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^3$. There is little ambiguity about how to parametrize these, and if the space of smooth conics were proper, there would be essentially no choice to make in part a above. But conics do degenerate, and so we have to choose a space that parametrizes at least some of these singular curves.

The space we choose, as we indicated already, is the Hilbert scheme. The Hilbert polynomial $p = p_C$ of a plane conic curve is $p(m) = 2m + 1$, and so specifically we consider the scheme $\mathcal{H} = \mathcal{H}_{2m+1}(\mathbb{P}^3)$ parametrizing all

subschemes of $\mathbb{P}^3$ with this Hilbert polynomial. Naturally, the first order of business to is to describe all such subschemes; this is the content of the following.

Proposition 11.14. *Any subscheme $C \subset \mathbb{P}^3$ with Hilbert polynomial $h_C(m) = 2m + 1$ is the complete intersection of a plane and a quadric.*

Proof. The key to this Proposition is this: If $C \subset \mathbb{P}^3$ is any subscheme with Hilbert polynomial $2m + 1$ then the Hilbert function of C is also equal to $2m + 1$. For under this hypothesis a general plane section $\Gamma = \{x = 0\} \cap C$ is a subscheme of degree 2 in the plane $H \cong \mathbb{P}^2$, and the Hilbert function of Γ is accordingly $h_\Gamma(m) = 2$ for all $m \geq 1$. Writing S_C for the homogeneous coordinate ring of C we have a surjective map $S_C \to S_\Gamma$ whose kernel contains xS_C, whence

$$h_C(m) - h_C(m-1) \geq h_\Gamma(m) = 2$$

for $m \geq 1$. Since $h_C(0) = 1$, it follows that $h_C(m) \geq 2m + 1$ for all m; and given that $h_C(m) = 2m + 1$ for large m it follows that we must have equality throughout.

Given this, we can read off the geometry of C from its Hilbert function: since $h_C(1) = 3$, C must be contained in a unique plane $H = V(L)$. Moreover, since $h_C(2) = 5$, C will lie on five linearly independent quadrics; since at most four of these can contain H, we see that C lies on a quadric $Q \subset \mathbb{P}^3$ not containing H. And finally, since $H \cap Q$ has Hilbert function $2m+1$ and $C \subset H \cap Q$ has the same, it follows that $C = H \cap Q$. □

Henceforth, by a "plane conic" in $\mathbb{P}^3$ we'll mean a complete intersection subscheme $C = H \cap Q \subset \mathbb{P}^3$. Note that among these are double lines; but a double line, like a smooth conic or a pair of incident lines, is contained in a unique plane. Note also that for any such subscheme, the cohomology $H^1(\mathcal{I}_C(k))$ of any twist of its ideal sheaf vanishes.

It follows from Proposition 11.14 that the Hilbert scheme $\mathcal{H} = \mathcal{H}_{2m+1}(\mathbb{P}^3)$ parametrizes plane conics in this sense, and it is easy to see what $\mathcal{H}$ looks like set-theoretically: Since each conic $C \subset \mathbb{P}^3$ lies in a unique plane $H_C \subset \mathbb{P}^3$, there is a set-theoretic map $\mathcal{H} \to \mathbb{P}^{3*}$ of $\mathcal{H}$ to the dual projective space, sending C to H_C. Moreover, the space of conics in a given plane H is the projectivization of the space $H^0(\mathcal{O}_H(2))$ of homogeneous quadratic polynomials on H, so $\mathcal{H}$ is set-theoretically a $\mathbb{P}^5$-bundle over $\mathbb{P}^{3*}$; specifically, it is the projectivization of the vector bundle $\operatorname{Sym}^2 S^*$, where S is the universal (rank 3) sub-bundle over $\mathbb{P}^{3*}$.

In fact, these assertions hold scheme-theoretically as well:

Proposition 11.15. *The Hilbert scheme $\mathcal{H} = \mathcal{H}_{2m+1}(\mathbb{P}^3)$ is isomorphic to the total space of the projective bundle $\mathbb{P} \operatorname{Sym}^2 S^*$ over $\mathbb{P}^{3*}$.*

Proof. To prove that these assertions hold scheme-theoretically we will use the definition of $\mathcal{H}$ as the base of the universal family of plane conics. That is, we will exhibit an isomorphism of functors from the category of schemes to the category of sets, between the functor that associates to any scheme B the set of subschemes $\mathcal{C} \subset B \times \mathbb{P}^3$, flat over B with relative Hilbert polynomial $2m+1$; and the functor $\mathrm{Mor}(\,\cdot\,, \mathbb{P}\,\mathrm{Sym}^2 S^*)$, which associates to B the set of morphisms from B to $\mathbb{P}\,\mathrm{Sym}^2 S^*$.

One direction is easy: there is a tautological family $\mathcal{X} \subset \mathbb{P}\,\mathrm{Sym}^2 S^* \times \mathbb{P}^3$ whose fiber over a point $(H, \xi) \in \mathbb{P}\,\mathrm{Sym}^2 S^*$ is the plane curve $C \subset H$ defined by the quadratic polynomials in $\xi \subset H^0(\mathcal{O}_H(2))$, so to a scheme B and a morphism $B \to \mathbb{P}\,\mathrm{Sym}^2 S^*$ we can associate the pullback family $B \times_{\mathbb{P}\,\mathrm{Sym}^2 S^*} \mathcal{X} \subset B \times \mathbb{P}^3$.

To see the correspondence in the other direction, let B be any scheme and suppose that $\mathcal{C} \subset B \times \mathbb{P}^3$ is any subscheme, flat over B, with relative Hilbert polynomial $2m+1$. We consider the exact sequence of sheaves on $B \times \mathbb{P}^3$:

$$0 \to \mathcal{I}_{\mathcal{C}}(1) \to \mathcal{O}_{B\times\mathbb{P}^3}(1) \to \mathcal{O}_{\mathcal{C}}(1) \to 0,$$

where in each case the twist "(1)" means tensor with the pullback of $\mathcal{O}_{\mathbb{P}^3}(1)$. Let $\pi : B \times \mathbb{P}^3 \to B$ be the projection. By our Baby Theorem on Cohomology and Base Change (Theorem 6.4), we see that $R^1\pi_*(\mathcal{I}_{\mathcal{C}}(1)) = 0$, and hence that, taking the pushforward π_* of this sequence, we have an exact sequence of vector bundles on B:

$$0 \to L \to H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to Q \to 0,$$

where the fiber L_b of L at each $b \in B$ is the (one-dimensional) space of linear forms vanishing on C_b, $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ is the eponymous trivial bundle of rank 4, and Q the rank 3 quotient. Dualizing, we have an epimorphism

$$H^0(\mathcal{O}_{\mathbb{P}^3}(1))^* \to L^*,$$

and hence a map $\varphi : B \to \mathbb{P}^{3*}$. Note also that if $\Phi \subset \mathbb{P}^3 \times \mathbb{P}^{3*}$ is the universal hyperplane described in Section 12.6.1, ****this used to be in the old Ch 3, and now it's moved to the chapter after this one. Should we replce the reference with a defn?**** then under this construction the curve $\mathcal{C} \subset B \times \mathbb{P}^3$ is actually contained in the pullback $\Phi_B = B \times_{\mathbb{P}^{3*}} \Phi \subset B \times \mathbb{P}^3$.

Next, we restrict to the pullback Φ_B of the universal hyperplane, and do the same thing with quadrics: that is, we take the direct image π_* of the sequence

$$0 \to \mathcal{I}_{\mathcal{C},\Phi_B}(2) \to \mathcal{O}_{\Phi_B}(2) \to \mathcal{O}_{\mathcal{C}}(2) \to 0.$$

By flatness and the constancy of the dimension of h^0 on the fibers, the direct image π_* of the sheaf $\mathcal{O}_{\Phi_B}(2)$ is a vector bundle, with fiber over $b \in B$ the space $H^0(H, \mathcal{O}_H(2))$, where $H = \varphi(b)$ is the unique plane containing the fiber C_b of $\mathcal{C}$ over b; in other words, the pullback $\varphi^*(\mathrm{Sym}^2 S^*)$. By the same

token, the pushforward of the sheaf $\mathcal{I}_{\mathcal{C},\Phi_B}(2)$ is the sub-line bundle M of $\pi_*\mathcal{O}_{\Phi_B}(2)$ whose fiber over b is the one-dimensional space of sections of $\mathcal{O}_H(2)$ vanishing on C_b. Dualizing, we have an epimorphism

$$\varphi^*(\mathrm{Sym}^2 S^*) \to M^*$$

and hence a lifting of the map $\varphi : B \to \mathbb{P}^{3*}$ to a map $\tilde{\varphi} : B \to \mathbb{P}\,\mathrm{Sym}^2 S^*$. □

11.4.2 Transversality

Having chosen our parameter space $\mathcal{H}$ for plane conics, we proceed to the next step: if, for any line $L \subset \mathbb{P}^3$, we let $D_L \subset \mathcal{H}$ be the cycle of plane conics meeting L, we have to show that for $L_1, \ldots, L_8$ general the cycles D_{L_i} intersect transversely in $\mathcal{H}$. We start with a little housekeeping, showing that the intersection of the D_{L_i} takes place entirely within the open subset $U \subset \mathcal{H}$ of smooth plane conics:

Lemma 11.16. *For a general choice of lines $L_1, \ldots, L_8 \subset \mathbb{P}^3$, no singular conic meets all 8.*

Proof. To start, we observe that no line will meet more than four of the eight lines $L_1, \ldots, L_8$; thus, if C is a singular conic meeting all 8, C must consist of the union of two incident lines, with one meeting four of the lines L_i and the other meeting the remaining four.

Choosing $L_1, \ldots, L_4$ generally, there will be only two lines $M, M' \subset \mathbb{P}^3$ meeting all four; and then choosing $L_5, \ldots, L_8$ generally no line will meet all four of these and either M or M'. The locus of 8-tuples $(L_1, \ldots, L_8)$ such that there exists a pair of incident lines with one meeting $L_1, \ldots, L_4$ and the other $L_5, \ldots, L_8$ is a proper subvariety of $\mathbb{G}(1,3)^8$; the union, over all permutations of $\{1, \ldots, 8\}$, of the corresponding loci is likewise a proper closed subset of $\mathbb{G}(1,3)^8$. □

We now need to describe the tangent spaces to the cycles D_L at points in U. This is where one of the virtues of the Hilbert scheme comes into play: we have already seen that the tangent space to $\mathcal{H}$ at the point $[C]$ corresponding to a conic C is the space $H^0(N_{C/\mathbb{P}^3})$ of sections of the normal bundle $N = N_{C/\mathbb{P}^3}$ of C in $\mathbb{P}^3$. In terms of this, it's likewise possible to describe the tangent spaces to D_L as well.

Proposition 11.17. *Let $L \subset \mathbb{P}^3$ be a line, and $D_L \subset \mathcal{H}$ the locus of conics meeting L. If $C \subset \mathbb{P}^3$ is a smooth plane conic such that $C \cap L = \{p\}$ is a single, reduced point, then D_L is smooth at $[C]$ with tangent space*

$$T_{[C]}D_L = \{\sigma \in H^0(N) : \sigma(p) \in \frac{T_pL + T_pC}{T_pC}\}$$

The proof of this proposition rests on a basic observation about first-order deformations; given its usefulness, we'll state it in general:

Lemma 11.18. *Let Z be any scheme and $X \subset Y \subset Z$ a pair of nested closed subschemes of Z. Let $\tilde{Y} \subset \operatorname{Spec} K[\epsilon]/(\epsilon^2) \times Z$ be a first-order deformation of Y in Z corresponding to the section $\sigma \in H^0(N_{Y/Z})$, and $\tilde{X} \subset \operatorname{Spec} K[\epsilon]/(\epsilon^2) \times Z$ likewise a first-order deformation of X in Z corresponding to the section $\tau \in H^0(N_{X/Z})$. In this case, $\tilde{X} \subset \tilde{Y}$ if and only if, under the maps*

$$\begin{array}{ccc} H^0(N_{X/Z}) & \xrightarrow{\alpha} & H^0(N_{Y/Z}|_X) \\ & & \beta \uparrow \\ & & H^0(N_{Y/Z}) \end{array}$$

we have

$$\beta(\sigma) = \alpha(\tau).$$

Proof. To begin with, the statement is local, so that we can assume Z is affine; let A be the coordinate ring of Z and $I_Y \subset I_X \subset A$ the ideals of Y and X. The sections σ and τ of the normal bundles of Y and X then correspond to module homomorphisms $I_Y \to A/I_Y$ and $I_X \to A/I_X$, which we'll also denote by σ and τ respectively.

The schemes $\tilde{Y}$ and $\tilde{X} \subset \operatorname{Spec} K[\epsilon]/(\epsilon^2) \times Z$ are given by the ideals

$$\tilde{I}_Y = \{f + \epsilon f' : f \in I_Y \text{ and } f' \equiv \sigma(f) \bmod I_Y\}$$

and

$$\tilde{I}_X = \{g + \epsilon g' : g \in I_X \text{ and } g' \equiv \tau(g) \bmod I_X\} \subset A[\epsilon](\epsilon^2).$$

Accordingly, we have $\tilde{X} \subset \tilde{Y}$—that is, $\tilde{I}_Y \subset \tilde{I}_X$—if and only if

$$\sigma(f) \equiv \tau(f) \bmod I_X \quad \text{for all } f \in I_Y,$$

which is the statement of the Lemma. □

Given Lemma 11.18, we can prove Proposition 11.17 by introducing (as usual!) an incidence correspondence:

Proof of Proposition 11.17. For $L \subset \mathbb{P}^3$ a line, we let

$$\Phi_L = \{(p, C) : p \in C\} \subset L \times \mathcal{H}.$$

The image of Φ_L under projection π_2 on the second factor is the cycle $D_L \subset \mathcal{H}$ of conics meeting L. Now suppose $[C] \in D_L$ is a point corresponding to a conic $C \subset \mathbb{P}^3$ intersecting L in a single reduced point p, with p a smooth

point of C. By Lemma 11.18, the tangent space to Φ_L at the point (p, C) is

$$T_{(p,C)}\Phi_L = \big\{(\nu, \sigma) \in T_pL \times H^0(N_{C/\mathbb{P}^3}) \mid \sigma(p) \equiv \nu \bmod T_pC\big\}.$$

In particular, Φ_L will be smooth at (p, C) and the projection π_2 will carry its tangent space injectively to the space of sections $\sigma \in H^0(N_{C/\mathbb{P}^3})$ such that $\sigma(p) \in (T_pL + T_pC)/T_pC$. Since the map π_2 is one-to-one over p, it follows that D_L is smooth at $[C]$ with this tangent space. □

Exercise 11.19. We've given the statement and proof of Proposition 11.17 in the specific context of the cycle of conics meeting a line, but the argument applies much more broadly. Suppose that $X \subset \mathbb{P}^n$ is a subscheme of dimension l, and $\mathcal{H}$ a component of the Hilbert scheme parametrizing subschemes of dimension $k < n - l$ in $\mathbb{P}^n$; let $[Y] \in \mathcal{H}$ correspond to a subscheme $Y \subset \mathbb{P}^n$ such that $Y \cap X = \{p\}$ is a single reduced point, and suppose moreover that p is a smooth point of both X and Y. Show that the cycle $\Sigma_X \subset \mathcal{H}$ of subschemes meeting X is smooth at $[Y]$, of the expected codimension $n - k - l$, with tangent space

$$T_{[Y]}\Sigma_X = \big\{\sigma \in H^0(N_{Y/\mathbb{P}^n}) : \sigma(p) \in \frac{T_pX + T_pY}{T_pY}\big\}.$$

Show also that this statement may be false if we replace $\mathbb{P}^n$ by an arbitrary smooth, projective variety of dimension n.

Next, we want to see that, for a given conic C and eight general lines meeting C the corresponding subspaces of $T_{[C]}\mathcal{H} = H^0(N)$ intersect transversely. This will follow from the following general statement:

Lemma 11.20. *Let X be any projective variety, E a vector bundle on X, $p_1, \ldots, p_k \in X$ general points and $V_i \subset E_{p_i}$ a general linear subspace of codimension 1 in the fiber E_{p_i} of E at p_i. If W is the vector space of global sections of E whose value at p_i lies in the subspace V_i for each i, then*

$$\dim W = \max\{0, h^0(E) - k\}.$$

Proof. To see this, we realize $H^0(E)$ as the space of sections of the line bundle $\mathcal{O}_{\mathbb{P}E^*}(1)$ on the projectivization $\mathbb{P}E^*$ of the dual of E. The subspaces $V_i \subset E_{p_i}$ correspond to points $(p_i, \xi_i) \in \mathbb{P}E^*$, and the condition that a section $\sigma \in H^0(E)$ have value $\sigma(p_i) \in V_i$ amounts to saying that the corresponding section $\tilde{\sigma} \in H^0(\mathcal{O}_{\mathbb{P}E^*}(1))$ vanish at (p_i, ξ_i). The Lemma now follows from the fact that general points of a variety impose independent conditions on the sections of a line bundle, up to the dimension of the space of sections. □

Exercise 11.21. Show that the analog of Lemma 11.20 is false if we don't specify that the V_i are codimension 1: in other words, $V_i \subset E_{p_i}$ is a general

linear subspace of codimension m_i, then the corresponding subspace $W \subset H^0(E)$ need not have dimension $\max\{0, h^0(E) - \sum m_i\}$.

Combining Proposition 11.17 and Lemma 11.20, we arrive at the

Proposition 11.22. *If $L_1, \ldots, L_8 \subset \mathbb{P}^3$ are eight general lines, then the cycles $D_{L_i} \subset \mathcal{H}$ intersect transversely.*

Proof. To start, we introduce the incidence correspondence

$$\Sigma = \{(L_1, \ldots, L_8; C) : C \cap L_i \neq \varnothing \ \forall i\} \subset \mathbb{G}(1,3)^8 \times \mathcal{H}.$$

Since the locus of lines $L \subset \mathbb{P}^3$ meeting a given smooth conic C is an irreducible hypersurface in the Grassmannian $\mathbb{G}(1,3)$, we see via projection to $\mathcal{H}$ that Σ is irreducible of dimension 32. We see from this in particular that the locus of conics meeting 8 general lines $L_1, \ldots, L_8$ is indeed finite.

Now, let $\Sigma_0 \subset \Sigma$ be the locus of $(L_1, \ldots, L_8; C)$ such that the cycles D_{L_i} fail to intersect transversely at $[C]$; this is a closed subset of Σ. As we've seen, if we start with C and choose eight general lines meeting C, the cycles D_{L_i} will meet transversely at $[C]$; thus Σ_0 is a proper closed subset of Σ. It follows that $\dim \Sigma_0 < 32$, and hence that Σ_0 cannot dominate $\mathbb{G}(1,3)^8$; in other words, for a general point $(L_1, \ldots, L_8) \in \mathbb{G}(1,3)^8$ the cycles D_{L_i} are transverse at every point of their intersection. □

11.4.3 The Chow Ring

Given the description of $\mathcal{H}$ as the projective bundle $\mathbb{P}\operatorname{Sym}^2 S^*$ over $\mathbb{P}^{3*}$ we can apply Theorem 11.4 to describe the Chow ring of $\mathcal{H}$. From the exact sequence

$$0 \to \operatorname{Sym}_2(S^*) \to \operatorname{Sym}^2(\mathcal{O}_{\mathbb{P}^n}^{n+1}) \to \mathcal{O}_{\mathbb{P}^n}^{n+1} \otimes \mathcal{O}_{\mathbb{P}^n}(1) \to 0$$

and the Whitney Sum formula we see that $c(\operatorname{Sym}_2(S^*)) = 1/(1+\zeta)^{n+1}$.

We can now use Theorem 11.4 to describe the Chow ring $A^*(\mathcal{H})$. As before, we'll let ω denote the hyperplane class on $\mathbb{P}^{3*}$ and its pullback to $\mathcal{H}$; we'll let T be the tautological sub-line bundle on $\mathcal{H} = \mathbb{P}\operatorname{Sym}^2 S^*$, $\mathcal{O}_{\mathbb{P}\operatorname{Sym}^2 S^*}(1)$ the dual of T, and $\zeta \in A^1(\mathcal{H})$ the first Chern class $c_1(\mathcal{O}_{\mathbb{P}\operatorname{Sym}^2 S^*}(1))$. To begin with, we have by Theorem 11.4

$$\begin{aligned} A^*(\mathcal{H}) &= A^*(\mathbb{P}^{3*})[\zeta]/(\zeta^6 + 4\omega\zeta^5 + 10\omega^2\zeta^4 + 20\omega^3\zeta^3) \\ &= \mathbb{Z}[\omega, \zeta]/(\omega^4, \zeta^6 + 4\omega\zeta^5 + 10\omega^2\zeta^4 + 20\omega^3\zeta^3) \end{aligned}$$

In anticipation of the calculation ahead, we'll take a moment out here to use this to calculate the values in $A^8(\mathcal{H}) \cong \mathbb{Z}$ of the monomials of top degree 8 in ω and ζ. To start with, we have $\omega^4 = 0$; and since ω^3 is the

class of a fiber of $\mathcal{H} \to \mathbb{P}^{3*}$ and ζ restricts to the hyperplane class on this fiber, we have

$$\omega^3\zeta^5 = 1.$$

To evaluate the next monomial $\omega^2\zeta^6$, we have to use the monic polynomial satisfied by ζ according to Theorem 11.4: this says that

$$\zeta^6 = -4\omega\zeta^5 - 10\omega^2\zeta^4 - 20\omega^3\zeta^3$$

and so

$$\begin{aligned}\omega^2\zeta^6 &= \omega^2(-4\omega\zeta^5 - 10\omega^2\zeta^4 - 20\omega^3\zeta^3)\\ &= -4.\end{aligned}$$

Next,

$$\begin{aligned}\omega\zeta^7 &= \omega\zeta(-4\omega\zeta^5 - 10\omega^2\zeta^4 - 20\omega^3\zeta^3)\\ &= -4\omega^2\zeta^6 - 10\omega^3\zeta^5\\ &= 16 - 10 = 6,\end{aligned}$$

and finally

$$\begin{aligned}\zeta^8 &= \zeta^2(-4\omega\zeta^5 - 10\omega^2\zeta^4 - 20\omega^3\zeta^3)\\ &= -4\omega\zeta^7 - 10\omega^2\zeta^6 - 20\omega^3\zeta^5\\ &= -24 + 40 - 20 = -4.\end{aligned}$$

11.4.4 The Cycle of Plane Conics Meeting a Line

We next compute the class δ of the divisor $D = D_L$. We'll employ the technique of undetermined coefficients: we know that $\delta = a\omega + b\zeta$ for some pair of integers a and b, and restricting to curves in $\mathcal{H}$ gives us linear relations on a and b. Specifically, we'll use the curve $\Gamma \subset \mathcal{H}$ corresponding to a general pencil $\{C_\lambda \subset H\}$ of conics in a general plane $H \subset \mathbb{P}^3$; and the curve $\Phi \subset \mathcal{H}$ consisting of a general pencil of plane sections $\{H_\lambda \cap Q\}$ of a fixed quadric Q.

Here is a table of intersection numbers between our divisor classes ω, ζ and δ, and the curves Γ and Φ:

	ω	ζ	δ
Γ	0	1	1
Φ	1	0	2

To compute $\zeta \cdot \Phi$ we first show that the tautological line bundle T is trivial on Φ. To see this, recall that a point of $\mathcal{H}$ is a pair (H, ξ), with H a plane in $\mathbb{P}^3$ and ξ a one-dimensional subspace of $H^0(\mathcal{O}_H(2))$; the fiber of T over the point (H, ξ) is the vector space ξ. Now, if $F \in H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ is the

homogeneous quadratic polynomial defining Q, we see that *the restrictions of F to the planes H_λ give an everywhere nonzero section of T over Φ*. Since $\mathcal{O}_{\mathbb{P}\,\mathrm{Sym}^2 S^*}(1)$ is the dual of T, we have

$$\zeta \cdot \Phi = -\deg(T|_\Phi) = 0.$$

The calculation of all of the other numbers in the table is elementary, and we leave it to the reader in Exercise 11.23.

Exercise 11.23. Calculate the remaining five intersection numbers in the table above.

Given the intersection numbers in the table above, we conclude that

$$\delta = 2\omega + \zeta.$$

The rest of the calculation—part (d) of our process, in other words—is purely mechanical. Given 8 general lines $L_1, \ldots, L_8 \subset \mathbb{P}^3$, the conics meeting all 8 represent the intersection of the corresponding divisors $D_{L_i} \subset \mathcal{H}$; since we've already shown that the intersection is contained in the open subset $U \subset \mathcal{H}$ parametrizing smooth conics, and that the intersection is transverse there, the number of conics meeting all 8 will be the product $\delta^8 \in A^8(\mathcal{H}) \cong \mathbb{Z}$. We have

$$\begin{aligned}(2\omega+\zeta)^8 &= \zeta^8 + 2\binom{8}{1}\omega\zeta^7 + 4\binom{8}{2}\omega^2\zeta^6 + 8\binom{8}{3}\omega^3\zeta^8 \\ &= -4 \;+\; 2\cdot 8\cdot 6 \;+\; 4\cdot 28\cdot -4 \;+\; 8\cdot 56 \\ &= 92\end{aligned}$$

In sum, we have proved:

Theorem 11.24. *There are 92 distinct plane conics in $\mathbb{P}^3$ meeting eight general lines, and each of them is smooth.*

In fact, the computation above yields a little more:

(a) For *any* eight lines $L_1, \ldots, L_8 \subset \mathbb{P}^3$, there will be at least one conic meeting all eight (here we have to include degenerate conics as well as smooth); and

(b) For any eight lines $L_1, \ldots, L_8 \subset \mathbb{P}^3$, if we assume that the number of conics meeting all eight (again including degenerate ones) is finite, then assigning to each such conic C a multiplicity equal to the scheme-theoretic degree of the component of the intersection $\cap D_{L_i} \subset \mathcal{H}$ supported at $[C]$, the total number of conics will be 92. In particular, there can't be more than 92 distinct conics meeting all eight lines.

We now look at some other problems involving conics in $\mathbb{P}^3$ and other applications of the techniques we've developed here. We start with some straightforward problems involving calculations in $A^*(\mathcal{H})$.

Exercise 11.25. Let $\Delta \subset \mathcal{H}$ be the locus of singular conics.

(a) Show that Δ is an irreducible divisor in $\mathcal{H}$.

(b) Express the class $\delta \in A^1(\mathcal{H})$ as a linear combination of ω and ζ.

(c) Use this to calculate the number of singular conics meeting each of 7 general lines in $\mathbb{P}^3$; and

(d) Verify your answer to the last part by calculating this number directly.

Exercise 11.26. Let $p \in \mathbb{P}^3$ be a point, and $F_p \subset \mathcal{H}$ the locus of conics containing the point p. Show that F_p is six-dimensional, and find its class in $A^2(\mathcal{H})$

Exercise 11.27. Use the result of the preceding exercise to find the number of conics passing through a point p and meeting each of 6 general lines in $\mathbb{P}^3$, the number of conics passing through two points p, q and meeting each of 4 general lines in $\mathbb{P}^3$, and the number of conics passing through three points p, q, r and meeting each of 2 general lines in $\mathbb{P}^3$. Verify your answers to the last two parts by direct examination.

Exercise 11.28. Find the class in $A^3(\mathcal{H})$ of the locus of double lines (note that this is five-dimensional, not four!)

The next few problems deal with an example of a phenomenon encountered in the preceding chapter: the possibility that the cycles in our parameter space corresponding to the conditions imposed in fact do not meet transversely, or even properly.

Exercise 11.29. Let $H \subset \mathbb{P}^3$ be a plane, and let $E_H \subset \mathcal{H}$ be the closure of the locus of smooth conics $C \subset \mathbb{P}^3$ tangent to H. Show that this is a divisor, and find its fundamental class $\beta \in A^1(\mathcal{H})$.

Exercise 11.30. Find the number of smooth conics in $\mathbb{P}^3$ meeting each of 7 general lines $L_1, \ldots, L_7 \subset \mathbb{P}^3$ and tangent to a general plane $H \subset \mathbb{P}^3$. More generally, find the number of smooth conics in $\mathbb{P}^3$ meeting each of $8-k$ general lines $L_1, \ldots, L_{8-k} \subset \mathbb{P}^3$ and tangent to a k general planes $H_1, \ldots, H_k \subset \mathbb{P}^4$, for $k = 1, 2$ and 3.

Exercise 11.31. Why don't the methods developed here work to calculate the number of smooth conics in $\mathbb{P}^3$ meeting each of $8-k$ general lines $L_1, \ldots, L_{8-k} \subset \mathbb{P}^3$ and tangent to a k general planes $H_1, \ldots, H_k \subset \mathbb{P}^4$, for $k \geq 4$? What can you do to find these numbers? (In fact, we have seen how to deal with this in Chapter 10)

Next, some problems involving conics in $\mathbb{P}^4$:

Exercise 11.32. Now let $\mathcal{K}$ be the space of conics in $\mathbb{P}^4$ (again, defined to be complete intersections of two hyperplanes and a quadric). Use the description of $\mathcal{K}$ as a $\mathbb{P}^5$-bundle over the Grassmannian $\mathbb{G}(2,4)$ to determine its Chow ring.

Exercise 11.33. In terms of your answer to the preceding problem, find the class of the locus D_Λ of conics meeting a 2-plane Λ, and of the locus E_L of conics meeting a line $L \subset \mathbb{P}^4$.

Exercise 11.34. Find the expected number of conics in $\mathbb{P}^4$ meeting each of 11 general 2-planes $\Lambda_1, \ldots, \Lambda_{11} \subset \mathbb{P}^4$.

Exercise 11.35. Prove that your answer to the preceding problem is in fact the actual number of conics by showing that for general 2-planes $\Lambda_1, \ldots, \Lambda_{11} \subset \mathbb{P}^4$ the corresponding cycles D_{Λ_i} intersect transversely.

Finally, here's a challenge problem:

Exercise 11.36. Let $\{S_t \subset \mathbb{P}^3\}_{t \in \mathbb{P}^1}$ be a general pencil of quartic surfaces (that is, take A and B general homogeneous quartic polynomials, and set $S_t = V(t_0 A + t_1 B) \subset \mathbb{P}^3$). How many of the surfaces S_t contain a conic?

We should indicate at least the steps in a solution to this. The key is to introduce a vector bundle E on the space $\mathcal{H}$ of conics in $\mathbb{P}^3$, with fiber

$$E_C = H^0(\mathcal{O}_C(4)),$$

and to find the Chern classes of E. To do this, we express E as a quotient F/G of bundles we know (or at least can get a hold of). First, we let F be the bundle on $\mathcal{H}$ whose fiber at a point of $\mathcal{H}$ corresponding to a conic C contained in the plane $H \subset \mathbb{P}^3$ is

$$F_C = H^0(\mathcal{O}_H(4));$$

we can recognize F as the pullback from $(\mathbb{P}^3)^*$ of the bundle $\operatorname{Sym}^4 S^*$, whose Chern classes we can calculate via the splitting principle. Next, let G be the bundle whose fiber over the same point in $\mathcal{H}$ is the space of quartics in H vanishing on C; that is,

$$G_C = H^0(\mathcal{I}_{C,H}(4)).$$

Now, the quartics in H vanishing on C are the multiples of the defining equation ξ of C in H by homogeneous quadrics in H; thus, if T is the tautological sub-line bundle on $\mathcal{H} = \mathbb{P} \operatorname{Sym}^2 S^* \to (\mathbb{P}^3)^*$, we see that

$$G \cong T \otimes \pi^* \operatorname{Sym}^2 S^*$$

and we can use this to find the Chern classes of G.

12

Segre Classes and Varieties of Linear Spaces

Keynote Questions

(a) At how many points of $\mathbb{P}^n$ do $2n$ general tangent vector fields all annihilate the same cotangent vector?

(b) If f is a general polynomial of degree $d = 2k - 1$ in one variable over a field of characteristic zero, then there is a unique way to write f as a sum of k d^{th} powers of linear forms (Exercise 12.13.) If f and g are general polynomials of degree $d = 2k$ in one variable, how many linear combinations of f and g are expressible as a sum of k d^{th} powers of linear forms?

(c) If $C \subset \mathbb{P}^3$ is a general rational curve of degree d over a field of characteristic zero, What is the degree of the surface swept out by the 3-secant lines to C? If $C \subset \mathbb{P}^4$ is a general rational curve of degree d over a field of characteristic zero, how many 3-secant lines does C have?

(d) Can a ruled surface contain more than one curve of negative self-intersection?

(e) Let $\{p_1, \ldots, p_4\}$, $\{q_1, \ldots, q_4\}$ and $\{r_1, \ldots, r_4\}$ be general points in the plane $\mathbb{P}^2$. For any point $p \in \mathbb{P}^2$ other than these, let C_p be the unique conic through $\{p_1, \ldots, p_4\}$ and p; let D_p be the unique conic through $\{q_1, \ldots, q_4\}$ and p and E_p the unique conic through $\{r_1, \ldots, r_4\}$ and p. For how many points $p \in \mathbb{P}^2$ are the conics C_p, D_p and E_p all tangent to one another at p?

12.1 Segre Classes

Our understanding of the Chow rings of projective bundles makes accessible the computation of another natural locus associated to a vector bundle. Suppose that E is a vector bundle on a scheme X that can be generated by global sections. How many global sections does it take to generate E? More generally, what sort of locus is it where a given number of general global sections fail to generate E locally?

We can get a feeling for these questions as follows. First, consider the case where E is a line bundle. In this case each section corresponds to a divisor of class $c_1(E)$. Given that E is generated by its global sections, the linear series of these divisors is base-point-free, so a general collection of i of them will intersect in a codimension i locus of class $c_1(E)^i$. That is, the locus where i general sections of E fail to generate E has "expected" codimension i and class $c_1(E)^i$.

Now suppose that E has rank $r > 1$. It is clear that fewer than r sections can never generate E locally anywhere. On the other hand, since we have assumed that E is generated by global sections, we can choose r global sections that are minimal generators of the fiber of E at a given closed point $p \in X$, and by Nakayama's Lemma (Eisenbud [1995] Theorem ***) these sections will generate the stalk of E at p, and thus will generated E locally at every point of an open set containing p. Of course for r sections to generate E locally at a point p is the same as for the r sections to be independent at p, so we know that the locus where this fails (for r general sections) is the first Chern class, $c_1(E)$, as before; in particular it has codimension 1.

Now given r general sections of E let X' be the codimension 1 subset of E consisting of points p where the sections do not generate E. One can hope that at a general point of X' the sections have only one dependency relation, so that, on some open set $U \subset X'$, they generate a co-rank 1 sub-bundle of $E' \subset E$, and the quotient E/E' is a line bundle on U. The sections of E yield sections of E/E' line bundle, so if it is a line bundle they will vanish in codimension 1 in U; that is, we should expect $r+1$ general sections of E to generate E away from a codimension 2 subset of X. Continuing in this way, it seems that $r+i-1$ sections of E might generate E away from a codimension i locus, and thus $r+\dim X-1$ sections might generate E locally everywhere.

It turns out that the construction of projective bundles gives us a much more effective way of reducing this question about vector bundles (and many others) to the case of line bundles, passing from E to the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$. To relate this line bundle to classes on X, we push forward its self intersections:

Definition 12.1. Let X be a smooth projective scheme let E be a vector bundle on X, and let $\zeta = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$. The *$i$-th Segre class* of E is the class

$$s_i(E) = \pi_*(\zeta^{r-1+i}) \in A^i(X).$$

and the the (total) *Segre class* of E is the sum

$$s(E) = 1 + s_1(E) + s_2(E) + \dots .$$

(For a definition of the Segre classes in greater generality see Fulton [1984] Chapter ****.)

The Segre classes give the answer to our question about generating vector bundles:

Proposition 12.2. *If E is a vector bundle of rank r that is generated by global sections, and X_i is the locus where a given set of $r+i-1$ general global sections fail to generate E, then X_i is the support of a positive cycle representing the i-th Segre class $s_i(E^*) = (-1)^i s_i(E)$.*

The Proposition shows an interesting parallel between the Chern classes and the Segre classes of a bundle:

- The i^{th} Chern class $c_i(E)$ is the locus of fibers where a suitably general bundle map
$$\mathcal{O}_X^{\oplus r+1-i} \to E$$
fails to be injective.
- The i^{th} Segre class $s_i(E)$ is $(-1)^i$ times the locus of fibers where a suitably general bundle map
$$\mathcal{O}_X^{\oplus r-1+i} \to E$$
fails to be surjective.

The Segre classes give a new way of defining a cycle class invariant of a vector bundle; but in fact they are essentially a different way of packaging the Chern classes. Postponing the proof of Proposition12.2 for a moment, we explain the remarkable relationship, which is a special case of the Porteous formula, as will be explained in Chapter 14.

Proposition 12.3. *The Segre class and the Chern class of a bundle E on X are inverses of one another in the Chow ring of X:*

$$s(E)c(E) = 1 \in A^*(X).$$

In particular,

$$s_i(E^*) = (-1)^i s_i(E).$$

Proof of Proposition12.3. If S and Q are the tautological sub- and quotient bundles on $\mathbb{P}E$ then by the Whitney Sum formula (Theorem7.3)

$$c(Q) = \frac{c(\pi^* E)}{c(S)} = c(\pi^* E)(1 + \zeta + \zeta^2 + \dots) \in A^*(\mathbb{P}E).$$

We may push the classes in this equation forward to X: For $i < r - 1$ the Chern classes $c_i(Q)$ is represented by a cycle of dimension $> \dim X$, so it maps to 0, while the top Chern class $c_{r-1}(Q)$ maps to the fundamental class of X. On the other hand, the push-pull formula tells us that

$$\begin{aligned} \pi_*\big(c(\pi^* E)(1 + \zeta + \zeta^2 + \dots)\big) &= c(E) \cdot \pi_*(1 + \zeta + \zeta^2 + \dots) \\ &= c(E)s(E) \end{aligned}$$

From this we deduce that

$$s(E) = \frac{1}{c(E)} \quad \text{or} \quad c(E)s(E) = 1$$

in $A^*(X)$. The last formula follows formally from this and the formula $c_i(E^*) = (-1)^i c_i(E)$. □

Proof of Proposition12.2. Let $\pi : \mathbb{P}(E^*) \to X$ be the projection morphism. The Theorem on Cohomology and Base Change, Theorem6.6, shows that the map $\pi^*(E) \to \mathcal{O}_{\mathbb{P}(E^*)}(1)$ that is dual to the tautological inclusion $S = \mathcal{O}_{\mathbb{P}(E^*)}(-1) \subset \pi^*(E^*)$ induces an isomorphism $E = \pi_*\pi^*(E) \to \pi_*\mathcal{O}_{\mathbb{P}(E^*)}(1)$, and thus an isomorphism on global sections

$$H^0(E) \to H^0(\mathcal{O}_{\mathbb{P}(E^*)}(1)).$$

The fiber of π over a point $p \in X$ is the projective space $\mathbb{P}(E_p^*)$, and the points $q \in \pi^{-1}(p)$ correspond to the different one-dimensional quotients of E_p. A collection of sections σ_i of E thus generates E_p if and only if the corresponding sections σ_i' of $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ generate that line bundle at every point $q \in \pi^{-1}(p)$. It follows that the locus in X where E is *not* generated by the σ_i is the image of the locus in $\mathbb{P}(E^*)$ where $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is not generated by the σ_i'. This last is precisely the intersection of the divisors

$$D_i = \{q \mid \sigma_i'(q) = 0\} \subset \mathbb{P}(E^*).$$

Since we have assumed that E is generated by global sections, the same will be true for $\pi^*(E)$, and thus of $\mathcal{O}_{\mathbb{P}(E^*)}(1)$. By the Principal Ideal Theorem, every component of the intersection of j divisors corresponding global sections of $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ will have codimension $\leq j$ in $\mathbb{P}(E^*)$. If we choose the sections $\sigma_1, \sigma_2, \dots$ generally, then the j-th section σ_j will be nonzero at a generic point of each component of the intersection $D_1 \cap \dots \cap D_{kj-1}$, and we see inductively that in fact every component of $D_1 \cap \dots \cap D_j$ will have codimension exactly j.

It follows from the Generalized Principal Ideal Theorem Eisenbud [1995] Theorem **** that every component of the locus X_{r-1+i} where $\sigma_1, \dots, \sigma_r -$

$1+i$ fail to generate E has codimension at most i. By what we have said, each of these components is the image of some component of $Y_{r-1+i} := D_1 \cap \cdots \cap D_{r-1+i}$. It follows that every component of X_{r-1+i} has codimension exactly i, and is covered by a generically finite map from some component of Y_{r-1+i}. Thus $[X_{r-1+i}] = \pi_*[Y_{r-1+i}]$ as claimed. □

We remark that, with notation as in the proof above, the locus in X where the fibers of the map $Y_{r-1+i} \to X_{r-1+i}$ have dimension k is precisely the locus in X where the rank of the map $\mathcal{O}_X^{r-1+i} \to E$ induced by the σ_i drops rank by k.

We can now answer Keynote Question **??**. A tangent vector field on $\mathbb{P}^n$ is a section of $T_{\mathbb{P}^n} = \Omega^*_{\mathbb{P}^n}$, so the question can be rephrased as: at how many points of $\mathbb{P}^n$ do $2n$ general sections of $T_{\mathbb{P}^n}$ fail to generate $T_{\mathbb{P}^n}$. By Proposition 12.2 this is -1^n times the degree of the Segre class $s_n(T_{\mathbb{P}^n})$. By Proposition12.3, $s(T_{\mathbb{P}^n}) = 1/c(T_{\mathbb{P}^n})$. From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}^{n+1} \to T_{\mathbb{P}^n} \to 0$$

and the Whitney Sum formula (Theorem**??**) we get $c(T_{\mathbb{P}^n}) = (1+\zeta)^{n+1}$, where ζ is the hyperplane class on $\mathbb{P}^n$. Putting this together,

$$s(T_{\mathbb{P}^n}) = \frac{1}{(1+\zeta)^{n+1}} = 1 - (n+1)\zeta + \binom{n+2}{2}\zeta^2 + \cdots,$$

so the answer is $\binom{2n}{n}$, the n-th Catalan number.

12.2 Universal hyperplane

Exercise 12.4. Consider the "universal hyperplane"

$$\Sigma = \{(p, H) : p \in H\} \subset \mathbb{P}^r \times \mathbb{P}^{r*}.$$

(a) Show that $\Sigma \to \mathbb{P}^r$ is the projectivization of the cotangent bundle of $\mathbb{P}^r$, and also of the dual of the universal quotient bundle Q on $\mathbb{P}^r$, and similarly for the projection $\Sigma \to \mathbb{P}^{r*}$. ****this was done in the old Ch 3.****

(b) For each of these four representations of Σ as a projective bundle, express the corresponding class $\zeta \in A^1(\Sigma)$ as a linear combination of the pullbacks of the hyperplane classes on $\mathbb{P}^r$ and $\mathbb{P}^{r*}$.

(c) Show that all four descriptions of the Chow ring $A^*(\Sigma)$ agree, and moreover they agree with the description of the Chow ring of Σ worked out in Chapter 3.

12.3 Varieties Swept Out by Linear Spaces

We can use Segre classes to calculate the degrees of some interesting varieties "swept out" by linear spaces in the following sense: Let $B \subset \mathbb{G}(k,n)$ be a subvariety of dimension m in the Grassmannian, and let

$$X = \bigcup_{b \in B} \Lambda_b \ \subset \ \mathbb{P}^n$$

be the union of the planes corresponding to the points of B. If a general point $x \in X$ lies on $d < \infty$ of the planes Λ_b, then we will say that X is *swept out d times* by the planes Λ_b.

Proposition 12.5. *Let S be the tautological subbundle over $\mathbb{G}(k,n)$. Let $B \subset \mathbb{G}(k,n)$ be a smooth subvariety, and let $E = S_B$ be the restriction of S to B. If*

$$X = \bigcup_{b \in B} \Lambda_b \ \subset \ \mathbb{P}^n$$

is swept out d times by the planes corresponding to points of B then

$$\deg(X) = s_k(E)/d.$$

Proof. We can realize X as the image of a projective bundle under a finite map that is generically d to 1. The projectivization of S is the universal k-plane

$$\mathbb{P}S = \{(\Lambda, p) : p \in \Lambda\} \subset \mathbb{G}(k,n) \times \mathbb{P}^n.$$

Let π and η be the projection maps to $\mathbb{G}(k,n)$ and $\mathbb{P}^n$ respectively, so that

$$X = \eta\big(\pi^{-1}(B)\big).$$

By hypothesis η is generically d to 1. Moreover, if $L \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ is a homogeneous linear form on $\mathbb{P}^n$, then L defines a section of E^* by restriction to each fiber of $E = S_B$, and hence a section σ_L of $\mathcal{O}_{\mathbb{P}E}(1)$. The preimage $\eta_B^{-1}(H)$ of the hyperplane $H = V(L) \subset \mathbb{P}^n$ given by L is the zero locus of σ_L. Thus the pullback of the hyperplane class on $\mathbb{P}^n$ under the map η_B is the tautological class $\zeta = c_1(\mathcal{O}_{\mathbb{P}E}(1))$ on $\mathbb{P}E$, and it follows that $d \cdot \deg(X) = \zeta^{m+k} = s_k(E)$ as required. □

12.3.1 Lines on hypersurfaces revisited

********this has been copied over from Chapter 8; it needs to be better integrated.********

Finally, we want to mention one other approach to the problem of finding the class of the variety of lines on a hypersurface; we'll describe this here just in the case of lines on a cubic surface, but the generalization to

hypersurfaces of any degree and dimension should be clear. To set it up, we introduce another bundle E' on the Grassmannian $\mathbb{G}(1,3)$, whose fiber over a point $L \in \mathbb{G}(1,3)$ is the subspace $H^0(\mathcal{I}_L(3)) \subset H^0(\mathcal{O}_{\mathbb{P}^3}(3))$—in other words, the kernel of the map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \longrightarrow E$$

from the trivial bundle on $\mathbb{G}(1,3)$ with fiber $H^0(\mathcal{O}_{\mathbb{P}^3}(3))$ to the bundle $E = \mathrm{Sym}_3(S^*)$ with fiber $H^0(\mathcal{O}_L(3))$. The incidence correspondence $\Phi \subset \mathbb{P}^{19} \times \mathbb{G}(1,3)$ of all pairs (X, L) with $L \subset X$ may then be realized as the projectivization of E'; and once we've studied the Chow rings of projective bundles we'll be able to use this realization of Φ to describe its enumerative geometry.

Exercise 12.6. Here is another approach to the problem of finding the class of the variety of lines on a hypersurface. We'll describe it in the case of lines on a cubic surface, but the generalization to hypersurfaces of any degree and dimension is easy.

Let E' on the Grassmannian $\mathbb{G}(1,3)$ be the kernel of the map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \longrightarrow E$$

from the trivial bundle on $\mathbb{G}(1,3)$ with fiber $H^0(\mathcal{O}_{\mathbb{P}^3}(3))$ to the bundle $E = \mathrm{Sym}^3(S^*)$ with fiber $H^0(\mathcal{O}_L(3))$. The fiber of E over a point $L \in \mathbb{G}(1,3)$ is the subspace $H^0(\mathcal{I}_L(3)) \subset H^0(\mathcal{O}_{\mathbb{P}^3}(3))$. Show that the incidence correspondence

$$\Phi = \{(S, L) : L \subset S\} \subset \mathbb{P}^{19} \times \mathbb{G}(1,3),$$

where $\mathbb{P}^{19}$ is the space of all cubic surfaces in $\mathbb{P}^3$, may be realized as the projectivization of E'.

Find the Chern classes of E', and use this to calculate the number of lines on a cubic surface.

****Move the following to Chapter 9****

Exercise 12.7. Show that under the restriction of the projection map $\alpha : \mathbb{P}^{19} \times \mathbb{G}(1,3) \to \mathbb{P}^{19}$ to Φ, the pullback of $\mathcal{O}_{P^{19}}(1)$ is the tautological bundle $\mathcal{O}_{\mathbb{P}E'}(1)$, and hence that the degree of the map $\alpha|_\Phi : \Phi \to \mathbb{P}^{19}$ is the Segre class $s_{19}(E')$. Calculate this, and show once more that the expected number of lines on a cubic surface is 27.

12.4 Secants of Rational Curves

12.4.1 Points on Secants and Sums of Powers

The reader is invited in Exercise12.13 to complete the proof that, if f is a general polynomial of degree $d = 2k - 1$ in one variable over a field of characteristic zero, then f can be written as a sum of k d^{th} powers of linear forms in a unique way. The next step is Keynote Question **??** : if f and g are general polynomials of degree $d = 2k$ in one variable over a field of characteristic zero, how many linear combinations of f and g are expressible as a sum of k d^{th} powers of linear forms?

These turn out to be questions about the secant loci of rational normal curves. We pause to discuss the definitions.

Definition 12.8. Let X be a subvariety of $\mathbb{P}^d$. The m^{th} *secant variety* of X, denoted $S_m(X)$ is the closure in $\mathbb{P}^d$ of the union of the planes spanned by m-tuples of distinct points of the variety.

The *rational normal curve of degree* d is the image in $\mathbb{P}^d$ of the d-th Veronese map $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ taking a point $(s,t) \in \mathbb{P}^1$ to the point $(s^d, s^{d-1}t, \ldots, t^d) \in \mathbb{P}^d$. Here is an elementary but very useful remark about the rational normal curve:

Lemma 12.9. *Let $C \subset \mathbb{P}^d$ be the rational normal curve. If $D \subset C$ is a divisor of degree $m \leq d$, then D is not contained in any linear subspace of $\mathbb{P}^d$ of dimension $\leq m - 1$.*

Informally: any finite subscheme of C of length $\leq d$ is linearly independent.

Proof. If D is a subset of a linear subspace L of dimension n, then adding $d-n-1$ general points to d we would arrive at a divisor $D' \subset C$ of degree $m+d-n-1$ contained in a hyperplane. Thus $m+d-n-1 \leq \deg C = d$, so $n+1 \leq m$. □

Even more naturally, we may think of $\mathbb{P}^d$ as the projective space

$$\mathbb{P}^d = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(d))$$

of homogeneous forms of degree d. (The same goes for nonzero inhomogeneous polynomials of degree $\leq d$; and we can do similar things for forms and polynomials in more variables.) The association of a homogeneous form in two variables to its roots leads to a useful identification.

Proposition 12.10. *Let G be the symmetric group on k letters. The map $(\mathbb{P}^1)^k \to \mathbb{P}^k$ given by the $k+1$ homogenized symmetric polynomials*

$$\sigma_i := \sum_{\substack{P\subset[k]\\|P|=i}} \prod_{p\in P} s_p \prod_{q\in[k]\setminus P} t_q$$

that are the coefficients of $\prod_p(s_pX_0 + t_pX_1)$ induces an isomorphism from the symmetric power $(\mathbb{P}^1)^{(k)} := (\mathbb{P}^1)^k/G$ to $\mathbb{P}^k$ where (s_p, t_p) are homogeneous coordinates on the p-th copy of $\mathbb{P}^1$, chosen to correspond under the action of G.

Proof. The given forms define a morphism on $(\mathbb{P}^1)^k$ because the polynomial ring $K[X_0, X_1]$ is a domain; the only way for the product $\prod_i(s_iX_0 + t_iX_1)$ to have coefficients that are all zero is for one of the factors to be zero, that is $s_i = t_i = 0$ for some i, and this does not represent a point of $\mathbb{P}^1$. Since the forms are invariant under the action of G, there is an induced morphism $\overline{\sigma} : (\mathbb{P}^1)^{(k)} \to \mathbb{P}^k$. Since the coefficients of a polynomial determine the roots up to permutation, $\overline{\sigma}$ is set-theoretically one-to-one. It is also birational, as one can see by examining the invariant open affine subset $\mathbb{A}^k \subset (\mathbb{P}^1)^k$ given by $s_0 = \cdots = s_k = 1$. On this affine set the σ_i are the elementary symmetric functions in the t_i, and thus generate the ring of invariants.

Since $\overline{\sigma}$ is projective it is finite as well as birational. Since it is set-theoretically one to one and the target $\mathbb{P}^k$ is normal, $\overline{\sigma}$ is an isomorphism. □

We also consider the map $\nu'_d : \mathbb{P}^1 \to \mathbb{P}^d$ taking a point (s, t) to the point whose coordinates are the coefficients of the polynomial $(sX - tY)^d$, that is,

$$\nu'_d : (s, t) \mapsto \left(s^d, ds^{d-1}t, \binom{d}{2}s^{d-2}t, \ldots, t^d\right) \in \mathbb{P}^d.$$

Thinking of $\mathbb{P}^d$ as the space of forms of degree d, the image of ν'_d as the set of forms that are d-th powers of linear forms; in terms of the identification with $(\mathbb{P}^1)^{(k)}$, the image of ν'_d is the image of the diagonal $\{(p, \ldots, p)\} \subset (\mathbb{P}^1)^k$ in $(\mathbb{P}^1)^{(k)}$.

A bit of caution is necessary if the ground field has positive characteristic: for example, if the characteristic p of the ground field does not divide any $\binom{d}{i}$ (for example, if $p > d$) then ν_d and ν'_d differ by an invertible change of variable in $\mathbb{P}^d$, and we may identify them and think of the rational normal curve as the set of d-th powers of linear forms in two variables. But when $d = p$, the map ν'_d is purely inseparable, and its image is a line, whereas ν_d is always separable (and Galois), and its image is never even contained in a hyperplane. Similar remarks apply to the Veronese maps of $\mathbb{P}^n$ for any n.

To simplify the exposition in this section, we will assume, when it makes a difference, that the characteristic of the ground field is zero.

A point $p \in \mathbb{P}^d$ lies on the plane spanned by points $q_1, \ldots, q_k$ if and only if any (or all) homogeneous coordinates for p can be expressed as a linear combination of any (or all) homogeneous coordinates of $q_1, \ldots, q_k$. Thus a form of degree d is a linear combination of k d-th powers of linear forms if and only if the point of $\mathbb{P}^d$ corresponding to that form lies in the union of the k-secant $k-1$ planes to $\nu'_d(\mathbb{P}^1)$. Thus in case the characteristic of our field is zero (or finite but $> d$) Keynote Question **??** becomes a question about the secants of the rational normal curve.

It is *not* the case that every point of $S_m(C)$ corresponds to a polynomial that is so expressible: for example, the tangent line to the rational normal curve $C \subset \mathbb{P}^d$ at the point corresponding to the polynomial $f(t) = (t-\lambda)^d$ is the set of linear combinations of f and $\partial f/\partial t$, or equivalently the set of polynomials that have $d-1$ roots equal to λ. But if $g(t) = a_1(t-\lambda_1)^d + a_2(t-\lambda_2)^d$ with $\lambda_1 \neq \lambda_2$ and g has $d-1$ roots equal to λ then g is itself a perfect d-th power (and one of λ_1 and λ_2 is zero.

We can easily estimate the dimension of the secant variety $S_m(X)$: for a general set of m points $p_1, \ldots, p_m \in X$, the span $\overline{p_1, \ldots, p_m} \subset \mathbb{P}^d$ is an $(m-1)$-plane contained in $S_m(X)$. The family X^m of such m-tuples has dimension mk, so the union of all these planes has dimension at most $\min(mk+m-1)$. Of course the dimension might be smaller; for example, it cannot exceed the ambient dimension, d. We will say that the *expected dimension* of $S_m(X)$ is $\min(mk+m-1, d)$. The actual dimension of $S_m(X)$ may still be less than the expected dimension. For example, if X is a surface in $\mathbb{P}^d$ for $d \geq 5$, the expected dimension of the secant variety $S_2(X)$ is 5-dimensional; but the secant variety of the quadratic Veronese variety is only 4-dimensional (Exercise 12.14.) Nevertheless, when X a curve and the characteristic is zero then the secant variety always has the expected dimension (Exercise 12.13.)

Exercise 12.11. (a) Let $d = 2k-1 > 0$, and suppose that $C \subset \mathbb{P}^d$ is the rational normal curve. Show that $S_k(C) = \mathbb{P}^d$.

(b) Suppose f is a general polynomial of degree $d = 2k-1$ in one variable over a field of characteristic zero—that is, a general point in $\mathbb{P}^d$. The number of ways of expressing f as a sum of k d^{th} powers of linear forms is the number of k-secant planes to the rational normal curve $C \subset \mathbb{P}^d$ containing the point f. Use Part Lemma**??** to deduce that this number is 1.

Exercise 12.12. Let $X \subset \mathbb{P}^d$ be a variety of dimension k, and $U \subset X^m$ the open subset of linearly independent m-tuples of points of smooth points of X; let Φ be the incidence correspondence

$$\Phi = \{(p_1, \ldots, p_m; q) : q \in \overline{p_1, \ldots, p_m}\} \subset U \times \mathbb{P}^d;$$

let $\pi : \Phi \to \mathbb{P}^d$ be projection on the second factor. Show that Φ is smooth of dimension $mk + m - 1$, and that at a point $(p_1, \ldots, p_m; q) \in \Phi$ such that q is not in the span of any proper subset of the p_i, the image of the differential $d\pi$ is the span $\overline{\mathbb{T}_{p_1} X, \ldots, \mathbb{T}_{p_m} X}$ of the tangent planes to X at the points $p_1, \ldots, p_m$.

Exercise 12.13. Let $C \subset \mathbb{P}^d$ be any irreducible, nondegenerate curve over a field of characteristic zero. Show that for any m with $2m - 1 \leq d$, the tangent lines to C at m general points $p_1, \ldots, p_m \in C$ are linearly independent; that is,

$$\dim \overline{\mathbb{T}_{p_1} X, \ldots, \mathbb{T}_{p_m} X} = 2m - 1.$$

Deduce that, whenever $2m-1 \leq d$, the dimension of the m^{th} secant variety to C is exactly $2m - 1$.

Exercise 12.14. Let $X \subset \mathbb{P}^5$ be the quadratic Veronese surface.

(a) Show that every point on the secant variety $S_2(X)$ in fact lies on a 1-dimensional family of secant lines, and deduce that $\dim S_2(X) < 5$.

(b) Show that every pair of tangent lines to X intersect, and deduce again that $\dim S_2(X) < 5$.

(c) Show that X may be realized as the locus of rank 1 matrices in the space $\mathbb{P}^5$ of all 3×3 symmetric matrices, and that in these terms the secant variety $S_2(X)$ is the locus of matrices of rank 2 or less; conclude once again that $\dim S_2(X) < 5$.

In fact, the Veronese is the *only* surface X, irreducible and nondegenerate in a projective space of dimension 5 or more, whose secant variety $S_2(X)$ has dimension less than 5. In general, though, the question of when m^{th} secant varieties have the expected dimension is a tricky one.A difficult theorem of Alexander and Hirschowitz [1995] shows that with the exception of the quadratic Veronese embeddings of any projective space and four other exceptional cases, the secant varieties of $\nu_d(\mathbb{P}^r) \subset \mathbb{P}^{\binom{d+r}{r}-1}$ all have the expected dimension.

Exercise 12.15. (a) Let $X = \nu_2(\mathbb{P}^r) \subset \mathbb{P}^{\binom{r+2}{2}-1}$ be the quadratic Veronese r-fold. Show that the secant variety $S_2(X)$ has dimension $2r$.

(b) Let $X = \nu_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$ be the quartic Veronese surface. Show that the secant variety $S_5(X)$ is a proper subvariety of $\mathbb{P}^{14}$.

(c) Let $X = \nu_4(\mathbb{P}^3) \subset \mathbb{P}^{34}$ be the quartic Veronese threefold. Show that the secant variety $S_9(X)$ is a proper subvariety of $\mathbb{P}^{34}$.

(d) Let $X = \nu_4(\mathbb{P}^4) \subset \mathbb{P}^{69}$ be the quartic Veronese fourfold. Show that the secant variety $S_{14}(X)$ is a proper subvariety of $\mathbb{P}^{69}$.

(e) Let $X = \nu_3(\mathbb{P}^4) \subset \mathbb{P}^{34}$ be the cubic Veronese fourfold. Show that the secant variety $S_7(X)$ is a proper subvariety of $\mathbb{P}^{34}$. (Warning: this one is substantially trickier than the others!)

12.4.2 Degrees of secant varieties to rational curves

Having established the basic facts above, we can rephrase and generalize our problem to this: *If $C \subset \mathbb{P}^d$ is a rational normal curve, what is the degree of the k^{th} secant variety $S_k(C)$?*

By Lemma12.9, for any divisor $D \subset \mathbb{P}^1$ of degree $k \leq d$, the span $\overline{D} \subset \mathbb{P}^d$ of the divisor is a $\mathbb{P}^{k-1}$, and the secant variety S is the union of these $(k-1)$-planes. Thus we may apply Proposition12.5.

As above, we regard $C \subset \mathbb{P}^d$ as the image of the natural map $\varphi : \mathbb{P}^1 \to \mathbb{P}H^0(\mathcal{O}_C(d))^*$, which sends a point $p \in C$ to the annihilator of the hyperplane $\Gamma_p \subset H^0(\mathcal{O}_C(d))$ of polynomials vanishing at p (in characteristic zero we may identify $H^0(\mathcal{O}_C(d))^*$ with $H^0(\mathcal{O}_C(d))$, and this annihilator becomes identified with the one-dimensional subspace of polynomials vanishing only at p.) In these terms, the span $\overline{D}$ of a divisor $D \subset C$ is the projectivization of the annihilator of the subspace $\Gamma_D \subset H^0(\mathcal{O}_C(d))$ of polynomials vanishing at D.

Let $\mathbb{P}^k = (\mathbb{P}^1)^{(k)}$ be the k-th symmetric power of $\mathbb{P}^1$, which we will identify with the space of effective divisors of degree k on $C \cong \mathbb{P}^1$. We want to define a vector bundle E on$\mathbb{P}^k$ such that the fiber at a divisor $D \subset \mathbb{P}^1$ is

$$E_D = H^0(\mathcal{O}_C(d))/H^0(\mathcal{I}_D(d)) = H^0(\mathcal{O}_D(d))$$

and such that any form of degree d on $\mathbb{P}^1$ gives rise to a section whose value at D is the image of the form in $H^0(\mathcal{O}_D(d))$. We can do this by forming an incidence correspondence and using Theorem6.4. Let $\mathcal{D} \subset \mathbb{P}^k \times \mathbb{P}^1$ be the universal divisor, the image of the "partial diagonal"

$$\{(p_1, \ldots, p_k; q) \in (\mathbb{P}^1)^k \times \mathbb{P}^1 \mid q = p_1\}.$$

Writing $\pi_1 : \mathcal{D} \to \mathbb{P}^k$ and $\pi_2 : \mathcal{D} \to \mathbb{P}^1$, we see that the sheaf $\mathcal{O}_\mathcal{D} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(d)$, restricted to a fiber $\pi_1^{-1}(D)$ is the space E_D. By Theorem**??** the sheaf

$$E := \pi_*\big(\mathcal{O}_\mathcal{D} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(d)\big)$$

is a vector bundle. If $F \in H^0(\mathcal{O}_C(d))$ is a homogeneous polynomial of degree d, we get a section σ_F of E by setting $\sigma_F(D)$ to be the image of $\pi_2^*(F)$.

We may also describe the fiber E_D^* of the dual bundle E^* at D to be the subspace, in the space of linear forms on the vector space $H^0(\mathcal{O}_C(d))$

of polynomials of degree d, of those linear forms vanishing on the subspace $H^0(\mathcal{I}_D(d)) \subset H^0(\mathcal{O}_C(d))$. The projectivization of this subspace is the linear space $\overline{D} \cong \mathbb{P}^{k-1}$ spanned by $D \subset C \subset \mathbb{P}^d = \mathbb{P}(H^0(\mathcal{O}_C(d))^*$. Our description of E above shows that E^* is naturally a sub-bundle of the trivial bundle

$$E^* \subset \left(\pi_*\big(\mathcal{O}_{\mathbb{P}^k} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(d)\big)\right)^*$$

with constant fiber $H^0(\mathcal{O}_{\mathbb{P}^1}(d))^*$. Under the corresponding inclusion $\mathbb{P}^k \to \mathbb{G}(k-1,d)$ the bundle E is the restriction of the tautological subbundle.

If $2k-1 \leq d$, then a point of $\mathbb{P}^d$ lies on at most one k-secant $k-1$ plane to the rational normal curve, since if it lay on more than one then the span of the union of two of them would be a linear space of dimension $m \leq 2(k-1) < d$ containing $2k > m+1$ points of the curve, contradicting Lemma12.9. Thus the projection map $\mathbb{P}E^* \to \mathbb{P}H^0(\mathcal{O}_C(d))^*$ is birational onto its image. It follows from Proposition12.5 that the degree of the k^{th} secant variety $S_k(C)$ to $C \subset \mathbb{P}H^0(\mathcal{O}_C(d))^*$ is the top Segre class $s_k(E^*)$. Thus if we know the Chern classes of the bundle E^* or E, we can work out the degree of $S_k(C)$.

12.4.3 Chern classes of E

As defined, E is a quotient of the trivial bundle on $\mathbb{P}^k$ with fiber $H^0(\mathcal{O}_C(d))$, and constant sections of the trivial bundle give rise to global sections of E. We will describe the Chern classes of E as degeneracy loci of appropriate collections of global sections: if $F_1, \ldots, F_{k-i+1} \in H^0(\mathcal{O}_C(d))$ are general polynomials of degree d then the degeneracy locus of the corresponding sections $\sigma_{F_1}, \ldots, \sigma_{F_{k-i+1}}$ of E will be the vanishing locus of $\sigma := \sigma_{F_1} \wedge \cdots \wedge \sigma_{F_{k-i+1}}$ as a section of $\wedge^{k-i+1}E$. This scheme is supported on the set of divisors D of degree k on C such that some nonzero linear combination of $F_1, \ldots, F_{k-i+1}$ vanishes on D. If this has codimension k and the zero scheme of s is reduced then the associated cycle represents $c_k(E)$.

Now, observe that $c_i(E) \in A^i(\mathbb{P}^k)$ is necessarily a multiple $\alpha\zeta^i$ of the i^{th} power of the hyperplane class $\zeta \in A^1(\mathbb{P}^k)$, and we can determine the coefficient α by restricting to an i-plane $\mathbb{P}^i \subset \mathbb{P}^k$. Here, rather than choosing a general i-plane, we'll take $p_1, \ldots, p_{k-i} \in C$ general points, and use the i-plane

$$\mathbb{P}^i = \{D \in \mathbb{P}^k : p_1, \ldots, p_{k-i} \in D\}.$$

The intersection

$$\mathbb{P}^i \cap V(\sigma_{F_1} \wedge \cdots \wedge \sigma_{F_{k-i+1}})$$

of this plane $\mathbb{P}^i$ with the degeneracy locus of $\sigma_{F_1}, \ldots, \sigma_{F_{k-i+1}}$ is the locus of divisors D such that

(a) some nonzero linear combination $\sum a_j F_j$ of $F_1, \ldots, F_{k-i+1}$ vanishes on D; and

(b) $p_1, \ldots, p_{k-i} \in D$.

Now, these two conditions imply that the sum $\sum a_j F_j$ vanishes at the points $p_1, \ldots, p_{k-i}$. But the points p_α being general, there is up to scalars only one linear combination $F = \sum a_j F_j$ that vanishes on them all. We can write the divisor of this polynomial as

$$(F) = p_1 + \cdots + p_{k-i} + q_1 + \cdots + q_{d-k+i};$$

and it follows that D must be of the form

$$D = p_1 + \cdots + p_{k-i} + q_{\alpha_1} + \cdots + q_{\alpha_i},$$

where $\{\alpha_1, \ldots, \alpha_i\} \subset \{1, \ldots, d-k+i\}$ is a subset of cardinality i. Conversely, and divisor D of this form is indeed a point of intersection of our i-plane $\mathbb{P}^i$ with the degeneracy locus of $\sigma_{F_1}, \ldots, \sigma_{F_{k-i+1}}$; and modulo checking transversality, we conclude that

$$c_i(E) = \binom{d-k+i}{i} \zeta^i \in A^i(\mathbb{P}^k).$$

****Make the following two exercises into text****

Exercise 12.16. Show that if $F = \sum a_j F_j$ is defined as above as the unique linear combination of $F_1, \ldots, F_{k-i+1}$ vanishing at $p_1, \ldots, p_{k-i}$, then the remaining zeroes of F are all simple, as asserted above.

Exercise 12.17. Verify that the sections $\sigma_{F_1}, \ldots, \sigma_{F_{k-i+1}}$ are transverse, that is, the degeneracy locus $V(\sigma_{F_1} \wedge \cdots \wedge \sigma_{F_{k-i+1}})$ is the locus of divisors D described above with multiplicity 1; and that this locus intersects the i-plane $\mathbb{P}^i \subset \mathbb{P}^k$ transversely.

It remains to derive the Segre classes of E^*. By the calculation above, we have:

$$c(E^*) = \sum (-1)^i \binom{d-k+i}{i} \zeta^i = \frac{1}{(1+\zeta)^{d-k+1}} \ \in \ A^*(\mathbb{P}^k)$$

so $s(E^*) = (1+\zeta)^{d-k+1}$. In sum we have proven:

Proposition 12.18. *If $C \subset \mathbb{P}^d$ is the rational normal curve of degree d, then the degree of the variety of k-secant $k-1$-planes is*

$$\deg S_k(C) = \binom{d-k+1}{k}.$$

□

It is as usual pleasant to check that this agrees with what we know in the easiest cases. When $k = 1$ we get d, the degree of $C = S_0(C)$ itself. For the case $k = 2$ note that the points of intersection of the secant variety $S_2(C)$ with a general $(d-3)$-plane Λ correspond to nodes of the projection map $\pi_\Lambda : C \to \mathbb{P}^2$; by the genus formula for plane curves, the number of these is $\binom{d-1}{2}$, so this is the degree of $S_2(C)$ as predicted. If $d = 2k - 1$, the formula gives us 1, which is also the result of Exercise 12.13.

The Proposition also answers Keynote Question **??**: given two general polynomials f and g of degree $d = 2k$, the linear combinations of the two that are expressible as sums of k d^{th} powers of linear forms correspond to points of intersection of the line $\overline{f, g} \subset \mathbb{P}^d$ with the k^{th} secant variety to the rational normal curve $C \subset \mathbb{P}^d$; by our calculation, the number of these is

$$\binom{k+1}{k} = k + 1.$$

********Add material computing the degree of the union of the trisecant lines in $\mathbb{P}^3$ for the general rational curve of degree d (currently part of Keynote Question **??** from $s_{(2,1)} \cap S_3(C)$ where C is the rational normal curve of degree d. (but only in char 0, where we can use Kleiman transversality.) We'd do this by expressing $s_{(2,1)}$ as a poly in the Chern classes we know. This and the following exe could then be another Keynote problem.********

********Do the following statement in the text – it's part of Keynote Question **??** or remove it from the Keynote Questions!********

Exercise 12.19. First, show by a naive dimension count that we'd expect a curve $C \subset \mathbb{P}^4$ to have a finite number of trisecant lines. Then, using the calculation of this section, show that this is correct, and find the number, in the case that C is a general rational curve of degree d over a field of characteristic zero.

It is instructive to ask whether we could extend these computations to curves other than rational normal curves. For nonsingular rational curves this is easy: any rational curve $C \subset \mathbb{P}^r$ of degree d is the linear projection $\pi(\tilde{C})$ of a rational normal curve $\tilde{C} \subset \mathbb{P}^d$, and its secant varieties $S_k(C)$ are correspondingly the projections of the corresponding $S_k(\tilde{C})$. If the center of projection doesn't meet $S_k(\tilde{C})$, and if the projection map $\pi : S_k(\tilde{C}) \to S_k(C)$ is birational, the formula above for the degree of $S_k(\tilde{C})$ applies to C as well.

A more difficult problem arises with a curve C of positive genus. We used the fact that the space of effective divisors of degree k on $\mathbb{P}^1$ is the variety $\mathbb{P}^k$, whose Chow ring we know. But except when C is rational or elliptic (see Exercise 12.20, The Chow rings $A^*(\text{Sym}_k(C))$ of symmetric powers of

C are unknown. But there is a way around this: we can describe a subring of the algebraic cohomology ring (see the discussion in Chapter 2) in which a calculation analogous to the one above can be made. This is explained in **?**] Chapter ****.

Exercise 12.20. If E is a smooth elliptic curve (over an algebraically closed field this means a curve of genus 1 with a chosen point), the addition law on E expresses the k^{th} symmetric power E_k as a $\mathbb{P}^{k-1}$ bundle over E. Verify this, and use it to give a description of $A^*(E_k)$.

Exercise 12.21. Using the preceding exercise, find the degrees of the secant varieties of an elliptic normal curve $E \subset \mathbb{P}^d$.

12.5 Subbundles of Projective Bundles

If $F \subset E$ are bundles on X then $\mathbb{P}(E)$ is naturally a subscheme of $\mathbb{P}(F)$ whose class is easy to compute. This computation is a crucial element in understanding the Chow ring of a blow-up (Section**??**), and will allow us to answer Keynote Questiond.

12.5.1 The Class of a Subbundle

Suppose that E is a vector bundle on a smooth projective variety X. Let $\pi : \mathbb{P}E \to X$ be the projection, and let $S \subset \pi^*E$ be the universal subbundle, and suppose that $F \subset E$ is a subbundle. Since the points of $\mathbb{P}E$ over a point $x \in X$ correspond to the rank one sub-bundles of E_x, the points of $p \in \mathbb{P}F \subset \mathbb{P}E$ are the ones where $S_p \subset (\pi^*F)_p$. In other words, $p \in \mathbb{P}F$ if and only if the map

$$\varphi : S \hookrightarrow \pi^*E \to \pi^*(E/F).$$

vanishes at p. We can view φ as a global section of the bundle

$$\operatorname{Hom}(S, \pi^*(E/F)) \cong S^* \otimes \pi^*(E/F) = \mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^*(E/F).$$

If we write everything in local coordinates then we see that in fact $\mathbb{P}F$ is scheme-theoretically defined by the vanishing of φ. (Exactly the same computation works only assuming that X is a scheme, but we will not use this.) Putting this together with the formula for the Chern class of the tensor product of a bundle with a line bundle of Proposition 7.6 we obtain:

Proposition 12.22. *If X is a smooth projective variety and $F \subset E$ are vector bundles on X of ranks s and r respectively, then*

$$[\mathbb{P}F] = c_{r-s}\big(S^* \otimes \pi^*(E/F)\big) = \zeta^{r-s} + \gamma_1\zeta^{r-s-1} + \cdots + \gamma_{r-s} \in A^{r-s}(\mathbb{P}E)$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}E}(1))$ *and* $\gamma_k = c_k(E/F)$, *which is the* k*-th coefficient of the power series* $c(E)/c(F)$. □

12.5.2 Ruled surfaces

A *ruled surface* $S = \mathbb{P}E \to X$ is the projectivization of a vector bundle E of rank 2 over a smooth curve X. As a first application we can answer Keynote Question d:

Proposition 12.23. *A ruled surface can contain at most one curve of negative self-intersection.*

Proof. Suppose that $S = \mathbb{P}E \to X$. A subbundle $\mathbb{P}F \subset \mathbb{P}E$, with $\operatorname{rank} F = 1$, gives rise to a section Σ of the ruled surface $S = \mathbb{P}E \to X$. Moreover, any section Σ may be written in this form: we can recover the dual of the line bundle F as $\pi_*(\mathcal{O}_{\mathbb{P}E}(1)|_\Sigma)$, and the restriction of the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$ to Σ pushes forward to a surjection

$$E^* = \pi_*\mathcal{O}_{\mathbb{P}E}(1) \to \pi_*(\mathcal{O}_{\mathbb{P}E}(1)|_\Sigma) =: F^*.$$

We will begin by calculating the class $\sigma := \Sigma \in A^1(S)$. By Theorem 11.4, we have

$$A^*(S) = A^*(X)[\zeta]/(\zeta^2 + e\zeta),$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}E}(1))$ and $e = c_1(E)$, the degree of E; that is, the self-intersection of the class ζ is $-e$. By Proposition 12.22,

$$\sigma = \zeta + c_1(E) - c_1(F),$$

so

$$\sigma^2 = \zeta^2 + 2e - 2f = e - 2f.$$

In particular, the self-intersection of a section of a ruled surface $S = \mathbb{P}E \to X$ is negative if and only if the degree of the corresponding sub-line bundle of E is strictly greater than one-half the degree of E.

It is easy to see that no vector bundle E of rank 2 on a curve can admit two distinct sub-line bundles whose degrees add up to more than the degree of E: if L and M are any two distinct sub-line bundles of E, the sheaf map

$$L \oplus M \to E$$

is injective, with cokernel $\mathcal{G}$ supported on a finite set of points, and thus

$$c_1(E) = c_1(L \oplus M) + c_1(\mathcal{G}) \geq c_1(L \oplus M) = c_1(L) + c_1(M).$$

In particular, a ruled surface $S \to X$ can have at most one section of negative self-intersection. (Exercise 12.24 strengthens this conclusion slightly.)

Exercise 12.24. Show more generally that if E is a vector bundle of rank 2 and degree e on a smooth projective curve X, and L and M sub-line bundles of degrees a and b corresponding to sections of $\mathbb{P}E$ with classes σ and τ, then

$$\sigma\tau = e - a - b$$

and

$$\sigma^2 + \tau^2 = 2e - 2a - 2b.$$

In particular, if L and M are distinct then $\deg \sigma^2 + \deg \tau^2 \geq 0$, with equality holding if and only if $E = L \oplus M$.

To complete the proof, observe that if $C \subset S = \mathbb{P}E \to X$ is any curve on a ruled surface having degree $d > 0$ over X, then after taking the fiber product with C over X we arrive at a ruled surface $\mu : S' = S \times_X C \to C$ with a section $C' \subset S'$. Moreover, if C has negative self-intersection on S then C' will have negative self-intersection on S': if we write

$$\mu^*(C) = C' + D$$

then we have

$$\begin{aligned}(C' \cdot C') &= (C' \cdot \mu^* C) - (C' \cdot D) \\ &\leq (C' \cdot \mu^* C) \\ &= (\mu_* C \cdot C) \\ &= (C \cdot C).\end{aligned}$$

□

Exercise 12.25. Let $C \subset \mathbb{P}^3$ be a smooth curve of degree n and genus g, and S and $T \subset \mathbb{P}^3$ two smooth surfaces containing C, of degrees d and e. At how many points of C are S and T tangent?

12.5.3 Self-intersection of the zero section of a vector bundle

As we explained in Chapter 6, a vector bundle E on a scheme X may itself be considered as a scheme $\operatorname{Spec}(\operatorname{Sym} E^*)$ over X. In this section we will write $|E|$ for this scheme.

It is useful to have a compactification of $|E|$; that is, a variety proper over X that includes $|E|$ as a dense open subset. It is natural to try to compactify each fiber by putting it inside a projective space of the same dimension, and we can do this globally by taking the projectivization of the direct sum $E \oplus \mathcal{O}_X$; that is, we set

$$\overline{E} := \mathbb{P}(E \oplus \mathcal{O}_X).$$

The subbundle $\mathbb{P}E \subset \overline{E}$ corresponding to the subbundle E of $E \oplus \mathcal{O}_X$ is a divisor, which we may call the "hyperplane at infinity;" its complement is the total space $|E|$.

Set $r := \operatorname{rank} E$, and write ζ for the class $c_1\mathcal{O}_{\mathbb{P}(E\oplus\mathcal{O}_X)}(1) \in A^*(\overline{E})$. Since $c(E \oplus \mathcal{O}_X) = c(E)$, we have

$$A^*(\overline{E}) = A^*(X)[\zeta]/(\zeta^{r+1} + c_1(E)\zeta^r + \cdots + c_r(E)\zeta).$$

By Proposition 12.22 the class of $\mathbb{P}E$ in $A^*(\overline{E})$ is ζ.

Note that the section of $\overline{E} \to X$ given by the subbundle $\mathcal{O}_X \subset E \oplus \mathcal{O}_X$ corresponds to the zero section of E and we will denote it by E_0. Using Proposition 12.22 again we see that its class $e = [E_0]$ is given by

$$e = \zeta^r + c_1(E)\zeta^{r-1} + \cdots + c_{r-1}(E)\zeta + c_r(E).$$

Suppose for a moment that E has a global section σ whose zero locus $Z = (\sigma) \subset X$ is reduced of the expected codimension r and let $E_\sigma \subset$—E—$\subset \overline{E}$ be the image of σ (E_σ may be identified with the sub bundle of $\mathcal{O} \oplus E$ that is the graph of the map $\mathcal{O}_X \to E$ sending 1 to σ.) ****The following isn't satisfactory. Put in a real proof. This is foundational for Chern classes. Also, do the non-reduced case.**** Using the formula of Proposition 12.22, or arguing directly, we see that $E_\sigma \sim E_0$, so

$$\pi_*(e^2) = \pi_*\big([E_0 \cap E_\sigma]\big) = [Z] = c_r(E).$$

We claim that this formula holds in general:

Proposition 12.26. *Let E be a vector bundle of rank r on a smooth variety X. Let $\pi : \overline{E} = \mathbb{P}(E \oplus \mathcal{O}_X) \to X$ be the compactification of the total space $|E|$ of E. If $e \in A^r(\overline{E})$ is the class of the zero section $E_0 \subset |E| \subset \overline{E}$ then*

$$\pi_*(e^2) = c_r(E).$$

Furthermore, for any class $\alpha \in A(X)$ we have

$$\iota^*\iota_*\alpha = \alpha c_r(E).$$

We can understand the formula for $\iota^*\iota_*\alpha$ as follows. Suppose that $N_{Z/X}$ has enough sections. For simplicity, suppose that α is the class of a subvariety A of Z. The class $\alpha \cdot c_m(N)$ is the zero locus of a general section $\sigma : Z \to N_{Z/X}$ restricted to A. The image A' of A under such a section, is rationally equivalent to A in $\overline{E}$, and meets the zero section generically transversely. Thus $\alpha \cdot c_m(N) = [A' \cap Z] = \iota^*\iota_*\alpha$. Of course the actual proof must deal with the fact that $N_{Z/X}$ may not have enough sections.

Proof. If we multiply the polynomial

$$p(\zeta) = \zeta^r + c_1(E)\zeta^{r-1} + \cdots + c_{r-1}(E)\zeta + c_r(E)$$

by ζ we get the defining relation of $A(\overline{E})$. By Proposition**??**, $p(\zeta)$ is the expression for e in $A(\overline{E})$, so we get $\zeta e = 0$. We can see this geometrically, too: the section $E_0 \subset \overline{E}$ is disjoint from the hyperplane at infinity $\mathbb{P}E \subset \mathbb{P}(E \oplus \mathcal{O}_X)$, which has class ζ.

Thus if we square the class e we get

$$e^2 = \big(\zeta^r + c_1(E)\zeta^{r-1} + \cdots + c_{r-1}(E)\zeta + c_r(E)\big)c_r(E) = ec_r(E) \in A(\overline{E})$$

where multiplication by $c_i(E)$ really means multiplication by $\pi^* c_i(E)$, as usual. Applying π_* to both sides and using the push-pull formula we get $\pi_*(e^2) = (\pi_* e)c_r(E) = c_r(E)$.

For the last assertion, we first assume that $\alpha = [X]$. In this case, $\iota_*\alpha = e$, so we must show $\iota^* e = c_r(E)$. Note that each term of the polynomial $p(\zeta)$ except for the last is naturally represented by a cycle that meets E_0 in the empty set. Thus $\iota^* e = i^*(\pi^*(c_r(E)) = c_r(E)$ as required.

The general case reduces at once to the situation when $\alpha = [A] \in A(X)$ for some irreducible variety $A \subset X$. We have $\iota_*\alpha = [\iota(A)] = [\pi^{-1}(A) \cap E_0]$. Since this last intersection is transverse, we have

$$\iota^*\iota_*\alpha = \iota^*([\pi^{-1}(A)][E_0]) = [A]\iota^* e = \alpha c_r(E),$$

as required. □

12.6 Dual varieties and conormal varieties

********the following section has been transported whole from Chapter 3; it needs some repotting—specifically, subsections, updated references, keynote questions, reconciliation with other discussions of the dual variety (e.g., of a smooth hypersurface) and most of all we need to add the formula for the degree of the dual of an arbitrary smooth $X \subset \mathbb{P}^n$ in terms of the Chern classes of X. Finally, this whole discussion could go in either 9 or 10; we should decide which.********

12.6.1 The universal hyperplane as a projective bundle

Let $\pi_1 : \Psi \to \mathbb{P}^n$ be the projection onto the first factor of $\mathbb{P}^n \times \mathbb{P}^{n*}$, and let π_2 be the projection to $\mathbb{P}^{n*}$. The fiber of π_1 over a point p is the set of hyperplanes containing p, and this is a hyperplane in $\mathbb{P}^{n*}$. Thus π_1 is the structure map of a bundle of $(n-1)$-dimensional projective spaces. Using the defining equation, we can identify this bundle.

Proposition 12.27. *$\pi_1 : \Psi \to \mathbb{P}^n$ is the projectivized cotangent bundle $\mathbb{P}(\Omega_{\mathbb{P}^n})$ of $\mathbb{P}^n$. The tautological sub-bundle*

$$L := \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n})}(-1) \subset \pi_1^*(\Omega_{\mathbb{P}^n})$$

is $\pi_1^*\mathcal{O}_{\mathbb{P}^n}(-1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^{n*}}(-1)$.

Proof. The bundle $\Omega_{\mathbb{P}^n}$ appears in the exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1}(-1) \xrightarrow{(Z_0,\dots,Z_n)} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

Thus its dual (the tangent bundle of $\mathbb{P}^n$) may be written as

$$T_{\mathbb{P}^n} = \frac{\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(1) \cdot A_i}{\sum_{i=0}^n A_i Z_i}$$

where the $A_0, \dots, A_n$ are basis elements. Consequently the projectivization of $\Omega_{\mathbb{P}^n}$ is

$$\operatorname{Proj}(\operatorname{Sym}(\Omega^*_{\mathbb{P}^n})) = \operatorname{Proj}\left(\frac{\mathcal{O}_{\mathbb{P}^n}[A_0, \dots, A_n]}{\sum A_i Z_i}\right),$$

as required.

To identify the tautological sub-bundle on Ψ, we take the dual, $\mathcal{O}_{\mathbb{P}(\Omega)}(1)$, a 1-quotient of $\pi_1^*(\Omega^*_{\mathbb{P}^n})$, which is itself a quotient of the bundle

$$\pi_1^*(\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}) = \pi_1^*(\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(1)A_i).$$

The section which is the image of $Z_i A_j \in H^0(\mathcal{O}_{\mathbb{P}^n}(1)A_j)$ is the restriction to Ψ of the corresponding section of

$$L := \pi_1^*(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^{n*}}(1))$$

on $\mathbb{P}^1 \times \mathbb{P}^1$, and thus the bundle $\mathcal{O}_{\mathbb{P}(\Omega)}(1)$ is the restriction of L. Dualizing again we obtain the desired formula. □

12.6.2 Conormal and dual varieties

One situation in which the universal hyperplane arises naturally is the construction of conormal varieties and dual varieties. SeeKleiman [1986] for a comprehensive account of the history of these constructions. Our account is based on that in Kleiman [1984].

Let $X \subset \mathbb{P}^n$ be a subvariety of dimension k. If X is smooth, we define the *conormal variety* $CX \subset \mathbb{P}^n \times \mathbb{P}^{n*}$ to be the incidence correspondence If X is singular, we define CX to be the closure in $\mathbb{P}^n \times \mathbb{P}^{n*}$ of the locus CX° of such pairs (p, H) where p is a smooth point of X. We define the *dual variety* $X^\star \subset \mathbb{P}^{n*}$ of X to be the image of CX under projection on the second factor. Whatever the dimension of X, the conormal variety CX

will have dimension $n-1$: it is the closure of the locus CX°, which maps onto the smooth locus of X with fibers of dimension $n-k-1$.

A fundamental and beautiful theorem about conormal and dual varieties is the following theorem, called *reflexivity*:

Theorem 12.28. *Let $X \subset \mathbb{P}^n$ be any variety and $X^* \subset \mathbb{P}^{n*}$ its dual. If the characteristic of the ground field is zero, or more generally if the map from CX to $X^\star$ is generically smooth, then the conormal varieties $CX \subset \mathbb{P}^n \times \mathbb{P}^{n*}$ is equal to $C(X^\star) \subset \mathbb{P}^{n*} \times \mathbb{P}^n$ with the factors reversed.*

As an immediate consequence, we have

Corollary 12.29. *If $X \subset \mathbb{P}^n$ is any variety in characteristic 0 then $(X^*)^* = X$—that is, the dual of the dual of X is X.*

This may seem puzzling. As we have seen, if X is smooth then $X^\star$ is a hypersurface, and in fact $X^\star$ is a hypersurface for any variety X that is a complete intersection ****give reference or argument**** or, more generally, has a singular hyperplane section that is singular only in dimension zero. Thus, we seem to be suggesting that there are virtually as many hypersurfaces as varieties of arbitrary dimension in $\mathbb{P}^n$! The discrepancy is due to the fact that the duals of smooth varieties tend to be highly singular. There are also many other results about the geometry of dual varieties and conormal varieties. We recommend in particular Kleiman [1986], and the surprising and beautiful theorems of Ein and Landman (Ein [1986]) and Zak (Zak [1991]). Ein and Landman prove, for example, that for any smooth variety $X \subset \mathbb{P}^n$ of dimension d in characteristic zero, the difference $(n-1) - \dim X^\star$ is congruent to $\dim X$ modulo 2.

Exercise 12.30. Show that the conclusion of Corollary 12.33 fails in characteristic $p > 0$:

(a) Let K be a field of characteristic 2, and consider the plane curve
$$C = V(X^2 - YZ) \subset \mathbb{P}^2.$$
Show that C is smooth, but that the dual curve $C^* \subset \mathbb{P}^{2*}$ is a line, so that $C^{**} \neq C$.

(b) Now let K be a field of characteristic p, set $q = p^e$ and consider the plane curve
$$C = V(YZ^q + Y^qZ - X^{q+1}) \subset \mathbb{P}^2.$$
Show that C is smooth, and that the dual curve $C^{**} = C$, but that $\mathcal{G}_C : C \to C^*$ is not birational!

As we mentioned earlier, it's relatively rare for the dual X^* of a smooth variety $X \subset \mathbb{P}^n$ to be other than a hypersurface. Here are two circumstances where it does happen:

Exercise 12.31. Let $X \subset \mathbb{P}^n$ be a k-dimensional *scroll*, that is, a variety given as the union

$$X = \bigcup \Lambda_b$$

of a one-parameter family of $(k-1)$-planes $\{\Lambda_b \cong \mathbb{P}^{k-1} \subset \mathbb{P}^n\}$.

(a) Show that if $H \subset \mathbb{P}^n$ is a hyperplane containing the tangent plane $\mathbb{T}_p X$ to X at a smooth point p then the hyperplane section $H \cap X$ is reducible; and

(b) Deduce that $\dim X^* \leq n - k + 2$.

Exercise 12.32. Let $X = \mathbb{G}(1,4) \subset \mathbb{P}^9$ be the Grassmannian of lines in $\mathbb{P}^4$, embedded in $\mathbb{P}^9$ by the Plücker embedding. Show that the dual of X is again projectively equivalent to X itself!

Corollary 12.33. *If X is a smooth hypersurface, or more generally a hypersurface whose dual is also a hypersurface, and the characteristic of the ground field is zero, then the Gauss map $\mathcal{G}_X : X \to X^\star$ is birational, with inverse $\mathcal{G}_{X^\star} : X^\star \to X$. Thus if $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d then $X^\star$ is a hypersurface of degree $d(d-1)^{n-1}$.*

Proof. If X and $X^\star$ are both hypersurfaces then both $\mathcal{G}_X$ and $CG_{X^\star}$ are defined. Since the graphs of CX and $C(X^\star)$ are equal after exchanging factors, the two rational maps are inverse to each other, and are thus birational. As already noted in Chapter 1.2.2, the degree computation follows from the birationality of $\mathcal{G}_X$. □

12.6.3 Proof of the duality theorem

Here are two proofs of Theorem12.28.

Proof of Theorem12.28. If $X \subset \mathbb{P}^n$ is any subvariety then, over the open set where X is smooth, the conormal variety $CX \subset \Psi = \mathbb{P}(\Omega_{\mathbb{P}^n}) = \operatorname{Proj} \operatorname{Sym} T_{\mathbb{P}^n}$ is the projectivized conormal bundle $\mathbb{P}(K) = \operatorname{Proj} \operatorname{Sym} K^*$, where

$$K := \operatorname{Ker}\big(\Omega_{\mathbb{P}^n}|_X \to \Omega_X\big).$$

The conormal variety of X itself is defined as the closure of this set. Over the open set where X is smooth, K^* is the cokernel of the map of bundles

$$T_X \to T_{\mathbb{P}^n}|_X$$

so $\operatorname{Sym} K^*$ is equal to $\operatorname{Sym} T_{\mathbb{P}^n}|_X$ modulo the ideal generated by T_X, thought of as contained in the degree 1 part $T_{\mathbb{P}^n}|_X$ of the graded algebra

$\operatorname{Sym} T_{\mathbb{P}^n}|_X$. Sheafifying, this means that the ideal sheaf of CX in $\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n})}$ is the image of the composite map u_1 in the following diagram:

$$\begin{array}{ccc} \pi_1^* T_X \otimes \mathcal{O}_{\Omega_{\mathbb{P}^n}}(1) & \longrightarrow & \pi_1^* T_{\mathbb{P}^n} \otimes \mathcal{O}_{\Omega_{\mathbb{P}^n}}(1) \\ & \searrow^{u} & \downarrow \\ & & \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n})}, \end{array}$$

where the vertical map is the dual of

$$\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n})}(-1) \xrightarrow{\sum A_i dZ_i} \pi_1^* \Omega_{\mathbb{P}^n},$$

the tautological inclusion, tensored with $\mathcal{O}_{\Omega\mathbb{P}^n}(1)$.

Having the equations, we can tell whether a given subvariety C of $\Psi \cap \pi_1^{-1}(X)$ is a subset of CX. Let $\iota : C \to \Psi$ be the inclusion. Set

$$v = u^* \otimes \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n})}(-1) : \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n})}(-1) \to \pi_1^* \Omega_X,$$

and consider the diagram

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n})}(-1)|_C & \xrightarrow{\sum A_i dZ_i} & \pi_1^* \Omega_{\mathbb{P}^n}|_C & \xrightarrow{d\pi_1} & \Omega_\Psi|_C \\ & \searrow^{v|_C} & \downarrow & & \downarrow d\iota \\ & & \pi_1^* \Omega_X|_C & \xrightarrow[d(\pi_1|_C)]{} & \Omega_C. \end{array}$$

From what we have said about the equations of the conormal variety we see that $C \subset CX$ if and only if $v|_C = 0$. If the projection $\pi_1|_C : C \to X$ is generically smooth (for example if the characteristic of the ground field is zero) then $d\pi_1|_C$ is generically injective, so in this case $C \subset CX$ if and only if the composition $d\pi_1|_C \circ v|_C$ is zero.

We must show that if C is also generically smooth over $X^\star$ then this condition is symmetric, so that $C \subset CX$ if and only if $C \subset C(X^\star)$ (with the factors reversed). Since Ψ is defined by a hypersurface of bi-degree $(1,1)$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^{n*}}(-1,-1) \xrightarrow{\varphi} \Omega_{\mathbb{P}^n \times \mathbb{P}^{n*}}|_\Psi \longrightarrow \Omega_\Psi \longrightarrow 0.$$

In coordinates, using the decomposition

$$\Omega_{\mathbb{P}^n \times \mathbb{P}^{n*}} = \pi_1^*(\Omega_{\mathbb{P}^n}) \oplus \pi_2^*(\Omega_{\mathbb{P}^{n*}})$$

this becomes

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^{n*}}(-1,-1)|_\Psi \xrightarrow{\sum A_i dZ_i, \sum Z_i dA_i} \pi_1^*(\Omega_{\mathbb{P}^n})|_\Psi \oplus \pi_2^*(\Omega_{\mathbb{P}^{n*}})|_\Psi \xrightarrow{(d\pi_1, d\pi_2)} \Omega_\Psi \longrightarrow 0.$$

Noting that $\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n})}(-1) = \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^{n*}}(-1,-1)|_\Psi$, we see that if $\iota : C \to \Psi$ is the inclusion of any subvariety then the composition

$$\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n})}(-1) \xrightarrow{\sum A_i dZ_i} \pi_1^*(\Omega_{\mathbb{P}^n}) \xrightarrow{d\pi_1} \Omega_\Psi$$

is the negative of the composition

$$\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n})}(-1) \xrightarrow{\sum Z_i dA_i} \pi_2^*(\Omega_{\mathbb{P}^{n*}}) \xrightarrow{d\pi_2} \Omega_\Psi.$$

It follows that the composition

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n})}(-1)|_C & \xrightarrow{\sum A_i dZ_i} & \pi_1^*\Omega_{\mathbb{P}^n}|_C & \xrightarrow{d\pi_1} & \Omega_\Psi|_C \\ & & & & \downarrow d\iota \\ & & & & \Omega_C. \end{array}$$

is zero if and only if the composition

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^{n*}})}(-1)|_C & \xrightarrow{\sum Z_i dA_i} & \pi_2^*\Omega_{\mathbb{P}^{n*}}|_C & \xrightarrow{d\pi_2} & \Omega_\Psi|_C \\ & & & & \downarrow d\iota \\ & & & & \Omega_C. \end{array}$$

is zero, where we have used the identifications

$$\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^{n*}})}(-1) = \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^{n*}}(-1,-1)|_\Psi = \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n})}(-1).$$

Thus if $C \subset \Psi$ and $CX \to X^\star$ is also generically smooth then we can repeat the argument above, replacing $X \subset \mathbb{P}^n$ with $X^\star \subset \mathbb{P}^{n*}$ and we see that $C \subset C(X^\star)$ if and only if (after exchanging factors) $C \subset CX$. In particular, if CX itself is generically smooth over $C(X^\star)$ then CX and $C(X^\star)$ are the same after exchanging factors. □

Proof of Theorem12.28. To prove Theorem12.28 we'll work not with projective varieties but with the cones over them in affine space. For the following, accordingly, we'll adopt the following notation. First, we let V be a vector space of dimension $n+1$, and V^* the dual vector space; let $\pi_1 : V \times V^* \to V$ and $\pi_2 : V \times V^* \to V^*$ be the projections. Next, let $\widehat{X} \subset V$ be a cone—that is, a variety invariant under scalar multiplication, or equivalently the zero locus of homogeneous polynomials. We set

$$C\widehat{X} = \{(v,w) \in V \times V^* \mid v \in X \text{ and } w|_{T_vX} = 0\}$$

(note that since X is a cone, the tangent space $T_vX \subset T_vV = V$ is a linear subspace of V). We'll also refer to $C\widehat{X}$ as the conormal variety of X; note that it's an $(n+1)$-dimensional subvariety of $V \times V^*$. Finally, we'll call the image $\widehat{X}^* = \pi_2(C\widehat{X}) \subset V^*$ the dual variety to $\widehat{X}$.

The relation between these constructions and the one in Theorem12.28 should be clear: if $X \subset \mathbb{P}^n = \mathbb{P}V$, $CX \subset \mathbb{P}^n \times \mathbb{P}^{n*}$ and $X^* \subset \mathbb{P}^{n*}$ are as in the Theorem, and we let $\widehat{X} \subset V$ be the cone over X, then $C\widehat{X}$ and $\widehat{X}^*$ are the cones over CX and X^* respectively; our task accordingly will be to prove that $C\widehat{X}$ is also the conormal variety of $\widehat{X}^*$.

Now, let $Z_0, \ldots, Z_n$ be linear coordinates on V, and $A_0, \ldots, A_n$ dual coordinates on V^* (so that, confusingly enough, $Z_0, \ldots, Z_n$ form a basis for V^* and $A_0, \ldots, A_n$ the dual basis for V), and consider the *contact form* on $V \times V^*$

$$\omega = \sum_{i=0}^{n} A_i dZ_i.$$

This is a differential 1-form on $V \times V^*$, that is, a linear form on the tangent spaces to $V \times V^*$; at a point $(v, w) \in V \times V^*$, its value on a tangent vector

$$(x, y) \in T_{(v,w)}(V \times V^*) = T_v V \oplus T_w V^* = V \oplus V^*$$

is

$$\omega(x, y) = \sum A_i(w) Z_i(x).$$

Note that ω does not depend on the choice of coordinates Z on V.

The key observation is now that *the conormal variety $C\widehat{X}$ is an integral manifold for the* 1*-form* ω. To see this, note that if we're at a point $(v, w) \in C\widehat{X}$, then we can choose coordinates Z with $T_v X$ the zero locus of $Z_{k+1}, \ldots, Z_n$. If the tangent vector $(x, y) \in T_{(v,w)} C\widehat{X}$ then $x \in T_v X$ and so the terms $i = k+1, \ldots, n$ in the sum above all vanish. But at the same time, by construction w will be a linear combination of $Z_{k+1}, \ldots, Z_n$, so the terms $i = 0, \ldots, k$ in the sum above also vanish.

The same argument also establishes a converse: if $\Gamma \subset V \times V^*$ is an $(n+1)$-dimensional integral manifold for the 1-form ω then Γ is the conormal variety of its image $X = \pi_1(\Gamma) \subset V$. To see this, suppose X has dimension $k+1$ and let $(v, w) \in \Gamma$ be a general point, so that the differential $d\pi_1 : T_{(v,w)}\Gamma \to T_v X$ is surjective. By our calculation, the kernel of $d\pi_1$ is contained in, and hence equal to, the subspace $\{0\} \times T_v X^\perp \subset V \times V^*$, where $T_v X^\perp \subset V^*$ is the annihilator of $T_v X$. Now, we know the fiber $\Gamma_v = \pi_1^{-1}(v)$ of $\pi_1 : \Gamma \to V$ over v has dimension $n - k$; given that its tangent space (at least on a Zariski open subset) is equal to $\{0\} \times T_v X^\perp$ it follows (in characteristic 0!) that $\Gamma_v = \{v\} \times T_v X^\perp$.

The next point is that, since the tangent space to X at a point $v \in X$ necessarily contains v, the conormal variety is contained in the hypersurface

$$\Psi = \{(v, w) \in V \times V^* \mid w(v) = 0\}.$$

In terms of coordinates $Z_0, \ldots, Z_n, A_0, \ldots, A_n$ on $V \times V^*$, Ψ is just the zero locus of the polynomial

$$f = \sum Z_i A_i$$

and it follows that the restriction to Ψ of the differential

$$df = \sum (A_i dZ_i + Z_i dA_i)$$

vanishes identically, and hence its restriction to $C\widehat{X} \subset \Psi$ is likewise 0. Thus we see that $C\widehat{X}$ is also an integral manifold for the differential

$$\omega^* = \sum Z_i dA_i,$$

and so by our previous remark it's also the cornormal variety of its image $\widehat{X}^* = \pi_2(C\widehat{X})$. □

12.6.4 Degree of the dual

Finally, we'd like to give a formula for the degree of the dual of a smooth, k-dimensional variety $X \subset \mathbb{P}^n$, in terms of the Chern classes of the tangent bundle to X and the hyperplane class $\zeta \in A^1(X)$.

This will be a straightforward application of the constructions we've made above, plus the notion of the *Segre class* introduced in the preceding Chapter. The key is to realize the conormal variety

$$CX = \{(p, H) \in \mathbb{P}^n \times \mathbb{P}^{n*} \mid p \in X \text{ and } H \supset \mathbb{T}_p X\}.$$

as a projective bundle $\mathbb{P}E$ over X, in such a way that the projection map $\pi_2 : CX \to \mathbb{P}^{n*}$ is given by the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$.

To do this, let $\mathcal{G}_X : X \to \mathbb{G}(k, n)$ be the Gauss map of X, sending each point $p \in X$ to its tangent plane $\mathbb{T}_p X \cong \mathbb{P}^k \subset \mathbb{P}^n$, and let $E = \mathcal{G}_X^*(Q)$, where Q is the universal quotient bundle on $\mathbb{G}(k, n)$. The fiber of the dual bundle E^* over a point $p \in X$ is the space of linear forms on $\mathbb{P}^n$ vanishing on $\mathbb{T}_p X$, from which we see that the projectivization

$$\mathbb{P}(E^*) \cong CX,$$

with $\pi_2^* \mathcal{O}_{\mathbb{P}^{n*}}(1) \cong \mathcal{O}_{\mathbb{P}E^*}(1)$. It follows that the degree of the dual is the k^{th} Segre class of the bundle E^*, and it's this we will proceed to calculate.

There are a number of ways to do this. One is to realize the bundle E as the twist $N(1)$ of the normal bundle $N = N_{X/\mathbb{P}^n}$ of X in $\mathbb{P}^n$. Another—more or less equivalent, but which will yield the answer in a form better suited to our question—is to look also at the pullback $F = \mathcal{G}^*(S)$ of the universal sub-bundle S on $\mathbb{G}(k, n)$. This is a bundle of rank $k + 1$, which includes the tautological line bundle $\mathcal{O}_X(-1)$ as a subbundle, with quotient the tangent bundle to X twisted by $\mathcal{O}_X(-1)$; in other words, we have an exact sequence

$$0 \to \mathcal{O}_X(-1) \to F \to T_X \otimes \mathcal{O}_X(-1) \to 0.$$

It follows that

$$c(F) = c\left(\mathcal{O}_X(-1)\right) c\left(T_X(-1)\right).$$

Now, from the exact sequence $0 \to S \to \mathcal{O}^{n+1} \to Q \to 0$ on $\mathbb{G}(k,n)$ pulled back, we have

$$c(E) = \frac{1}{c(F)}$$

and so the Segre class is

$$s(E) = \frac{1}{c(E)} = c(F).$$

In sum, the degree of the dual is the k^{th} graded piece of the Segre class of E^*, which is

$$\begin{aligned}\deg(X^*) &= (-1)^k \Big[c\left(\mathcal{O}_X(-1)\right) c\left(T_X(-1)\right) \Big]_k \\ &= (-1)^k \Big[(1-\zeta) c\left(T_X(-1)\right) \Big]_k \\ &= (-1)^k c_k(T_X(-1)) + (-1)^{k-1} c_{k-1}(T_X(-1)).\end{aligned}$$

Now. from our formula for the Chern classes of the tensor product with a line bundle, we have

$$c_k(T_X(-1)) = c_k(T_X) - \zeta c_{k-1}(T_X) + \cdots + (-1)^k \zeta^k$$

and

$$\zeta c_{k-1}(T_X(-1)) = \zeta c_{k-1}(T_X) - 2\zeta^2 c_{k-2}(T_X) + \cdots + k(-1)^{k-1}\zeta^k$$

Adding these up (or rather taking $(-1)^k$ times the first, plus $(-1)^{k-1}$ times the second, we arrive at the

Proposition 12.34. *Let $X \subset \mathbb{P}^n$ be a smooth, k-dimensional projective variety, and $X^* \subset \mathbb{P}^{n*}$ its dual. If X^* has dimension $n-1$, its degree is*

$$\deg(X^*) = (k+1)\zeta^k - k\zeta^{k-1} c_1(T_X) + \cdots + (-1)^k c_k(T_X)$$

where $\zeta \in A^1(X)$ is the hyperplane class. If X^* has dimension strictly less than $n-1$, the expression on the right is necessarily 0.

By way of examples, consider first the case $k = 1$, that is, $X \subset \mathbb{P}^n$ is a smooth curve of genus g. Since (in general) ζ^k is the degree of X and $c_k(T_X)$ its Euler characteristic, the formula is

$$\deg(X^*) = 2\deg(X) + 2g - 2,$$

as we've seen before in Chapter 9. Similarly, for a surface $X \subset \mathbb{P}^n$ we have

$$\deg(X^*) = 3\deg(X) + 2\zeta K_X + \chi(X);$$

again, this coincides with the formula for the number of singular elements in a pencil of hyperplane sections found in Chapter 9.

Exercise 12.35. Applying Proposition 12.34, verify the formula of Corollary 12.33 for the degree of the dual of a smooth hypersurface.

12.6.5 Pencils of hypersurfaces

********This was moved from the original Chapter 3; it still uses the notation introduced there for the generators of the Chow groups of the universal hyperplane (based on the affine stratification). We need to rewrite all the formulas in terms of the generators introduced in this Chapter********

As a variant of this construction—and one that will allow us to solve Keynote Questione—suppose now that we have a pencil $\mathcal{D} = \{X_t\}_{t\in\mathbb{P}^1}$ of hypersurfaces X_t of degree d in $\mathbb{P}^n$. We can then form an n-dimensional cycle $\Gamma_{\mathcal{D}} \subset \Psi$ by taking the closure of the locus of pairs (p, H) with p a smooth point of the hypersurface X_t and $H = \mathbb{T}_p X_t$; again, we ask for the class $\gamma_{\mathcal{D}}$ of this locus.

We'll do this just in the case $n = 2$ of pencils of plane curves, leaving the general case as an exercise. Here, we have a priori

$$\gamma_{\mathcal{D}} = a\varphi_{1,0} + b\varphi_{0,1}$$

for some integers a and b, which we can evaluate as intersection numbers of $\gamma_{\mathcal{D}}$ with cycles of complementary dimension. Specifically, we have

$$b = (\gamma_{\mathcal{D}} \cdot \varphi_{2,1}) = \#(\Gamma_{\mathcal{D}} \cap \Phi_{2,1})$$

where $\Phi_{2,1} = \{(p, H) : p = p_0\}$ for general $p_0 \in \mathbb{P}^2$; since there'll be a unique curve X_t through p_0, and it'll be smooth at p_0, the intersection consists of ther single point $(p_0, \mathbb{T}_{p_0} X_t) \in \Psi$ and we conclude that $b = 1$. Similarly,

$$a = (\gamma_{\mathcal{D}} \cdot \varphi_{1,2} = \#(\Gamma_{\mathcal{D}} \cap \Phi_{1,2})$$

where $\Phi_{2,1} = \{(p, H) : H = H_0\}$ for general line $H_0 \subset \mathbb{P}^2$. Now, $\mathcal{D}$ will cut on H_0 a pencil of divisors of degree d, giving a map $\pi : H_0 \to \mathbb{P}^1$ of degree d. Moreover, since each curve X_t has only finitely many flex lines, no curve in the pencil has a point of contact of order 3 or more with H_0; thus π has only simple ramification points p_α, and by the Riemann-Hurwotz formula there will be exactly $2d - 2$ of them. The intersection $\Gamma_{\mathcal{D}} \cap \Phi_{1,2}$ will thus consist of the $2d - 2$ points $(p_\alpha, H_0) \in \Psi$; we conclude that $a = 2d - 2$ and in sum

$$\gamma_{\mathcal{D}} = (2d - 2)\varphi_{1,0} + \varphi_{0,1} = (2d - 2)\alpha + \beta.$$

Exercise 12.36. Find the class of the cycle $\Gamma_{\mathcal{D}}$ for $n > 2$.

As promised, we can use this to answer Keynote Question (e), and indeed it's almost a direct application: if we let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be the pencil of conics through $\{p_1, \ldots, p_4\}$, $\{q_1, \ldots, q_4\}$ and $\{r_1, \ldots, r_4\}$ respectively, we are essentially asking for the cardinality of the intersection of the cycles $\Gamma_{\mathcal{C}}$, $\Gamma_{\mathcal{D}}$ and $\Gamma_{\mathcal{E}}$. Note that GL_{n+1} acts transitively on Ψ. By Kleiman's Theorem

5.20, these will intersect transversely, so that the answer to our question is

$$\begin{aligned}\#(\Gamma_{\mathcal{C}} \cap \Gamma_{\mathcal{D}} \cap \Gamma_{\mathcal{E}}) &= (\gamma_{\mathcal{C}} \cdot \gamma_{\mathcal{D}} \cdot \gamma_{\mathcal{E}}) \\ &= (2\alpha + b)^3 \\ &= 18.\end{aligned}$$

Actually, we're not quite done: the hedging in the last paragraph is because the statement of the question excludes the point p being one of the base points of one of the three pencils, and the definition of the cycles $\Gamma_{\mathcal{C}}$, $\Gamma_{\mathcal{D}}$ and $\Gamma_{\mathcal{E}}$ involves taking a closure, which means for example there will be points $(p, H) \in \Gamma_{\mathcal{C}}$ with p a singular point of the conic in the pencil $\mathcal{C}$ through p. We have to check that neither of these contributes to the cardinality 18 of the intersection.

Fortunately, this is straightforward: for example, if p_i is one of the base points of the pencil $\mathcal{C}$, then since the p_i are general, the (unique) curves of the pencils $\mathcal{D}$ and $\mathcal{E}$ through p_i will not be tangent to one another there. Likewise, if p is a singular point of any one of the pencils, we can check that the elements of the other two pencils passing through p will not be tangent there. Thus none of the 18 points of $\Gamma_{\mathcal{C}} \cap \Gamma_{\mathcal{D}} \cap \Gamma_{\mathcal{E}}$ will be excluded.

13

The Tangent Bundle of a Projective Bundle and Relative Principal Parts

(a) Given a general quintic surface $S \subset \mathbb{P}^3$, how many lines $L \subset \mathbb{P}^3$ meet S in only one point?

(b) If $\mathcal{D}$ is a net of curves on a smooth surface S, how many of the curves $C \in \mathcal{D}$ will have cusps?

(c) If $\{C_t = V(t_0F + t_1G)\} \subset \mathbb{P}^2$ is a general pencil of quartic plane curves, how many of the curves C_t will have hyperflexes?

If $\pi : Y \to X$ is a smooth map, then the induced map of tangent bundles $T_Y \to \pi^* T_X$ is surjective. It's kernel is called the *relative tangent bundle* of π, denoted T_π or, when there is no ambiguity, as $T_{Y/X}$; it's local sections are the vector fields on Y that are everywhere tangent to a fiber. Thus, for example, if $x \in X$ then the restriction $T_{Y/X}|_{\pi^{-1}}(x)$ is the tangent bundle to the smooth variety $\pi^{-1}(x)$.

We can also define the *bundles of relative principal parts* $\mathcal{P}^m_{Y/X}(\mathcal{F})$ of a sheaf of a sheaf $\mathcal{F}$ on Y. Just as in the absolute case treated in ***, these bundles are iterated extensions involving symmetric powers of the relative tangent bundle tensored with the sheaf $\mathcal{F}$. They In this Chapter we will compute this bundle in the case of a projective bundle, and explain some of the applications of this computation.

Theorem 13.1. *If E is a vector bundle on the smooth variety X. then the relative tangent bundle $T_{\mathbb{P}E/X}$ is the cokernel of a natural map $\mathcal{O}_{\mathbb{P}E} \to$*

$(\pi^*T_X)(1)$. *Setting* $\zeta = c_1(\mathcal{O}_{\mathbb{P}E}(1)$ *as usual, we have:*

$$c_k(T_{\mathbb{P}E/X}) = \sum_{i=0}^{k} \binom{r+1-i}{k-i} c_i(E)\zeta^{k-i}.$$

Although we will not need it here, it is worth knowing the generalization to Grassmannian bundles: with notation as in the Theorem, the relative tangent bundle of the Grassmannian $G(k, E)$ is

$$T_{Gr(k,E)/X} = Hom(S, \pi^*E/S) = S^* \otimes \pi^*E/S.$$

Over an open subset where E is trivial this is an immediate consequence of the computation of the tangent bundle of the ordinary Grassmannian $G(k, n)$, and an easy argument in local coordinates shows that it glues together properly. We will give a proof in the case of projective space along different lines, and the general case can also be treated in this way.

Proof. Given an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}E} \to (\pi^*T_X)(1) \to T_{\mathbb{P}E/X} \to 0$$

we see that $c(T_{\mathbb{P}E/X}) = c((\pi^*T_X)(1))$, and the formula of Proposition 7.6 yields the formula for the Chern classes.

The desired exact sequence, which is a globalization of the usual Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \to T_{\mathbb{P}^n} \to 0,$$

can be derived using local charts, as in the □

13.0.6 Chern Classes of Relative Tangent Bundles

But wait! (as they say in late-night TV commercials)—there's more! In fact, the identification $T_\Lambda G = \mathrm{Hom}(\Lambda, V/\Lambda)$ can be carried out simultaneously not just for all Λ in an n-dimensional vector space V, but for all subspaces of a *family* of such vector spaces, that is, for subspaces of the fibers of a vector bundle.

To set this up, suppose now that X is any variety and E a vector bundle of rank n on X. Let $G = G(k, E) \to X$ be the Grassmann bundle of k-planes in the fibers of E, as introduced in the last Chapter. Let $T^v_{G/X}$ denote the *relative tangent bundle* on G; that is, the bundle whose fiber at each point of G is the tangent space to the fiber of the map $\pi : G \to X$ through that point, or equivalently the kernel of the differential $d\pi : T_G \to \pi^*T_X$. We have already introduced, in the last Chapter, the tautological sub- and quotient bundles S and Q on G; we have the isomorphism

$$T^v_{G/X} \cong \mathrm{Hom}(S, Q).$$

From the exact sequence

$$0 \to T^v_{G/X} \to T_G \to \pi^* T_X \to 0$$

this will allow us to determine the Chern classes of the tangent bundle T_G if we know those of E and of the tangent bundle T_X; though in many cases—including all the applications below—it is actually the relative tangent bundle we're primarily interested in.

Since we'll be using this just in the case $k = 1$ of a projective bundle, and since the result has a relatively simple expression in that case, let's write out the resulting formula for the Chern classes of the relative tangent bundle to a projective bundle. To be consistent with the notation of the preceding section, let E be a vector bundle of rank $r + 1$ on X, and $\pi : \mathbb{P}E \to X$ the corresponding $\mathbb{P}^r$-bundle. As before, we let ζ be the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}_e}(1) \cong S^*$.

Now, we can take the sequence

$$0 \to S \to \pi^* E \to Q \to 0$$

and tensor with S^* to arrive at the sequence

$$0 \to \mathcal{O}_{\mathbb{P}E} \to S^* \otimes \pi^* E \to S^* \otimes Q = T^v_{\mathbb{P}E/X} \to 0$$

(again, this is just the usual Euler sequence, applied to a family of projective spaces); we conclude that

$$c(T^v_{\mathbb{P}E/X}) = c(S^* \otimes \pi^* E).$$

Applying the formula ** intref ** for the Chern class of a tensor product of a rank $r + 1$ bundle and a line bundle, we arrive at the answer to our question:

Theorem 13.2. *Let X be any variety, E a vector bundle of rank $r+1$ on X. The Chern classes of the relative tangent bundle $T^v_{\mathbb{P}E/X}$ to the associated projective bundle $\mathbb{P}E \to X$ are given by*

$$c_k(T^v_{\mathbb{P}E/X}) = \sum_{i=0}^{k} \binom{r+1-i}{k-i} c_i(E) \zeta^{k-i}.$$

13.1 Application: Contact problems

As a first application of this discussion, let's take up the first of the keynote questions of this Chapter: given a general quintic surface $S \subset \mathbb{P}^3$, how many lines $L \subset \mathbb{P}^3$ meet S in only one point?

We start, as we should with any enumerative problem, by asking ourselves why we would expect a finite number in the first place. To a 19^{th}

century geometer, this would be clear: a general line meets S in 5 points; to require them all to coincide is 4 conditions, and there's a 4-dimensional family of lines in $\mathbb{P}^3$. As usual, we can make this rigorous by introducing an incidence correspondence. First, we introduce the flag variety

$$\Phi = \{(L, p) : p \in L\} \subset \mathbb{G}(1, 3) \times \mathbb{P}^3.$$

Next, we look at pairs consisting of a point $(L, p) \in \Phi$ and a surface $S \subset \mathbb{P}^3$ such that the line L has contact of order at least 5 with S at p (or is contained in S):

$$\Gamma = \{(L, p, S) : m_p(S \cdot L) \geq 5\} \subset \Phi \times \mathbb{P}^{55}.$$

Now, the fiber of Γ over any point $(L, p) \in \Phi$ is just a linear subspace $\mathbb{P}^{50} \subset \mathbb{P}^{55}$ of codimension 5 in the space $\mathbb{P}^{55}$ of quintic surfaces; it follows that Γ is irreducible of dimension 55, and hence that the fiber of Γ over a general point $[S] \in \mathbb{P}^{55}$ is finite.

One way to think of this construction is to say that we are, as we have many times previously, *linearizing* the problem: the condition that a line L and a surface $S \subset \mathbb{P}^3$ meet in only one point is not a linear condition on either the surface or the line; but if we specify both the line L and the point $p \in L$ where S is to have contact of order 5, it becomes a linear condition on S.

Another way to think of this situation—largely useless for purposes of either computation or proof, but fun to try to visualize—is as follows. Suppose S is a general surface of degree d in $\mathbb{P}^3$. There is a 3-parameter family of tangent lines to S: at each point $p \in S$, there is a pencil of lines through p and lying in $\mathbb{T}_p S$. At a general point $p \in S$, the intersection $C = S \cap \mathbb{T}_p S$ of S with its tangent plane is a curve with a node at p, showing that there are two lines through p—the tangent lines to the two branches of C at p—having contact of order 3 with S at p. Now, in codimension one—that is, along a curve in S—one of the branches of C at p will have a flex at p; this gives rise to a line having contact of order 4 with S at p. And finally, at finitely many points of this curve, a branch of C will actually have a hyperpflex at p, giving rise to a line with contact of order 5 with S.

Exercise 13.3. Let $S \subset \mathbb{P}^3$ be a general surface of degree $d \geq 3$.

(a) Show that for a general point $p \in S$, the intersection $S \cap \mathbb{T}_p S$ has a node at p.

(b) Show that there is a curve $P \subset S$ such that for $p \in P$, $S \cap \mathbb{T}_p S$ has a cusp at p.

(c) Show that there are finitely many points $p \in S$ such that $S \cap \mathbb{T}_p S$ has a tacnode.

(d) Show that no plane section of S has a triple point.

Exercise 13.4. This is tangential (so to speak) but interesting: show that the conclusion of part (a) of the preceding exercise holds for an arbitrary smooth surface $S \subset \mathbb{P}^3$, if we assume characteristic zero. What can we say about parts (b) and (c)?

Exercise 13.5. Let S be a general surface of degree $d \geq 5$ and $L \subset \mathbb{P}^3$ any line having contact of order 5 with S at a point p. Show that the intersection $C = S \cap \mathbb{T}_p S$ of S with its tangent plane at p has a node at p, with one of the branches having contact of order 4 with L and the other transverse.

13.1.1 Contact bundles

We return to the enumerative problem, with one change: as is suggested by the above, we may as well take S to be a general surface of any degree $d \geq 5$ and ask how many lines L have a point of contact of order 5 with S. Now, as is often the case, the proof that for general S there are finitely many lines $L \subset \mathbb{P}^3$ having a point p of contact of order 5 with S also suggests a way to calculate the number. Since specifying the line L and the point p makes the condition $m_p(L \cdot S) \geq 5$ into a linear condition, we should introduce a vector bundle E on the space Φ of such pairs (L, p) that expresses that condition. Specifically, we want to take the vector bundle E on Φ with fiber

$$E_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathcal{I}^5_{p,L}(d)).$$

Such a bundle is called a *contact bundle*; we'll indicate in Exercise 13.6 below how to construct them.

If $A \in H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ is the homogeneous polynomial of degree d defining the surface S then the restrictions of A to each line $L \subset \mathbb{P}^3$ yield a global section σ_A of the bundle E (again, we'll verfiy this in Exercise 13.6 below), whose zeroes are exactly the pairs (L, p) with $m_p(L \cdot S) \geq 5$; and the number of such pairs (counting multiplicity, and as always assuming there are only finitely many) is thus $c_5(E) \in A^5(\Phi) \cong \mathbb{Z}$.

Exercise 13.6. Verifications: let

$$Z = \Phi \times_{\mathbb{G}(1,3)} \Phi = \{(L, p, q) : p, q \in L\} \subset \mathbb{G}(1,3) \times \mathbb{P}^3 \times \mathbb{P}^3;$$

let $\Delta \subset Z$ be the diagonal, let α be the projection map $Z \to \Phi$ sending (L, p, q) to (L, p) and $\beta : Z \to \mathbb{P}^3$ the projection on the last factor, sending (L, p, q) to q.

(a) Show that the bundle E we want is

$$E = \alpha_* \left(\beta^* \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{O}_Z / \mathcal{I}^5_{\Delta,Z} \right);$$

that is, show that the fiber of this bundle over $(L, p) \in \Phi$ may be naturally identified with $H^0(\mathcal{O}_L(5)/\mathcal{I}^5_{p,L}(5))$, as desired.

(b) Show that the pushforward of the quotient map

$$\beta^*\mathcal{O}_{\mathbb{P}^3}(d) \;\to\; \beta^*\mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{O}_Z/\mathcal{I}^5_{\Delta,Z}$$

on Z gives a map from the trivial bundle F on Φ with fiber $H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ to the bundle E of the first part, and we can take σ_A the image of the constant section A of F

13.1.2 Chern classes of contact bundles

The problem is now to calculate the Chern class $c_5(E)$ in the Chow ring of Φ. We start by writing down this ring: since

$$\Phi = \mathbb{P}S \to \mathbb{G}(1,3)$$

is the projectivization of the universal subbundle on $\mathbb{G}(1,3)$, and we have

$$c_1(S) = -\sigma_1 \quad \text{and} \quad c_2(S) = \sigma_{11},$$

we have by Theorem 11.4

$$A^*(\Phi) = A^*(\mathbb{G}(1,3))[\zeta]/(\zeta^2 - \sigma_1\zeta + \sigma_{11}).$$

Note that if M is a homogeneous linear form on $\mathbb{P}^3$, vanishing on a hyperplane $H \subset \mathbb{P}^3$, the restrictions of M to each point in each line in $\mathbb{P}^3$ give a global section σ_M of the dual of the universal subbundle on $\Phi = \mathbb{P}S$, that is, of $\mathcal{O}_{\mathbb{P}S}(1)$; it vanishes at (L,p) exactly when $p \in H$. We have thus

$$\zeta = \beta^*\omega$$

where $\beta : \Phi \to \mathbb{P}^3$ is the projection $(L,p) \mapsto p$ on the second factor, and $\omega \in A^1(\mathbb{P}^3)$ is the hyperplane class.

Next, the best way to calculate the Chern classes of E is by filtering it: we introduce a sequence of subbundles

$$0 = E_5 \subset E_4 \subset E_3 \subset \cdots \subset E_0 = E$$

where the fiber of E_i at (L,p) is the subspace $H^0(\mathcal{I}^i_{p,L}(d)/\mathcal{I}^5_{p,L}(d)) \subset H^0(\mathcal{O}_L(d)/\mathcal{I}^5_{p,L}(d))$. The successive quotients E_{i+1}/E_i will be line bundles, with fibers

$$\begin{aligned}(E_i/E_{i+1})_{(L,p)} &= H^0(\mathcal{I}^i_{p,L}(d)/\mathcal{I}^{i+1}_{p,L}(d)) \\ &= H^0(\mathcal{I}^i_{p,L}/\mathcal{I}^{i+1}_{p,L}) \otimes \mathcal{O}_{\mathbb{P}^3}(d)_p.\end{aligned}$$

Now, the vector space $H^0(\mathcal{I}_{p,L}/\mathcal{I}^2_{p,L})$ is the cotangent space to L at p; and the space $H^0(\mathcal{I}^i_{p,L}/\mathcal{I}^{i+1}_{p,L})$ is its i^{th} symmetric or tensor power. We can thus identify the successive quotients in our filtration of E as

$$E_i/E_{i+1} = \mathrm{Sym}^i(T^v_{\Phi/\mathbb{G}(1,3)})^* \otimes \beta^*\mathcal{O}_{\mathbb{P}^3}(d),$$

and we know how to write down the Chern classes of these: by Theorem 13.2, we have

$$c_1(T^v_{\Phi/\mathbb{G}(1,3)}) = 2\zeta - \sigma_1,$$

and as we remarked above,

$$c_1(\beta^*\mathcal{O}_{\mathbb{P}^3}(d)) = d\zeta.$$

Thus

$$c_1(E_i/E_{i+1}) = (d-2i)\zeta + i\sigma_1$$

and altogether

$$c_5(E) = d\zeta\cdot((d-2)\zeta+\sigma_1)\cdot((d-4)\zeta+2\sigma_1)\cdot((d-6)\zeta+3\sigma_1)\cdot((d-8)\zeta+4\sigma_1).$$

13.1.3 Counting lines with contact of order 5

It remains now to evaluate this product in $A^*(\Phi)$. To do this, note that we can evaluate the monomials of degree 5 in ζ and σ_1 by applying Theorem 11.4. Since the Grassmannian $\mathbb{G}(1,3)$ is four-dimensional, $\sigma_1^5 = 0$; and from our description of the Chow ring of $\mathbb{G}(1,3)$ **ref** we see that σ_1^4 is the class of two fibers of $\Phi \to \mathbb{G}(1,3)$. Thus

$$\zeta\cdot\sigma_1^4 = 2.$$

For the rest, we use the relation $\zeta^2 - \sigma_1\zeta + \sigma_{11} = 0$ satisfied by ζ: this gives us

$$\begin{aligned}\zeta^2\sigma_1^3 &= \sigma_1^3(\zeta\sigma_1 - \sigma_{11})\\ &= 2\end{aligned}$$

($\sigma_1^3\sigma_{11} = 0$ again because $\mathbb{G}(1,3)$ is only four-dimensional.) Next,

$$\begin{aligned}\zeta^3\sigma_1^2 &= \zeta\sigma_1^2(\zeta\sigma_1 - \sigma_{11})\\ &= \zeta^2\sigma_1^3 - \zeta\sigma_1^2\sigma_{11}\\ &= 2 - 1 = 1\end{aligned}$$

and

$$\begin{aligned}\zeta^4\sigma_1 &= \zeta^2\sigma_1(\zeta\sigma_1 - \sigma_{11})\\ &= \zeta^3\sigma_1^2 - \zeta^2\sigma_1\sigma_{11}\\ &= 1 - 1 = 0\end{aligned}$$

since $\sigma_1\sigma_{11} = \frac{1}{2}\sigma_1^3$. Note that we could see this directly: since ζ is the pullback of the hyperplane class on $\mathbb{P}^3$ under the projection $b : \Phi \to \mathbb{P}^3$, its fourth power is necessarily 0; so at least we haven't made a mistake so far. It follows likewise that $\zeta^5 = 0$.

To evaluate $c_5(E)$ we write

$$\begin{aligned} c_5(E) &= d\zeta \cdot ((d-2)\zeta + \sigma_1) \cdot ((d-4)\zeta + 2\sigma_1) \cdot ((d-6)\zeta + 3\sigma_1) \cdot ((d-8)\zeta + 4\sigma_1) \\ &= 24d\zeta\sigma_1^4 \\ &\quad + d(50d - 192)\zeta^2\sigma_1^3 \\ &\quad + d(35d^2 - 200d + 240)\zeta^3\sigma_1^2 \\ &= 35d^3 - 200d^2 + 240d. \end{aligned}$$

Our conclusion, pending verification of transversality, is that if $S \subset \mathbb{P}^3$ is a general surface of degree d, the number of lines having a point of contact of order 5 with S is $35d^3 - 200d^2 + 240d$; in particular, if S is a general quintic, we expect that there will be exactly 575 lines meeting S in only one point.

At this point, we should mention one aspect of enumerative geometry that every practitioner of the art knows from experience, but that is seldom stated in textbooks: we should always look for ways to check our answer. In this case, consider the case $d = 4$: we know that a general quartic surface $S \in \mathbb{P}^3$ contains no lines, and so there can be no line in $\mathbb{P}^3$ having a point of contact of order 5 with S. The formula should thus yield the answer 0 in case $d = 4$—and, happily, it does.

Finally, to conclude this analysis we need to know that for a general quintic surface S all the lines having a point of contact of order 5 with S "count with multiplicity 1"—that is, the all the zeroes of the corresponding section of the bundle E on Φ are simple zeroes. To do this, we invoke the irreducibility of the incidence correspondence

$$\Gamma = \{(L, p, S) : m_p(S \cdot L) \geq 5\} \subset \Phi \times \mathbb{P}^{55}$$

introduced earlier in the discussion. By virtue of the irreducibility of Γ, it's enough to show that at just one point $(L, p, S) \in \Gamma$, the section of E corresponding to S has a simple zero at $(L, p) \in \Phi$: given this, the locus of (L, p, S) for which this is not the case, being a proper closed subvariety of Γ, will have strictly smaller dimension, and so cannot dominate $\mathbb{P}^{55}$. As for locating such a triple (L, p, S), the following exercise suggests one.

Exercise 13.7. Let $L \subset \mathbb{P}^3$ be the line $X_2 = X_3 = 0$, and let $p \in L$ be the point $[1, 0, 0, 0]$. By trivializing the bundle E in a neighborhood of $(L, p) \in \Phi$ and writing everything in local coordinates, show that the section of E coming from the polynomial $X_1^5 + X_0^4X_2 + X_0^2X_1^2X_3$ has a simple zero at (L, p).

Note, by the way, that we are proving a general quintic has exactly 575 such flex lines by looking at a singular surface that in fact has infinitely many! In any event, we have now established the

Theorem 13.8. *If $S \subset \mathbb{P}^3$ is a general quintic surface, there will be exactly 575 lines meeting S in only one point.*

Exercise 13.9. Say that a point p on a smooth surface $S \subset \mathbb{P}^3$ is a *flat point* of S if the intersection $S \cap \mathbb{T}_p S$ has a triple point at p.

(a) Show that a general surface $S \subset \mathbb{P}^3$ of degree d has no flat points, and that the locus of smooth surfaces that do is an open subset of an irreducible hypersurface $\Psi \subset \mathbb{P}^{\binom{d+3}{3}-1}$ in the space of all surfaces.

(b) Find the degree of the hypersurface Ψ.

13.2 The Case of Negative Expected Dimension

In this section, we'll describe an application of the contact calculus developed here to another class of enumerative problems—one we haven't encountered up to now. To start, we'll take a moment for a discussion of the sort of enumerative problems that are susceptible to exact answers, and those that aren't.

13.2.1 Predestination versus Free Will

The reader has by now come to appreciate the fact that the number of solutions of some systems of polynomial equations is determined by their structure. A cubic surface, for example, must contain 27 lines, properly counted. There is an escape clause: if the locus of lines is positive-dimensional, enumerative geometry a priori says nothing about that locus (though as we'll see in Chapter **excess int**, there are still some constraints). But if it has finitely many, the number must be 27—it's preordained.

On the other hand, all bets are off in case the expected dimension of the solution set is negative—that is, if by a dimension count we expect no solutions at all. In this situation, the system of equations is apparently free to have any number of solutions it likes. For example, we don't expect a surface $S \subset \mathbb{P}^3$ of degree $d \geq 4$ to contain any lines, and indeed a general such surface doesn't. But some quartics do and some contain several; in these circumstances, in other words, a quartic is seemingly free to contain as many lines as it likes.

Well, not entirely: letting $\mathbb{P}^N$ be the space of surfaces of degree $d \geq 4$, since the incidence correspondence

$$\Phi = \{(S, L) : L \subset S\} \subset \mathbb{P}^N \times \mathbb{G}(1, 3)$$

is a closed subset of $\mathbb{P}^N \times \mathbb{G}(1, 3)$, its fibers over $\mathbb{P}^{34}$ can have only finitely many cardinalities. (We should remark that a surface in $\mathbb{P}^3$ of degree $d > 2$

that is swept out by lines is necessarily singular, so over the open set $U \subset \mathbb{P}^N$ of smooth surfaces all fibers are finite.)

From the above, we see that for each d there are only finitely many m such that there exists a surface $S \subset \mathbb{P}^3$ of degree d containing exactly m lines; and we can ask what those values of m are. In particular, we can pose the natural question:

Question 13.10. *What is the largest number $M(d)$ of lines that a smooth surface $S \subset \mathbb{P}^3$ of degree d can have?*

Remarkably, we don't know the answer to this for general d, and the enumerative methods we've developed thus far are seemingly mute on the subject. But we do have a way of at least bounding the number $M(d)$ using our techniques, and we'll describe this in the next subsection.

In fact, the situation here is very typical: there is a large range of quasi-enumerative problems where the actual number is indeterminate because the expected dimension of the solution set is negative. In general, almost every time we have an enumerative problem there are analogous "negative-expected-dimension" variants—how many conics in $\mathbb{P}^2$ can be tangent to each of 6 conics; how many conics in $\mathbb{P}^3$ can meet each of 9 lines, and so on. These are not always interesting, but many are: for example, we can ask

Question 13.11. (a) *How many (isolated) singular points can a hypersurface $X \subset \mathbb{P}^n$ of degree d have?*

(b) *How many tritangents can a plane curve $C \subset \mathbb{P}^2$ of degree d have? How many hyperflexes?*

(c) *How many cuspidal curves can a pencil of plane curves of degree d have? How many reducible ones?*

We can even go all the way back to Bezout, and ask: How many isolated points of intersection can $n+1$ linearly independent hypersurfaces of degrees d in $\mathbb{P}^n$ have? Again, this is not known, though there is a conjectured answer, described in Eisenbud et al. [1996]. We'll come back to this in subsection 13.2.3.

****do you have the Lazarsfeld reference?****

13.2.2 Lines on surfaces of degree $d \geq 4$ in $\mathbb{P}^3$

As we said, the exact value of the maximum possible number $M(d)$ of lines on a smooth surface $S \subset \mathbb{P}^3$ of degree d isn't known. But we can use intersection theory to give us an upper bound on the number, as we'll show now.

The trick is to solve a related enumerative problem—one whose solutions do occur in the expected (non-negative) dimension, and that tells us something about the locus of lines on S. That problem is to describe the *flecnodal curve* $C \subset S$ of S, which we'll describe now.

First of all, we'll say that a point $p \in S$ is *flecnodal* if there exists a line $L \subset \mathbb{P}^3$ having contact of order 4 or more with S at p; let $\mathcal{F} \subset S$ be the locus of such points. (The reason for the name comes from the analog of Exercise 13.5 above: for a general surface S, a general flecnodal point $p \in S$ will be one such that the intersection $S \cap \mathbb{T}_p S$ has a flex at p, with one branch of the node having a flex; such a node was classically called a *flecnode*.) The expected dimension of the flecnodal locus is 1 (if this is not apparent, it will be when we do the enumerative calculation), and a crucial fact is the

Proposition 13.12. *If $S \subset \mathbb{P}^3$ is any smooth surface of degree $d \geq 3$, the flecnodal locus is a curve; that is, the general point of S is not flecnodal.*

To describe the locus of flecnodes on S, we let as before Φ be the incidence correspondence

$$\Phi = \{(L, p) : p \in L\} \subset \mathbb{G}(1, 3) \times \mathbb{P}^3;$$

we let ζ and $\sigma_1 \in A^1(\Phi)$ be the pullbacks of the corresponding classes on $\mathbb{P}^3$ and $\mathbb{G}(1, 3)$ as before. This time, though, we focus on a slightly different bundle: we let F be the bundle of rank 4 on Φ whose fiber at a point (L, p) are sections of $\mathcal{O}_L(d)$ modulo those vanishing to order at least 4 at p; that is,

$$F_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathcal{I}^4_{p,L}(d)).$$

If $A \in H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ is the homogeneous polynomial of degree d defining the surface S then the restrictions of A to each line $L \subset \mathbb{P}^3$ yield a global section σ_A of the bundle F, whose zeroes are exactly the pairs (L, p) with $m_p(L \cdot S) \geq 4$. Thus we expect the locus Γ of such pairs to be one-dimensional, and if it is its class will be given as the top Chern class

$$[\Gamma] = c_4(F) \in A^4(\Phi).$$

We calculate this class as before: filtering the bundle F by order of vanishing, we see that

$$c_4(F) = d\zeta \cdot ((d-2)\zeta + \sigma_1) \cdot ((d-4)\zeta + 2\sigma_1) \cdot ((d-6)\zeta + 3\sigma_1).$$

Now, suppose we want to find the degree of the curve $C \subset S$ of flecnodal points; that is, the image of Γ under the projection to $\mathbb{P}^3$. This (assuming the projection is generically one-to-one) is just the intersection of Γ with the class ζ, which we can evaluate as before:

$$\begin{aligned} \deg(C) &= d\zeta^2 \cdot ((d-2)\zeta + \sigma_1) \cdot ((d-4)\zeta + 2\sigma_1) \cdot ((d-6)\zeta + 3\sigma_1) \\ &= d(11d - 24). \end{aligned}$$

In any event, we can conclude—given Proposition 13.12—that *the locus of flecnodes on S is a curve of degree at most $d(11d-24)$*. If $L \subset S$ is any line contained in S then $(L,p) \in \Gamma$ for every $p \in L$—in other words, the flecnodal curve C contains the union of all lines lying in S, and we have proved the

Proposition 13.13. *The number of lines lying on a smooth surface $S \subset \mathbb{P}^3$ of degree $d \geq 3$ is at most $d(11d-24)$.*

This argument was originally given by ****I thought this was done by Ziv Ran, but I can't find it under his name in MathSciNet. Do you know who did it, and where?****. Note that when $d = 3$ the number is 27, which is exactly right: on a cubic surface, a point is flecnodal if and only if it lies on a line of the surface. The bound is not sharp in general: for $d = 4$ it yields the bound $M(4) \leq 80$, but it's known that the maximum number of lines on a smooth quartic surface is 64.

Exercise 13.14. For $S \subset \mathbb{P}^3$ a general surface of degree d, find the degree of the surface swept out by the lines in $\mathbb{P}^3$ having a point of contact of order at least 4 with S.

Exercise 13.15. Let $S \subset \mathbb{P}^3$ be a general surface of degree d.

(a) Find the first Chern class of the bundle F above.

(b) Show that the curve Γ is smooth, and that the projection $\Gamma \to C$ is generically one-to-one.

(c) Using the preceding parts, find the genus of the curve Γ.

(d) Show, on the other hand, that the flecnodal curve of S is the intersection of S with a surface of degree $11d-24$, and use this to calculate the arithmetic genus of C.

(e) Can you describe the singularities of the curve C? Do these account for the discrepancy between the genera of Γ and of C?

Exercise 13.16. Let $\mathbb{P}^N$ be the space of surfaces of degree $d \geq 4$ and $\Psi \subset \mathbb{P}^N$ the locus of surfaces containing a line. Show that the maximum possible number $M(d)$ of lines on a smooth surface $S \subset \mathbb{P}^3$ of degree d is at most the degree of Ψ, by considering the pencil spanned by S and a general second surface T. Is this bound better or worse than the one we've just derived?

13.2.3 Other configurations with negative expected dimension

We return now to some of the other problems posed in Question 13.11.

To begin with, in many cases it's easy to write down an upper bound for the number in question. For example, as we saw in Chapter **, a hyperflex counts for at least two flexes; thus the number of hyperflexes on a smooth plane curve of degree d is bounded above by $3d(d-2)/2$. (Of course, this is highly unlikely to be sharp, though it is for $d = 4$: there do exist plane quartics with 12 hyperflexes, like the Fermat quartic $X^4 + Y^4 + Z^4 = 0$.) Similarly, the number of tritangents is bounded above by one-third the number of bitangents, though in this case we don't even know if the bound has the right order of growth: the number of bitangents grows as a polynomial of degree 4 in d, while the largest known number of tritangent to a smooth plane curve of degree d so far is only quadratic in d.

Other cases may require more cleverness. For example, to bound the number of singular points of a surface $S \subset \mathbb{P}^3$ with isolated singularities, we can consider the dual surface. For a smooth surface of degree d, the dual will have degree $d(d-1)^2$; but this number drops by at least 2 for each isolated singular point of a surface, since these points are base points of the linear series spanned by the partial derivatives of the defining equation of S.

Exercise 13.17 (For those with an interest in the topology of surfaces)**.** Let $S \subset \mathbb{P}^3$ be a surface of degree d with ordinary nodes, and $\tilde{S}$ the resolution of singularities of S, obtained by blowing up S at its nodes. Show that the exceptional divisors of the resolution are linearly independent in $\mathrm{Pic}(\tilde{S})$, and use the calculation in Chapter ** of the Euler characteristic of S to obtain a bound on the number of nodes.

13.3 Flexes, again

****The last question of Chapter9 has now moved to this chapter.**** In Chapter9, we considered the *flexes* of a plane curve $C \subset \mathbb{P}^2$, and by means of a calculation based on C (or, rather, its normalization $\tilde{C}$) we counted their number. Here we'll use the tools developed in this and the last Chapter to answer the question another way—a way that will allow us to answer Keynote Question c

13.3.1 The Cartesian view of flexes

In our initial discussion of flexes in Section **ref**, we gave the curve C in question *parametrically*—that is, as the image of a map $\nu : \tilde{C} \to \mathbb{P}^2$ from a smooth curve $\tilde{C}$ to $\mathbb{P}^2$. We defined flexes likewise in these terms, as points $p \in \tilde{C}$ such that for some line $L \subset \mathbb{P}^2$ the multiplicity $m_p(\nu^* L) \geq 3$.

In the present discussion, we'll take a different tack: we'll give the curve in *Cartesian* form—that is, as the zero locus of a homogeneous polynomial F on $\mathbb{P}^2$—and we'll define flexes accordingly. This means that we'll define a *flex* of C to be a point $p \in \mathbb{P}^2$ and a line $L \subset \mathbb{P}^2$ through p such that the restriction of F to L vanishes to order 3 or more at p; that is, the set Γ of flexes is the locus

$$\Gamma = \{(p, L) : \text{mult}_p(C \cdot L) \geq 3\} \subset \mathbb{P}^2 \times \mathbb{P}^{2^*}.$$

The two notions of flexes agree when the curve C is smooth, but differ when C has singularities: for example, if C is a general curve with a node at p, the tangent lines to the two branches will appear as flexes in our new, Cartesian definition, but not in the sense of Section **ref**. (This necessitates, unfortunately, a different name: we'll call the objects described here "flex lines" rather than flexes.)

The calculation of the number of flex lines of a plane curve C is a variant of the one we used in subsections (13.1.1)-(13.1.3). To begin with, we take Φ to be the incidence correspondence

$$\Phi = \{(L, p) : p \in L\} \subset (\mathbb{P}^2)^* \times \mathbb{P}^2,$$

and introduce the contact bundle E on Φ with fiber

$$E_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathcal{I}^3_{p,L}(d)).$$

As in that setting, a homogeneous polynomial F of degree d on $\mathbb{P}^2$ gives rise to a section σ_F of E, and that the zeroes of this section correspond to the flex lines of the corresponding plane curve $C = V(F)$; thus the number we seek is $c_3(E) \in A^3(\Phi) \cong \mathbb{Z}$.

We start by writing down the Chow ring of Φ: first, let $\sigma \in A^1(\mathbb{P}^{2^*})$ be the hyperplane class on $\mathbb{P}^{2^*}$, and its pullback to Φ. Now, since the projection on the first factor expresses Φ as the projectivization

$$\Phi = \mathbb{P}S \to \mathbb{P}^{2^*}$$

of the universal subbundle on $\mathbb{P}^{2^*}$, and we have

$$c_1(S) = -\sigma \quad \text{and} \quad c_2(S) = \sigma^2,$$

we have by Theorem 11.4

$$A^*(\Phi) = \mathbb{Z}[\sigma, \zeta]/(\sigma^3, \zeta^2 - \sigma\zeta + \sigma^2).$$

Exercise 13.18. Let $\beta : \Phi \to \mathbb{P}^2$ be the projection $(L, p) \mapsto p$ on the second factor, and $\omega \in A^1(\mathbb{P}^3)$ the hyperplane class. Show that

$$\zeta = \beta^*\omega$$

and verify that $\zeta^3 = 0$ in $A^*(\Phi)$.

Next, as before we calculate the Chern classes of E by filtering it: we introduce a sequence of subbundles

$$0 = E_3 \subset E_2 \subset E_1 \subset E_0 = E$$

where the fiber of E_i at (L, p) is the subspace $H^0(\mathcal{I}^i_{p,L}(d)/\mathcal{I}^3_{p,L}(d)) \subset H^0(\mathcal{O}_L(d)/\mathcal{I}^3_{p,L}(d))$. The successive quotients E_{i+1}/E_i will be

$$E_i/E_{i+1} = \mathrm{Sym}^i(T^v_{\Phi/\mathbb{P}^{2*}})^* \otimes \beta^*\mathcal{O}_{\mathbb{P}^2}(d),$$

so that

$$c_1(E_i/E_{i+1}) = (d-2i)\zeta + i\sigma$$

and altogether

$$\begin{aligned} c_3(E) &= d\zeta \cdot ((d-2)\zeta + \sigma) \cdot ((d-4)\zeta + 2\sigma) \\ &= 2d(d-2) + d(d-4) + 2d \\ &= 3d(d-2). \end{aligned}$$

As we indicated, the Cartesian approach is useful when we're considering a family of curves, as in the last keynote problem of Chapter 6. To recall the situation, we can define a *hyperflex* of a plane curve $C \subset \mathbb{P}^2$ is a line with a point of contact of order at least 4 with C, that is, a pair (L, p) such that $m_p(L \cdot C) \geq 4$. Denote by $\mathbb{P}^N$ the space of plane curves of degree d, and consider the incidence correspondence

$$\Psi = \{(C, L, p) : m_p(L \cdot C) \geq 4\} \subset \mathbb{P}^N \times \Phi.$$

When $d \geq 3$, the fibers of the projection $\Psi \to \Phi$ are linear spaces of dimension $N-4$, from which we see that Ψ is irreducible of dimension $N-1$; in particular, it follows that *a general curve $C \subset \mathbb{P}^2$ of degree $d > 1$ has no hyperpflexes*; since for $d \geq 4$ the general fiber of the projection $\Psi \to \mathbb{P}^N$ is finite **exercise?** we also see that in this case the locus $\Xi \subset \mathbb{P}^N$ of curves that do admit a hyperflex is a hypersurface. In this context, the last keynote question of Chapter 6—if $\{C_t = V(t_0F + t_1G)\}_{[t_0,t_1]\in\mathbb{P}^1}$ is a general pencil of quartic plane curves, how many will have hyperflexes?—asks for the degree of this hypersurface in case $d = 4$.

To answer it, we introduce another contact bundle H on Φ, this time of rank 4: we take

$$H_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathcal{I}^4_{p,L}(d)).$$

The pair F, G of homogeneous polynomials of degree d on $\mathbb{P}^2$ defining our pencil gives rise to a pair of sections σ_F, σ_G of H, and the set of pairs (L, p) that are hyperflexes of some element of our pencil is just the locus where these sections fail to be linearly independent; to count their number we want to evaluate $c_3(H) \in A^3(\Phi) \cong \mathbb{Z}$.

We do this exactly as before: filtering the bundle H by order of vanishing, we arrive at the expression

$$c(H) = \big(1 + d\zeta\big)\big(1 + (d-2)\zeta + \sigma\big)\big(1 + (d-4)\zeta + 2\sigma\big)\big(1 + (d-6)\zeta + 3\sigma\big).$$

In particular,

$$\begin{aligned} c_3(H) = {} & d\zeta\big((d-2)\zeta + \sigma\big)\big((d-4)\zeta + 2\sigma\big) \\ & + d\zeta\big((d-2)\zeta + \sigma\big)\big((d-6)\zeta + 3\sigma\big) \\ & + d\zeta\big((d-4)\zeta + 2\sigma\big)\big((d-6)\zeta + 3\sigma\big) \\ & + \big((d-2)\zeta + \sigma\big)\big((d-4)\zeta + 2\sigma\big)\big((d-6)\zeta + 3\sigma\big) \end{aligned}$$

and collecting terms we have

$$\begin{aligned} c_3(H) &= \big(18d^2 - 88d + 72\big)\zeta^2\sigma + \big(22d - 36\big)\zeta\sigma^2 \\ &= 18d^2 - 66d + 36 \\ &= 6(d-3)(3d-2). \end{aligned}$$

Note that this gives zero when $d = 3$, as it should: a cubic with a hyperflex is necessarily reducible, and a general pencil of plane cubics won't include any reducible ones.

Exercise 13.19. Verify that for a general pencil $\{C_t = V(t_0F + t_1G)\}_{[t_0,t_1]\in\mathbb{P}^1}$ of plane curves of degree d, if (L, p) is a hyperflex of some element C_t of the pencil then

(a) $m_p(C_t \cdot L) = 4$; that is, no line has a point of contact of order 5 or more with any element of the pencil;

(b) p is a smooth point of C_t; and

(c) p is not a base point of the pencil.

Using these facts, show that the degeneracy locus of the sections σ_F and σ_G of H is reduced.

From this last exercise, combined the preceding calculation, we deduce our answer to the keynote question: *in a general pencil of plane quartic curves, exactly 60 members will have hyperflexes.* It's worth observing that the approach to counting flexes on a plane curve C taken in Chapter 6—introducing a bundle on C whose fiber at a point $p \in C$ is the space $H^0(\mathcal{O}_C(1)/\mathcal{I}^3_{p,C}(1))$—can't be readily adapted to answer this question: we can form a bundle on the total space

$$\mathcal{C} = \{(t, p) : p \in C_t\} \subset \mathbb{P}^1 \times \mathbb{P}^2$$

whose fiber at a point $(t, p) \in \mathcal{C}$ with C_t smooth at p is the space $H^0(\mathcal{O}_{C_t}(1)/\mathcal{I}^4_{p,C_t}(1))$, but this bundle doesn't extend over the points $(t, p) \in \mathcal{C}$ where p is a singular point of C_t.

We can also use the Cartesian approach to answer another question about flexes in pencils, one that shed some more light on how flexes behave in families. Again, suppose that $\{C_t = V(t_0F + t_1G)\}_{[t_0,t_1]\in\mathbb{P}^1}$ is a general pencil of plane curves of degree d. The general member C_t of the pencil will have, as we've seen, $3d(d-2)$ flex points, and as t varies these points will sweep out another curve B in the plane. We can ask: what are the degree and genus of this curve? What is the geometry of this curve around singular points of curves in the pencil?

To answer these, we need to look at the product $\mathbb{P}^1 \times \Phi$, and at the curve

$$\Gamma = \{(t, L, p) : m_p(L \cdot C_t) \geq 3\} \subset \mathbb{P}^1 \times \Phi.$$

We can describe Γ as the (transverse) zero locus of a section of a vector bundle on $\mathbb{P}^1 \times \Phi$; this will allow us to determine not only its class (which will give us the degree of its image B under the projection $\mathbb{P}^1 \times \Phi \to \Phi \to \mathbb{P}^2$) but its genus as well.

The set-up here is similar to that of Section **ref** of Chapter 5. First, we let E be the contact bundle on Φ as before. If V is the two-dimensional vector space spanned by F and G then we have a map of bundles

$$V \otimes \mathcal{O}_\Phi \to E.$$

We now pull this map back to $\mathbb{P}^1 \times \Phi$ via the projection $\nu : \mathbb{P}^1 \times \Phi \to \Phi$. If $\mathbb{P}^1 = \mathbb{P}V$ is the projective line parametrizing our pencil, we also have a natural inclusion

$$\mathcal{O}_{\mathbb{P}V}(-1) \hookrightarrow V \otimes \mathcal{O}_{\mathbb{P}V}$$

which we can pull back to the product $\mathbb{P}^1 \times \Phi$ via the projection $\mu : \mathbb{P}^1 \times \Phi \to \mathbb{P}^1$. Composing these, we arrive at a map

$$\rho : \mu^*\mathcal{O}_{\mathbb{P}^1}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^1\times\Phi} \to \nu^*E;$$

over the point $(t, L, p) \in \mathbb{P}^1 \times \Phi$, this is just the map that takes a scalar multiple of t_0F+t_1G to its restriction to L (mod sections of $\mathcal{O}_L(d)$ vanishing to order 3 at p) . In particular, *the zero locus of this map is just the incidence correspondence* Γ.

Now, tensoring with the line bundle $\mu^*\mathcal{O}_{\mathbb{P}^1}(1)$, we can think of ρ as a section of the bundle $\mu^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^*E$; the class of Γ is thus given by the Chern class

$$[\Gamma] = c_3\big(\mu^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^*E\big) \in A^3(\mathbb{P}^1 \times \Phi).$$

Denoting by $\eta \in A^1(\mathbb{P}^1)$ the class of a point, and suppressing the symbols μ^* and ν^*, we have

$$c\big(\mu^*\mathcal{O}_{\mathbb{P}^1}(1)\otimes\nu^*E\big) = \big(1+\eta+d\zeta\big)\big(1+\eta+(d-2)\zeta+\sigma\big)\big(1+\eta+(d-4)\zeta+2\sigma\big);$$

in particular,

$$\begin{aligned} c_3(\mu^*\mathcal{O}_{\mathbb{P}^1}(1)\otimes\nu^*E) &= \big(\eta+d\zeta\big)\big(\eta+(d-2)\zeta+\sigma\big)\big(\eta+(d-4)\zeta+2\sigma\big) \\ &= (3d^2-8d)\zeta^2\sigma+2d\zeta\sigma^2 \\ &\quad +\eta\big((3d^2-12d+8)\zeta^2+(6d-8)\zeta\sigma+2\sigma^2\big). \end{aligned}$$

We can recover the number of flexes on a general curve from this by intersecting with the class η of a fiber; since $\eta^2=0$ while the degrees of $\eta\zeta^2\sigma$ and $\eta\zeta\sigma^2$ are 1, this yields our original answer $(3d^2-8d)+2d=3d(d-2)$. To find the degree of the curve $B\subset\mathbb{P}^2$ swept out by the flex points of members of the family, we intersect with the (pullback of the) class ζ of a line $L\subset\mathbb{P}^2$; we get

$$\begin{aligned} \deg(B) &= \zeta\cdot c_3(\mu^*\mathcal{O}_{\mathbb{P}^1}(1)\otimes\nu^*E) \\ &= 6d-6 \end{aligned}$$

Note that this yields the answer 6 in case $d=2$, which makes sense: a smooth conic has no flexes, while a singular one consists of two lines and so is comprised of nothing but; thus the flex points of a general pencil of conics form the union of the three singular elements of the pencil.

Exercise 13.20. Let $p\in\mathbb{P}^2$ be a general point. How many flex lines to members of our pencil $\{C_t\}$ pass through p?

Next, we can use the same set-up to find the genus of the curve Γ: as we observed in Chapter 5, the normal bundle to Γ in the product $\mathbb{P}^1\times\Phi$ is just the restriction to Γ of the bundle $\mu^*\mathcal{O}_{\mathbb{P}^1}(1)\otimes\nu^*E$. Now, since $\Phi\subset\mathbb{P}^{2^*}\times\mathbb{P}^2$ is a hypersurface of bidegree $(1,1)$, its canonical class is

$$-c_1(T_\Phi)=K_\Phi=K_{\mathbb{P}^{2^*}\times\mathbb{P}^2}+\zeta+\sigma=-2\zeta-2\sigma;$$

it follows that

$$K_{\mathbb{P}^1\times\Phi}=-2\eta-2\zeta-2\sigma.$$

By the calculation above,

$$c_1(E)=3\eta+(3d-6)\zeta+3\sigma$$

and so we have

$$K_\Gamma=\big(\eta+(3d-8)\zeta+\sigma\big)|_\Gamma.$$

Now, we've seen that the degree of $\eta|_\Gamma$ is $3d(d-2)$, and $\deg(\zeta|_\Gamma)=6d-6$; similarly, we can calculate

$$\deg(\sigma|_\Gamma)=(3d^2-12d+8)+(6d-8)=3d^2-6d$$

Altogether, we have

$$\begin{aligned} 2g(\Gamma)-2 &= \deg(K_\Gamma) \\ &= 3d(d-2)+(3d-8)(6d-6)+3d(d-2) \\ &= 24d^2-78d+48 \end{aligned}$$

and so

$$g(\Gamma) = 12d^2 - 39d + 25.$$

In fact, we can see this in two other ways, and in so doing shed some light on the geometry of flexes in families; we'll outline these in the following Exercises. The first has to do with the geometry of the plane curve B traced out by the flex points of the curves C_t—that is, the image of the curve Γ under projection to $\mathbb{P}^2$.

Exercise 13.21. First, show that Γ is indeed smooth, by showing that the "universal flex"

$$\Psi = \{(C, L, p) : m_p(C \cdot L) \geq 3\} \subset \mathbb{P}^N \times \Phi$$

(where $\mathbb{P}^N$ is the space parametrizing all plane curves of degree d) is smooth, and invoking Bertini. Can you give explicit conditions on the pencil equivalent to the smoothness of Γ?

Exercise 13.22. Show that the curve B has an ordinary triple point at each base point of the pencil $\{C_t\}$, and an ordinary double point at each singular point of a curve C_t. Show that these are all the singularities of the curve B, and rederive the formula above for the genus of Γ (that is, the geometric genus of B).

The second way of looking at the curve Γ is as a cover of the line $\mathbb{P}^1$ parametrizing our pencil, of degree $3d(d-2)$. The idea is to observe that this cover is branched only over points $t \in \mathbb{P}^1$ such that C_t is singular, or has a hyperflex, describe the branching there, and apply Riemann-Hurwitz.

Exercise 13.23. Let C_t be an element of our pencil with a hyperflex (L, p). Show that the map $\Gamma \to \mathbb{P}^1$ is simply ramified at (t, L, p), and simply branched at t.

Exercise 13.24. Let C_t be an element of our pencil with a node p; let L_1 and L_2 be the tangent lines to the two branches of C_t at p. Show that $(t, p, L_i) \in \Gamma$, and that these are ramification points of weight 2 of the map $\Gamma \to \mathbb{P}^1$ (that is, each of the lines L_i is a limit of three flex lines of nearby smooth curves in our pencil, and these three are cyclically permuted by the monodromy in the family). Conclude that t is a branch point of multiplicity 4 for the cover $\Gamma \to \mathbb{P}^1$.

Exercise 13.25. Verify once again the formula above for the genus of the curve Γ by applying Riemann-Hurwitz to the map $\Gamma \to \mathbb{P}^1$.

It's worth going back to Exercise **ref** of Chapter 6, where we observed that the nodal plane curves in our pencil each have six fewer flexes than the smooth ones, and asked what happened to the other six; based on the Exercises above, we now have a pretty good idea of the answer.

13.4 Cusps of plane curves

As a final application of some of the ideas introduced in this Chapter and earlier ones, we will work out the answer to the second keynote question of this Chapter: if S is a smooth surface, L a line bundle on S and $\mathcal{D} \subset |L|$ a two-dimensional linear system of curves, how many of the curves $C \in \mathcal{D}$ will have cusps?

The first thing we need to do, naturally, is to say what we mean by a cusp. Suppose S is a surface, $p \in S$ a smooth point and $C \subset S$ a curve given in a neighborhood of p as the zero locus of a regular function $f \in \mathcal{O}_{S,p}$. Assume that C is singular at p, that is, $f \in m_p^2$ is in the square of the maximal ideal at p. We consider the class $\overline{f}$ of f mod m_p^3; that is,

$$\overline{f} \in m_p^2/m_p^3 = \mathrm{Sym}^2(m_p/m_p^2) = \mathrm{Sym}^2 T_p^* S.$$

We will say that C *has a cusp at* p if $f \in m_p^2$, and $\overline{f}$ is a square in $\mathrm{Sym}^2 T_p^* S$; more specifically, if $\xi \subset T_p S$ is a tangent line to S at p—that is, a one-dimensional subspace of $T_p S$—we say that C *has a cusp at* p *with tangent line* ξ if $\overline{f}$ is a multiple of the square of the linear form on $T_p S$ vanishing on ξ (note that in both cases $\overline{f}$ may be 0). If $\xi = T_p D$ is the tangent line to a curve $D \subset S$ through p, we can express this condition by saying that

$$f \in m_p^3 + I_D^2 \subset \mathcal{O}_{S,p}.$$

Observe that we are, by this definition, including as "honorary cusps" singularities that are specializations of what is normally called a cusp (that is, singularities locally analytically isomorphic to $y^2 - x^3$, which we will call *traditional cusps*): higher-order double points, such as tacnodes and ramphoid cusps, are included, as are all triple points. We do this so as to make the condition of "having a cusp" a closed one. It's a relatively harmless deviation from the regular definition, since a general net in a sufficiently ample linear series on a smooth surface will have no members satisfying our condition except those possessing a traditional cusp, as the following exercise asks you to verify for plane curves:

Exercise 13.26. Let $\mathbb{P}^N$ be the projective space parametrizing plane curves of degree d; let $A \subset \mathbb{P}^N$ be the locus of curves having a point of multiplicity 3 or greater, or a double point locally analytically isomorphic to $y^2 - x^n$ for $n \geq 4$, or y^2. Show that A has codimension 3 in $\mathbb{P}^N$, and conclude that if $\mathcal{D} = \{C_t = V(t_0 F + t_1 G + t_2 H)\}_{t \in \mathbb{P}^2}$ is a general net of plane curves of degree d, the only singularities of members of $\mathcal{D}$ are nodes and traditional cusps.

(The statement for a general net in the third or higher power of a very ample linear series on a smooth surface is proved similarly, by introducing

an incidence correspondence and estimating the dimension of the locus of curves having each type of singularity at a specified point $p \in S$)

As in the case of the much simpler problem of counting singular elements of a pencil of curves, the first thing we need to do is linearize the problem. The difficulty arises from the fact that even after we specify a point $p \in S$, it's not a linear condition on the curves in our linear system to have a cusp at p. *It becomes linear, though, if we specify both the point p and the tangent line to S at p along which we require our curve to have a cusp.* We have to introduce, accordingly, the space of tangent directions to S, which is just the projectivization of the tangent bundle T_S to S: we set

$$B = \mathbb{P}T_S = \{(p, \xi) : \xi \subset T_pS \text{ is a 1-dimensional subspace}\}.$$

As always, Theorem 11.4 gives us a description of the Chow ring $A^*(B)$. By way of notation, we'll introduce symbols for the classes we're going to be dealing with: we'll write $\alpha = c_1(L)$, $\omega = c_1(T_S^*)$ and $\chi = c_2(T_S^*)$; as usual, we let $\zeta = c_1(\mathcal{O}_{\mathbb{P}T_S}(1))$, so that we have

$$A^*(B) = A^*(S)[\zeta]/(\zeta^2 - \omega\zeta + \chi = 0)$$

Next, for a given point $(p, \xi) \in B$, we want to express the condition that the curve $C = V(\sigma)$ associated to a section σ of a line bundle L on S have a cusp at p in the direction ξ. This suggests that we introduce for each (p, ξ) the ideal $J_{(p,\xi)}$ of functions whose zero locus has such a cusp; that is, we set

$$J_{(p,\xi)} = m_p^3 + I_D^2$$

where D is any smooth curve through p with tangent line ξ (equivalently, if we think of the tangent direction ξ to S at p as a subscheme $\Gamma \subset S$ of degree 2 supported at p, we can take $J_{(p,\xi)} = m_p^3 + I_\Gamma^2$). Now, we let E be the vector bundle on B whose fiber at a point (p, ξ) is

$$E_{(p,\xi)} = H^0(L/L \otimes J_{(p,\xi)}).$$

A global section of the line bundle L gives rise to a section of E by restriction. Given a net $\mathcal{D}$ corresponding to a three-dimensional vector space $V \subset H^0(L)$ we get three sections of E, and the locus in B where they fail to be independent—that is, where some linear combination is zero—is the locus of (p, ξ) such that some element of the net has a cusp at p in the direction ξ. In sum, the answer to our question is the Chern class $c_3(E)$, and in the remainder of this section will calculate this.

The first step is to relate the bundle E to a bundle whose Chern classes we know. What we'll do is to express E as a quotient of a simpler (if higher-rank) bundle: we let F be the rank 6 bundle on B with fiber

$$F_{(p,\xi)} = H^0(L/L \otimes m_p^3).$$

(Note that F is in fact a pullback from S.) We have seen how to find the Chern classes of F in Chapter **: we consider the filtration

$$0 \subset F_2 \subset F_1 \subset F_0 = F$$

where

$$(F_i)_{(p,\xi)} = H^0(L \otimes m_p^i / L \otimes m_p^3).$$

The successive quotients F/F_1, F_1/F_2 and F_2 are, respectively, the line bundle L; the tensor product $L \otimes T_S^*$ and the tensor product $L \otimes \operatorname{Sym}^2 T_S^*$. Using the splitting principle, we can calculate the Chern class of each of these bundles to arrive at

$$c(F) = 1 + (4\omega + 6\alpha) + (15\alpha^2 + 20\alpha\omega + 5\omega^2 + 5\chi).$$

How does the bundle F relate to E? We have a quotient map $F \to E$, whose kernel is the line bundle M on B with fibers

$$M_{(p,\xi)} = H^0(L \otimes J_{(p,\xi)} / L \otimes m_p^3).$$

Now, the quotient

$$J_{(p,\xi)}/m_p^3 \subset m_p^2/m_p^3 = \operatorname{Sym}^2(T_pS^*)$$

is just the one-dimensional space of squares of linear forms vanishing on $\xi \subset T_pS$; that is, it's the square of the dual of the quotient $Q = T_pS/\xi$. From the sequence

$$0 \to S \to \pi^* T_S \to Q \to 0$$

we have

$$c_1(Q) = \zeta - \omega,$$

so that $c_1(\operatorname{Sym}^2 Q^*) = 2\omega - 2\zeta$, and hence

$$c_1(M) = 2\omega - 2\zeta + \alpha.$$

We have

$$\begin{aligned} c(E) &= \frac{c(F)}{c(M)} \\ &= \frac{1 + (4\omega + 6\alpha) + (15\alpha^2 + 20\alpha\omega + 5\omega^2 + 5\chi)}{1 + 2\omega - 2\zeta + \alpha} \\ &= 1 + (4\omega + 6\alpha) + (15\alpha^2 + 20\omega\alpha + 5\omega^2 + 5\chi) \\ &\quad \cdot \big(1 - (2\omega - 2\zeta + \alpha) + (2\omega - 2\zeta + \alpha)^2 - (2\omega - 2\zeta + \alpha)^3\big) \end{aligned}$$

The third Chern class of E is thus

$$\begin{aligned} c_3(E) &= -(2\omega - 2\zeta + \alpha)^3 + (2\omega - 2\zeta + \alpha)^2(4\omega + 6\alpha) \\ &\quad - (2\omega - 2\zeta + \alpha)(15\alpha^2 + 20\alpha\omega + 5\omega^2 + 5\chi) \\ &= 8\omega^2 - 8\chi \\ &\quad - 16\omega^2 - 40\alpha\omega - 24\alpha^2 \\ &\quad + 10\omega^2 + 10\chi + 40\alpha\omega + 30\alpha^2 \\ &= 12\alpha^2 + 12\alpha\omega + 2\omega^2 + 2\chi. \end{aligned}$$

In sum, *the number of cuspidal elements of a net $\mathcal{D} \subset |L|$ of curves on a surface S is*

$$12c_1(L)^2 - 12c_1(L)c_1(T_S) + 2c_1(T_S)^2 + 2c_2(T_S).$$

As always, this number is subject to the usual caveats: it is meaningful only if the number of cuspidal curves in the net is in fact finite; and in this case it represents the number of cuspidal curves counted with multiplicity (with multiplicity defined as the degree of the component of the zero-scheme of the corresponding section of E supported at (p, ξ)). In the special case of plane curves of degree d, we get the answer $12d^2 - 36d + 24$; note that this yields 0 in the cases $d = 1$ and 2, as it should.

13.4.1 Another approach to the cusp problem

There is another approach to the problem of counting cuspidal curves in a linear system that gives a beautiful picture of the geometry of nets. It's not part of the overall logical structure of this book, so we'll run through the sequence of steps involved without proof; the reader who's interested can view supplying the verifications as an extended exercise.

To begin with, let S be a smooth projective surface and L a very ample line bundle; let $\mathcal{D} \subset |L|$ be a general two-dimensional sub-series, corresponding to the 3-dimensional vector subspace $V \subset H^0(L)$. We have a natural map

$$\varphi : S \to \mathbb{P}^2 = \mathbb{P}V^*$$

to the projectivization of the dual V^*; the preimages $\varphi^{-1} \subset S$ of the lines $L \subset \mathbb{P}V^*$ are the divisors $C \subset S$ of the linear system $\mathcal{D}$. If we want, we can think of the complete linear system as giving an embedding of S in the larger projective space $\mathbb{P}^n = \mathbb{P}H^0(L)^*$, and the map φ as the projection of S corresponding to the $(n-3)$-plane $\mathbb{P}\mathrm{Ann}(V)$.

Now, the geometry of generic projections of smooth varieties is well-understood in low dimensions; in this case, it's the case **ref** that

- The ramification divisor $R \subset S$ of the map φ is a smooth curve; and

- The branch divisor $B \subset \mathbb{P}V^*$ is the birational image of R, and has only nodes and ordinary cusps as singularities.

In fact, étale locally around any point $p \in S$, one of three things is true: either

i. the map is étale (if $p \notin R$);

ii. the map is simply ramified, that is, of the form $(x, y) \mapsto (x, y^2)$ (if p is a point of R not lying over a cusp of B); or

iii. the surface S is given, in terms of local coordinates (x, y) on $\mathbb{P}V^*$ around $\varphi(p)$, by the equation

$$z^3 - xz - y = 0.$$

(This is the picture around a point where 3 sheets of the cover come together; in a neighborhood of $\varphi(p)$ the branch curve is the zero locus of the discriminant $4x^3 - 27y^2$, and in particular has a cusp at $\varphi(p)$.)

Now, the interesting thing about this set-up is that we have two plane curves associated to it, lying in dual projective planes: we have

(a) the branch curve $B \subset \mathbb{P}V^*$ of the map φ, and

(b) in the dual space $\mathbb{P}V$ parametrizing the divisors in the net $\mathcal{D}$, we have the *discriminant curve* $\Delta \subset \mathbb{P}V$, that is, the locus of singular elements of the net.

What ties everything together is the observation that *the discriminant curve $\Delta \subset \mathbb{P}V$ is the dual curve of the branch curve $B \subset \mathbb{P}V^*$*. To see this, note that if $L \subset \mathbb{P}V^*$ is a line transverse to B (in particular, not passing through any of the singular points of B), then the preimage $\varphi^{-1}(L) \subset S$ will be smooth: this is certainly true away from points of B, where the map φ is étale; and at a point $p \in L \cap B$ we can take local coordinates (x, y) on $\mathbb{P}V^*$ with L given by $y = 0$ and B by $x = 0$; at a point of $\varphi^{-1}(p)$ the cover $S \to \mathbb{P}V^*$ will either be étale or given by $z^2 = x$. A similar calculation shows conversely that if L is tangent to B at a smooth point, then $\pi^{-1}(L)$ will be singular.

This, and the Plücker formulas, are all we need; all we have to do is write down everything we know about the curves R, B and Δ. To begin with, we invoke the Riemann-Hurwitz formula for finite covers $f : X \to Y$: if η is a rational canonical form on Y with divisor D, the divisor of the pullback $f^*\eta$ will be the preimage of D, plus the ramification divisor $R \subset X$; thus

$$K_X = f^*K_Y + R \in A^1(X).$$

In our present circumstances, this says that

$$K_S = \varphi^*K_{\mathbb{P}V^*} \otimes \mathcal{O}_S(R);$$

since the pullback $\varphi^*\mathcal{O}_{\mathbb{P}V^*}(1) = L$, we can write this as

$$K_S = L^{-3}(R),$$

or in terms of the notation $\omega = c_1(T_S^*)$ and $\alpha = c_1(L)$ introduced in the last section, the class of R is

$$[R] = \omega + 3\alpha \in A^1(S).$$

Among other things, this tells us the genus g of the curve R: since R is smooth, by adjunction we have

$$\begin{aligned} g &= \frac{R \cdot (R + K_S)}{2} + 1 \\ &= \frac{(\omega + 3\alpha)(2\omega + 3\alpha)}{2} + 1 \\ &= \frac{9\alpha^2 + 9\alpha\omega + 2\omega^2}{2} + 1. \end{aligned}$$

It also tells us the degree d of the branch curve $B = \varphi(R) \subset \mathbb{P}V^*$: this is the intersection of R with the preimage of a line, so that

$$d = \alpha(\omega + 3\alpha) = 3\alpha^2 + \alpha\omega.$$

Finally, we also know the degree e of the discriminant curve $\Delta \subset \mathbb{P}V$: this is the number of singular elements in a pencil, which we calculated back in Chapter **; we have

$$e = 3\alpha^2 + 2\alpha\omega + \chi.$$

Readers familiar with the Plücker formulas for plane curves will recognize that at this point we have enough information to determine the number of cusps of Δ (readers not familiar should probably become so before continuing **ref**). Let δ and κ denote the number of nodes and cusps of Δ respectively. First off, the geometric genus of Δ is given by

$$g = \frac{(e-1)(e-2)}{2} - \delta - \kappa;$$

and the degree d of the dual curve is

$$d = e(e-1) - 2\delta - 3\kappa.$$

Subtracting twice the first equation from the second yields

$$\begin{aligned} \kappa &= 2g - d + 2(e-1) \\ &= 9\alpha^2 + 9\alpha\omega + 2\omega^2 + 2 - (3\alpha^2 + \alpha\omega) + 2(3\alpha^2 + 2\alpha\omega + \chi - 1) \\ &= 12\alpha^2 + 12\alpha\omega + 2\omega^2 + 2\chi, \end{aligned}$$

agreeing with our previous calculation. Note that this method also gives us a geometric sense of when a cusp "counts with multiplicity one;" in particular, if all the hypotheses above about the geometry of the map φ are satisfied, the count is exact.

Note that this also gives us a formula for the number of curves C in the net with two nodes. This is the number δ of nodes of the curve Δ, which we get by subtracting three times the equation for g above from the equation for d: this yields

$$\begin{aligned}\delta &= d - 3g - e(e-1) + \frac{3}{2}(e-1)(e-2)\\ &= (3\alpha^2 + \alpha\omega) - \frac{3}{2}(9\alpha^2 + 9\alpha\omega + 2\omega^2 + 2)\\ &\quad + \frac{1}{2}(3\alpha^2 + 2\alpha\omega + \chi - 1)(3\alpha^2 + 2\alpha\omega + \chi - 6)\\ &= \frac{1}{2}\big[9(\alpha^2)^2 + 12(\alpha^2)(\alpha\omega) + 6(\alpha^2)\chi + 4(\alpha\omega)^2 + 4(\alpha\omega)\chi\\ &\quad - 42\alpha^2 - 39\alpha\omega - 6\omega^2 + \chi^2 - 7\chi\big].\end{aligned}$$

(Note that in this formula, when we write "$(\alpha^2)^2$" or χ^2 we mean the square of the integer α^2 or χ, not a class in $A^4(S)$!) In the case of a net of plane curves, this says that *the number of binodal curves in a general net of curves of degree d in $\mathbb{P}^2$ is*

$$\frac{1}{2}\left(9d^4 - 36d^3 + 12d^2 + 81d - 66\right).$$

Exercise 13.27. Verify the last formula

(a) in case $d = 1$ and $d = 2$, by checking that the formula yields 0; and

(b) in case $d = 3$, by finding the degree of the locus of reducible cubics in the $\mathbb{P}^9$ of all cubics.

The following three exercises describe a way of deriving the formula for the number of binodal curves in a net via linearization. We begin by introducing a smooth, projective compactification of the space of unordered pairs of points $p, q \in \mathbb{P}^2$: we set

$$\tilde{\Phi} = \{(L, p, q) : p, q \in L\} \subset \mathbb{P}^{2*} \times \mathbb{P}^2 \times \mathbb{P}^2$$

and let Φ be the quotient of $\tilde{\Phi}$ by the involution $(L, p, q) \mapsto (L, q, p)$. To put it differently, Φ consists of pairs (L, D) with $L \subset \mathbb{P}^2$ a line and $D \subset L$ a subscheme of degree 2; or differently still, Φ is the Hilbert scheme of subschemes of $\mathbb{P}^2$ with Hilbert polynomial 2. (Compare this with the description in Chapter 9 of the Hilbert scheme of conic curves in $\mathbb{P}^3$—this is the same thing, one dimension lower.)

Exercise 13.28. Observe that the projection $\Phi \to \mathbb{P}^{2*}$ expresses Φ as a projective bundle over $\mathbb{P}^{2*}$, and use this to calculate its Chow ring.

Exercise 13.29. Viewing Φ as the Hilbert scheme of subschemes of $\mathbb{P}^2$ of dimension 0 and degree 2, construct a vector bundle E on Φ whose fiber at a point D is the space

$$E_{(L,p,q)} = H^0(\mathcal{O}_{\mathbb{P}^2}(d)/\mathcal{I}_D^2(d)).$$

(What would go wrong if, instead of using the Hilbert scheme Φ as our parameter space, we used the Chow variety—that is, the symmetric square of $\mathbb{P}^2$?) Express the condition that a curve $C = V(F) \subset \mathbb{P}^2$ be singular at p and q in terms of the vanishing of an associated section σ_F of E on $\mathcal{H}$ at (L, p, q)

Exercise 13.30. Calculate the Chern classes of this bundle, and derive accordingly the formula for the number of binodal curves in a net.

14

Porteous' Formula

Keynote Question

(a) Let $C \subset \mathbb{P}^3$ be a rational curve of degree d. How many lines $L \subset \mathbb{P}^3$ meet C four times?

14.1 Porteous' formula

********We have promised a proof of Theorem 7.9 for this section. Make it explicit. Also, be careful about the non-reduced case.********

We have seen in Chapters **** and **** that the Chern and Segre classes of a vector bundle F on a variety X can be characterized as the classes of the loci where a (suitably general) collection of sections of E failed to be independent, or failed to span F; in other words, the loci where a bundle map

$$\mathcal{O}_X^{\oplus m} \to F$$

failed to have maximal rank. There are two particularly useful generalizations of this situation. First, instead of a map $\mathcal{O}_X^{\oplus m} \to F$ we can consider a map from an arbitrary vector bundle E to F. Second, instead of considering the locus where such a bundle map has less than maximal rank, we can consider the locus where the map has any rank $\leq k$ for some given k. We shall see that the classes of these loci can be expressed in terms of the Chern classes of the bundles in question using *Porteous' formula*.

More precisely, suppose that E and F are vector bundles of ranks m and n on a variety X, and $\varphi : E \to F$ a vector bundle map. We consider the loci

$$M_k = M_k(\varphi) = \{x \in X : \operatorname{rank}(\varphi_x) \leq k\}.$$

M_k may be given the structure of subscheme of X by taking it to be the zero locus of the ideal generated by the $(k+1) \times (k+1)$ minors of a matrix representation of φ, in terms of any local trivialization of the bundles E and F. Generalizing the Principal Ideal Theorem, the components of the subscheme $M_k \subset X$ can have codimension at most $(m-k)(n-k)$ ****give a forward reference or at least a literature reference****; if M_k has this codimension then Porteous' formula expresses the class $[M_k] \in A^{(m-k)(n-k)}(X)$ as a polynomial in the Chern classes of E and F.

In the simplest (and most common) special case the formula is simple to state and easy to make plausible: To express is we write $[\alpha]_r$ for the component in dimension r of a class $\alpha \in A(X)$. The following is a Corollary of Theorem 14.3:

Corollary 14.1. *Suppose that $\varphi : E \to F$ is a map of vector bundles on a smooth variety X, with $\operatorname{rank} E \leq \operatorname{rank} F$. If φ fails to be a monomorphism in exactly the "expected" codimension $s = \operatorname{rank} F - \operatorname{rank} E + 1$, then the degeneracy scheme $M_{\operatorname{rank} E - 1}$ has class*

$$[M_{\operatorname{rank} E-1}] = \left[\frac{c(F)}{c(E)}\right]_s.$$

□

****I'm not so sure we've proven this. See the "fix" just before Theorem 14.3.**** One way to see that this formula is plausible is to note that if the degeneracy locus was empty then F/E would be a vector bundle of rank equal to $\operatorname{rank} F - \operatorname{rank} E = s-1$, so the r Chern class of F/E would be zero. By the Whitney sum formula, $c(F/E) = c(F)/c(E)$. Thus the class in the Corollary measures the failure of F/E to be a vector bundle, and Porteous' formula identifies this measurement as the class of the degeneracy scheme.

To derive Porteous' formula we linearize. Since it is given by determinants, the condition that a linear map $\varphi : V \to W$ of vector spaces has rank k or less is not a linear condition on $\varphi \in \operatorname{Hom}(V, W)$ (unless $k = 0$); but the condition that $\operatorname{Im}(\varphi) \subset \Lambda$ for a given k-plane $\Lambda \subset W$ is linear. (Likewise, the condition that $\Gamma \subset \operatorname{Ker}(\varphi)$, that is, $\varphi(\Gamma) = 0$, is linear.) To make use of this, we approach the problem not on X, but on the space of pairs (x, Λ), with $x \in X$ and $\Lambda \subset F_x$ a k-plane in the fiber of F at x—in other words, the Grassmann bundle $G = G(k, F)$. The class of the corresponding locus in the Grassmannian is easy to express, and we can then push it forward to the variety X. This solves the problem in principle, but the fact that

there is a nice formula in terms of the Chern classes of E and F seems to us a matter of glorious good luck.

To carry this out, let $G = G(k, F) \to X$ be the Grassmannian of k-planes in the fibers of F; let S be the universal subbundle on G, with fiber

$$S_{(x,\Lambda)} = \Lambda \subset F_x,$$

and Q the universal quotient bundle π^*F/S. We take the pullback

$$\pi^*\varphi : \pi^*E \to \pi^*F$$

of the map φ, and compose it with the quotient map

$$\pi^*F \to Q$$

to arrive at a map $\gamma : \pi^*E \to Q$. The zero locus $Z \subset G$ of the map γ is the locus of pairs (x, Λ) such that $\mathrm{Im}(\varphi_x) \subset \Lambda$, so that the image $\pi(Z) \subset X$ of Z is the desired locus $M_k \subset X$.

Thus passage to the Grassmann bundle G converts a problem about the rank of the original bundle map φ into a problem about the zero locus of a section $\gamma \in \mathrm{Hom}(\pi^*E, Q)$ of a vector bundle, whose class is just the top Chern class of this bundle. To get an expression for $[M_k] \in A^*(X)$

(a) Describe the Chow ring $A^*(G)$ and the pullback and pushforward maps $A^*(X) \to A^*(G)$ and $A^*(G) \to A^*(X)$, at least insofar as we need to;

(b) Find the class z of Z in $A^*(G)$. Since Z is the zero locus of the bundle $\mathrm{Hom}(\pi^*E, Q) = \pi^*E^* \otimes Q$, this means calculating the top Chern class $c_{m(n-k)}(\pi^*E^* \otimes Q))$;

(c) Determine the pushforward $\pi_*(z) \in A^*(X)$; and finally

(d) Say very carefully what we have proved—that is, under what conditions the class $\pi_*(z)$ really is the class of the subscheme $M_k \subset X$.

It's clear that each of these steps can be carried out in theory: we have given, in Section **, a description of the Chow ring of Grassmann bundles, and we can describe the pushforward map in terms of this; by the splitting principle, the Chern classes of a tensor product of two vector bundles are expressible in terms of the Chern classes of the bundles, and so on. But we emphasize again, we will have to be extremely lucky to carry these out in practice and arrive at a closed form expression for the answer.

For the first part of the program, we really don't need much. We do have a description of the Chow ring of the Grassmann bundle G:

$$A^*(G) = A^*(X)[\zeta_1, \ldots \zeta_k] / \left(\left[\frac{c(F)}{1 + \zeta + \zeta^2 + \ldots} \right]_l, l > n - k \right)$$

where $\zeta_1, \ldots, \zeta_k$ are the Chern classes of the tautological subbundle S on G. From this we can deduce the pushforward of any monomial in the ζ_i,

in much the same way as we did for projective bundles in Chapter **. But thankfully we don't need to do this in general: all we're going to need are a couple of elementary observations, based just on degree:

Remark. (a) If $\zeta^A = \zeta_1^{a_1} \cdot \cdots \cdot \zeta_k^{a_k}$ is a monomial of weighted degree

$$|A| = \sum ia_i < k(n-k),$$

then the pushforward $\pi_*(\zeta^A) = 0$; and

(b) If ζ^A is a monomial of weighted degree $|A| = k(n-k)$, then the pushforward

$$\pi_*(\zeta^A) = \alpha[G]$$

where α is the intersection of ζ^A with a fiber of π; that is, the product σ^A of the corresponding Schubert cycles $\sigma_{1,1,\ldots,1}$ in the Grassmannian $G(k,n)$.

14.2 The top Chern class of a tensor product

Our next task is to calculate the top Chern class $c_{m(n-k)}(E^* \otimes Q))$, in terms of the Chern classes $\mu_i = c_i(E)$ and $\zeta_j = c_j(S)$. Now, if we formally factor

$$c(E^*) = \prod_{i=1}^{m}(1-\alpha_i) \qquad \text{and} \qquad c(Q) = \prod_{j=1}^{n-k}(1+\beta_j)$$

then the splitting principle in theory tells us all the Chern classes of the tensor product $E^* \otimes Q$. In practice, though, it's extremely difficult to express these simply in terms of the classes μ_i and ζ_j, with a few exceptions: when the rank of either bundle is 1, for example. Fortunately, another exception is the top Chern class. This is given as

$$c_{m(n-k)}(E^* \otimes Q) = \prod_{i=1}^{m}\prod_{j=1}^{n-k}(\beta_j - \alpha_i)$$

Now, this is clearly symmetric with respect to permutations of the variables $\{\alpha_1, \ldots, \alpha_m\}$ and $\{\beta_1, \ldots, \beta_{n-k}\}$, and so in theory expressible as a polynomial in the elementary symmetric functions of the α_i and of the β_j; in practice, it may not be so clear how. But we can do it. To see how, we'll need to know about Sylvester's Determinant, which we'll now derive.

14.2.1 Sylvester's determinant

Let's reset the counter on m and n, and consider the following question. Let $\mathbb{P}^m$ and $\mathbb{P}^n$ be the spaces of polynomials of degree m and n on $\mathbb{P}^1$, and

consider the locus

$$\Psi = \{(F, G) : F \text{ and } G \text{ have a common zero}\} \subset \mathbb{P}^m \times \mathbb{P}^n.$$

This is pretty clearly a divisor in $\mathbb{P}^m \times \mathbb{P}^n$—for one thing, it's the image of the incidence correspondence

$$\{(F, G, p) : F(p) = G(p) = 0\} \subset \mathbb{P}^m \times \mathbb{P}^n \times \mathbb{P}^1$$

which is the complete intersection of the preimages of the universal divisors in $\mathbb{P}^m \times \mathbb{P}^1$ and $\mathbb{P}^n \times \mathbb{P}^1$. From the same description, we can see as well that it's of bidegree (n, m); or we can arrive at this by observing that Ψ intersects a general line in a fiber of the first (respectively, second) projection m (resp., n) times. It's thus the zero locus of a bihomogeneous polynomial of bidegree (n, m) on $\mathbb{P}^m \times \mathbb{P}^n$—the question is, which one?

To answer this, we consider linear combinations $AF + BG$ of F and G, of degree $m + n - 1$, and ask: is every polynomial of degree $m + n - 1$ expressible as such a linear combination? Well, if F and G have a common zero p, the answer is clearly no; any linear combination of the two will also have to vanish at p. The key observation is that the converse is also true: if F and G have no common zero, then every polynomial of degree $m+n-1$ is indeed a linear combination of the two. To see this, just observe that the vector spaces S_{n-1} and S_{m-1} of polynomials of degrees $n - 1$ and $m - 1$ have dimensions n and m respectively. The map

$$\begin{aligned} \mu_{F,G} : S_{m-1} \oplus S_{n-1} &\xrightarrow{(F,G)} S_{m+n-1} \\ (A, B) &\mapsto AF + BG \end{aligned}$$

is thus a map between vector spaces of the same dimension $m + n$; in particular, it's surjective if and only if it's injective. But if F and G have no common zero, it must be injective: if $(A, B) \in \mathrm{Ker}(\mu)$, so that we have

$$AF + BG = 0$$

then A must vanish everywhere G does, and B wherever F does, and to at least the same order; since $\deg A < \deg G$ and $\deg B < \deg F$, it follows that $A = B = 0$. In sum: polynomials F and G have no common zero if and only if every polynomial of degree $m + n - 1$ is a linear combination of the two.

Now, it's straightforward to say when this is the case. Write, in terms of an affine coordinate t on $\mathbb{P}^1$,

$$F(t) = a_0 + a_1 t + \cdots + a_m t^m \quad \text{and} \quad G(t) = b_0 + b_1 t + \cdots + b_n t^n.$$

Then the space of linear combinations of F and G of degree $m + n - 1$ is spanned by

$$F, tF, t^2F, \ldots, t^{n-1}F, G, tG, \ldots, t^{n-1}G$$

and so taking the coefficients of these $m+n$ polynomials as the rows of a square matrix, we see that F and G will have a common zero if and only if the determinant

$$\begin{vmatrix}
a_0 & a_1 & \dots & \dots & a_m & 0 & 0 & \dots & \dots & 0 \\
0 & a_0 & a_1 & \dots & a_{m-1} & a_m & 0 & \dots & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\
0 & 0 & \dots & \dots & a_0 & a_1 & \dots & \dots & \dots & a_m \\
b_0 & b_1 & \dots & \dots & b_{n-1} & b_n & 0 & 0 & \dots & 0 \\
0 & b_0 & b_1 & \dots & b_n & b_{n-1} & b_n & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & \vdots \\
0 & 0 & \dots & \dots & \dots & b_{n-m+1} & \dots & \dots & & \dots b_n
\end{vmatrix} = 0.$$

This is called *Sylvester's determinant*. Note that it is indeed bihomogenous of bidegree (n, m) in the variables a_i and b_j; by what we have said, the divisor $\Psi \subset \mathbb{P}^m \times \mathbb{P}^n$ is scheme-theoretically the zero locus of this polynomial.

If we assume that $a_0 = b_0 = 1$, we can reduce the size of the matrix by (in effect) carrying out a column reduction on the first n rows: we write

$$\frac{1}{F(t)} = 1 + d_1 t + d_2 t^2 + \dots$$

then multiply on the right by the $(m+n) \times (m+n)$ matrix

$$\begin{pmatrix}
1 & d_1 & d_2 & d_3 & d_4 & \dots & \dots & d_{m+n-1} \\
0 & 1 & d_1 & d_2 & d_3 & \dots & \dots & d_{m+n-2} \\
0 & 0 & 1 & d_1 & d_2 & \dots & \dots & d_{m+n-3} \\
0 & 0 & 0 & 1 & d_1 & \dots & \dots & d_{m+n-4} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots \\
0 & 0 & \dots & \dots & & 1 & & d_1 \\
0 & 0 & \dots & \dots & & 0 & & 1
\end{pmatrix}$$

to arrive at the determinant

$$\begin{vmatrix}
1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \dots & \dots & 1 & 0 & 0 & \dots & 0 \\
1 & c_1 & \dots & \dots & \dots & c_n & c_{n+1} & \dots & c_{m+n-1} \\
0 & 1 & c_1 & \dots & \dots & c_{n-1} & c_n & \dots & c_{m+n-2} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \dots & \dots & 0 & c_{n-m+1} & \dots & \dots & c_n
\end{vmatrix}$$

where

$$\sum_{i=0}^{\infty} c_i t^i = \frac{G(t)}{F(t)}.$$

This calls for some notation. For any ring R and any power series

$$c(t) = c_0 + c_1 t + c_2 t^2 + \cdots \in R[[t]],$$

we'll set

$$\Delta_{n,m}(c) = \begin{vmatrix}
c_n & c_{n+1} & \dots & c_{m+n-1} \\
c_{n-1} & c_n & \dots & c_{m+n-2} \\
\vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
c_{n-m+1} & \dots & \dots & c_n
\end{vmatrix} \in R.$$

What we have proved is that two polynomials $f(t)$ and $g(t)$ of degrees m and n, with constant terms equal to 1, have a common zero if and only if

$$\Delta_{n,m}\left(\frac{g(t)}{f(t)}\right) = 0.$$

Equivalently, in the polynomial ring $\mathbb{C}[\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n]$, if $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_n\}$ are the elementary symmetric functions in the α and the β, we have the identity

$$\prod_{i=1}^{m} \prod_{j=1}^{n} (\beta_j - \alpha_i) = \lambda \cdot \Delta_{n,m}\left(\frac{\sum b_j t^j}{\sum a_i t^i}\right)$$

for some scalar $\lambda \in \mathbb{C}$. It remains only to find the scalar λ, and we do this simply by comparing the coefficient of $b_n^m = (\prod \beta_j)^m$ on both sides to conclude $\lambda = 1$. Note that we could reverse the roles of the α and the β to arrive at an alternative expression:

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(\beta_j - \alpha_i) = (-1)^{mn} \cdot \Delta_{m,n}\left(\frac{\sum a_i t^i}{\sum b_j t^j}\right).$$

This in turn translates, via the splitting principle, into a statement about the top Chern class of a tensor product. One last bit of language: if E is a vector bundle on a variety X, we define the *Chern polynomial* $c_t(E)$ to be the polynomial

$$c_t(E) - 1 + c_1(E)t + c_2(E)t^2 + \ldots \quad \in A^*(X)[t].$$

In these terms, we have the

Proposition 14.2. *If E and F are vector bundles of ranks m and n on a variety X, then*

$$\begin{aligned} c_{mn}(E^* \otimes F) &= \Delta_{n,m}\left(\frac{c_t(F)}{c_t(E)}\right) \\ &= (-1)^{mn}\Delta_{m,n}\left(\frac{c_t(E)}{c_t(F)}\right). \end{aligned}$$

14.3 Back to our original problem

We return now to the situation we started with: we have a map $\varphi : E \to F$ of vector bundles of ranks m and n, and in order to describe the locus $M_k(\varphi) = \{x \in X : \text{rank}(\varphi_x) \leq k\}$ we form the Grassmannian bundle $\pi : G = G(k, F) \to X$ over X and consider the composite map $\gamma : \pi^*E \to Q$. The locus $M_k \subset X$ will then be just the image of the zero locus $Z \subset G$ of the section $\gamma \in \text{Hom}(\pi^*E, Q)$; and (assuming the codimension of Z in G is the expected $m(n-k)$), the class $z \in A^*(G)$ of Z will be given as

$$\begin{aligned} z &= c_{m(n-k)}(\pi^*E^* \otimes Q) \\ &= \Delta_{n-k,m}\left(\frac{c_t(Q)}{c_t(\pi^*E)}\right) \\ &= (-1)^{m(n-k)}\Delta_{m,n-k}\left(\frac{c_t(\pi^*E)}{c_t(Q)}\right) \end{aligned}$$

Our goal now is to evaluate the Gysin image $\pi_*(z)$ of this class. At first glance this seems daunting: when we expand out either of the determinantal expressions above for z, we get monomials of high degree (bigger than

$k(n-k)$) in the Chern classes of the tautological bundles on G, and as we've said it's hard to calculate their Gysin images explicitly. But once more we're in luck: the exact sequence

$$0 \to S \to \pi^* F \to Q \to 0$$

gives

$$c_t(Q) = \frac{c_t(\pi^* F)}{c_t(S)},$$

and applying this to the second expression for z above we have

$$z = (-1)^{m(n-k)} \Delta_{m,n-k} \left(\frac{c_t(\pi^* E)}{c_t(\pi^* F)} c_t(S) \right).$$

The key observation now is this: this is the determinant of an $(n-k) \times (n-k)$ matrix, and each entry of the matrix is a linear combination, with coefficients in $A^*(X)$, of the Chern classes $\zeta_1, \ldots, \zeta_k$ of S. When we expand out the determinant, then, the weighted degree in the ζ_i of each product is at most $k(n-k)$, with the term of highest degree $k(n-k)$ coming from taking the ζ_k term in each entry. Now, when we take the Gysin image, by the Remark in the initial section of this Chapter, any term of weighted degree strictly less than $k(n-k)$ maps to zero; so *only the ζ_k term in each entry contributes to the Gysin image of the determinant.*

We can thus apply the push-pull formula $\pi_*(\pi^*\alpha \cdot \beta) = \alpha \cdot \pi_*\beta$ to write

$$\pi_* \begin{vmatrix} \left(\frac{c_t(\pi^*E)}{c_t(\pi^*F)} c_t(S)\right)_{t^m} & \cdots & \cdots & \left(\frac{c_t(\pi^*E)}{c_t(\pi^*F)} c_t(S)\right)_{t^{m+n-k+1}} \\ \vdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ \left(\frac{c_t(\pi^*E)}{c_t(\pi^*F)} c_t(S)\right)_{t^{m-n+k+1}} & \cdots & \cdots & \left(\frac{c_t(\pi^*E)}{c_t(\pi^*F)} c_t(S)\right)_{t^m} \end{vmatrix}$$

$$= \pi_* \begin{vmatrix} \left(\frac{c_t(\pi^*E)}{c_t(\pi^*F)}\right)_{t^{m-k}} \zeta_k & \cdots & \cdots & \left(\frac{c_t(\pi^*E)}{c_t(\pi^*F)}\right)_{t^{m+n-2k+1}} \zeta_k \\ \vdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ \left(\frac{c_t(\pi^*E)}{c_t(\pi^*F)}\right)_{t^{m-n+1}} \zeta_k & \cdots & \cdots & \left(\frac{c_t(\pi^*E)}{c_t(\pi^*F)}\right)_{t^{m-k}} \zeta_k \end{vmatrix}$$

$$= \begin{vmatrix} \left(\frac{c_t(E)}{c_t(F)}\right)_{t^{m-k}} & \cdots & \cdots & \left(\frac{c_t(E)}{c_t(F)}\right)_{t^{m+n-2k+1}} \\ \vdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ \left(\frac{c_t(E)}{c_t(F)}\right)_{t^{m-n+1}} & \cdots & \cdots & \left(\frac{c_t(E)}{c_t(F)}\right)_{t^{m-k}} \end{vmatrix} \cdot \pi_*(\zeta_k^{n-k})$$

$$= \Delta_{m-k,n-k}\left(\frac{c_t(E)}{c_t(F)}\right) \cdot \pi_*(\zeta_k^{n-k}).$$

Finally, to evaluate $\pi_*(\zeta_k^{n-k})$, recall that the k^{th} Chern class $c_k(S^*)$ of the dual of the universal subbundle on a single Grassmannian $G(k,n)$ is just the class $\sigma_{1,1,\ldots,1}$ of the Schubert cycle of k-planes lying in a hyperplane; the $(n-k)^{\text{th}}$ power of this class is thus the class of a single point. Again by our remark,

$$\pi_*(\zeta_k^{n-k}) = (-1)^{k(n-k)}[X]$$

and we conclude finally that

$$\begin{aligned} \pi_*(z) &= (-1)^{(m-k)(n-k)} \Delta_{m-k,n-k}\left(\frac{c_t(E)}{c_t(F)}\right) \\ &= \Delta_{n-k,m-k}\left(\frac{c_t(F)}{c_t(E)}\right). \end{aligned}$$

In sum, we have

********In fact we are missing a statement about Chern Classes, indicated at the end of sect 3 (Lemma **??** of the Chern class chapter, that is necessary

in the case the degen locus is of correct codim but not reduced—the proof below works only in the reduced case til this statment is proven.****

Theorem 14.3 (Porteous' theorem)**.** *Let $\varphi : E \to F$ be a map of vector bundles of ranks m and n on a variety X. If the scheme $M_k(\varphi) \subset X$ where φ has rank k or less has the expected dimension* $\dim X - (m-k)(n-k)$*, then its class is given by*

$$\begin{aligned}[M_k(\varphi)] &= \Delta_{n-k,m-k}\left(\frac{c_t(F)}{c_t(E)}\right) \\ &= (-1)^{(m-k)(n-k)}\Delta_{m-k,n-k}\left(\frac{c_t(E)}{c_t(F)}\right).\end{aligned}$$

****at one time we worried about how to prove that a single section of a bundle F, vanishing in the correct codim, computes the chern class. This is implied by the version of Porteous here, since it is the degeneracy locus of the map $\mathcal{O}_X \to F$. Is it proven?****

14.3.1 One last wrinkle

****admit at the beginning that the argument will be first to prove a slightly weaker case, rather than depending on the reader's visual acuity.**** The sharp-eyed reader will note one thing here: the calculation above seems to prove something infinitesimally weaker than Porteous' theorem as stated above, in that for it to apply we require not just that the locus M_k we're looking has the expected dimension, but that the associated scheme $Z \subset G(k, F)$ does as well. It's possible that M_k might have the expected dimension, but Z not, in which case the formula would not apply.

In fact, we can fix this with one further argument. To set this up, let X, E, F and $\varphi : E \to F$ be as above. We introduce another Grassmann bundle:

$$\pi : G = G(m, E \oplus F) = \{(x, \Lambda) : \Lambda^m \subset (E \oplus F)_x\} \to X,$$

and we consider the subscheme $\Phi \subset G$ defined by

$$\Phi = \{(x, \Lambda); \dim(\Lambda \cap E_x) \geq m - k\}$$

where S is the universal subbundle on G. This can be characterized as the locus $M_k(\psi)$ where the bundle map given as the composition

$$\psi : S \hookrightarrow \pi^*(E \oplus F) \to \pi^* F$$

has rank k or less. Moreover, in this setting both $M_k(\psi)$ and the associated variety Z introduced in the proof of Porteous' formula have the expected

dimension; so we can conclude by the calculation above that the class

$$[M_k(\psi)] = \Delta_{n-k,m-k}\left(\frac{c_t(\pi^*F)}{c_t(S)}\right).$$

The point is twofold. First, we observe that we have a section

$$\begin{aligned}\alpha : X &\to G\\ x &\mapsto (x, \Gamma_{\varphi_x}),\end{aligned}$$

where $\Gamma_{\varphi_x} \subset E_x \oplus F_x$ is simply the graph of the linear map $\varphi_x : E_x \to F_x$; and by construction the preimage

$$\alpha^{-1}(M_k(\psi)) = M_k(phi)$$

as subschemes of X. Moreover, we see that

$$\alpha^*(S) = E$$

and of course since α is a section of $\pi : G \to X$, we have

$$\alpha^*(\pi^*F) = F.$$

Thus, subject only to the hypothesis that $M_k(\varphi) = \alpha^{-1}(M_k(\psi))$ has the same codimension $(m-k)(n-k)$ as $\Phi = M_k(\psi)$, we have established that

$$\begin{aligned}[M_k(\varphi)] &= \alpha^*[M_k(\psi)]\\ &= \alpha^*\left(\Delta_{n-k,m-k}\left(\frac{c_t(\pi^*F)}{c_t(S)}\right)\right)\\ &= \Delta_{n-k,m-k}\left(\frac{c_t(F)}{c_t(E)}\right),\end{aligned}$$

as desired.

14.4 Applications

14.4.1 Degrees of determinantal varieties

The first application of Porteous' formula is a classic one. Let $\mathbb{P}^{mn-1}$ be the projective space of all nonzero $m \times n$ matrices, mod scalars, and let $M_k \subset \mathbb{P}^{mn-1}$ be the locus of matrices of rank k or less. M_k is clearly a closed subvariety of $\mathbb{P}^{mn-1}$, and a standard argument shows that it's irreducible of codimension $(m-k)(n-k)$. We ask now: what's the degree of M_k?

As an application of Porteous' formula, this is completely straightforward: if we let $\{X_{i,j}\}$ be the homogeneous coordinates on $\mathbb{P}^{mn-1}Z$, then multiplication by the matrix $X = (X_{i,j})$ defines a vector bundle map

$$(\mathcal{O}_{\mathbb{P}^{mn-1}})^{\oplus m} \longrightarrow (\mathcal{O}_{\mathbb{P}^{mn-1}}(1))^{\oplus n}$$

and we are asking just for the class of the locus where this map has rank k or less. Letting $\omega \in A^1(\mathbb{P}^{mn-1})$ be the hyperplane class, we have

$$c(F) = \sum_{r=0}^{n} \binom{n}{r} \omega^r$$

from which we conclude that the class of M_k is given by

$$[M_k] = \begin{vmatrix} \binom{n}{n-k}\omega^{n-k} & \cdots & \binom{n}{n+m-2k-1}\omega^{n+m-2k-1} \\ \vdots & & \vdots \\ \binom{n}{n-m+1}\omega^{n-m+1} & \cdots & \binom{n}{n-k}\omega^{n-k} \end{vmatrix}$$

$$= \begin{vmatrix} \binom{n}{n-k} & \cdots & \binom{n}{n+m-2k-1} \\ \vdots & & \vdots \\ \binom{n}{n-m+1} & \cdots & \binom{n}{n-k} \end{vmatrix} \omega^{(m-k)(n-k)}.$$

The degree of M_k is thus simply the last determinant.

Well, "simply" may not be exactly the right word; it remains to be seen if we can evaluate the determinant in closed form. In fact, we are once again in luck, and can do it, though it takes a certain amount of manipulation. To begin with, we make a series of column operations: first, we replace each column, starting with the second, with the sum of it and the column to its left, to arrive at

$$\begin{vmatrix} \binom{n}{n-k} & \binom{n}{n-k+1} & \cdots & \binom{n}{n+m-2k-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \binom{n}{n-m+1} & \binom{n}{n-m+2} & \cdots & \binom{n}{n-k} \end{vmatrix}$$

$$= \begin{vmatrix} \binom{n}{n-k} & \binom{n+1}{n-k+1} & \cdots & \binom{n+1}{n+m-2k-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \binom{n}{n-m+1} & \binom{n+1}{n-m+2} & \cdots & \binom{n+1}{n-k} \end{vmatrix}$$

Now we do the same thing again, this time starting with the third column; then again, starting with the fourth, and so on, to arrive at the determinant

$$\begin{vmatrix} \binom{n}{n-k} & \binom{n+1}{n-k+1} & \cdots & \binom{n+m-k-1}{n+m-2k-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \binom{n}{n-m+1} & \binom{n+1}{n-m+2} & \cdots & \binom{n+m-k-1}{n-k} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n!}{k!(n-k)!} & \frac{(n+1)!}{k!(n-k+1)!} & \cdots & \frac{(n+m-k-1)!}{k!(n+m-2k-1)!} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{n!}{(m-1)!(n-m+1)!} & \frac{(n+1)!}{(m-1)!(n-m+2)!} & \cdots & \frac{(n+m-k-1)!}{(m-1)!(n-k)!} \end{vmatrix}$$

Now, we can pull a factor of $n!$ from the first column, $(n+1)!$ from the second, and so on; similarly, we can pull a $k!$ from the denominators in the first row, a $(k+1)!$ from the denominators in the second row, and so on. We arrive at the product

$$\prod_{i=0}^{m-k-1} \frac{(n+i)!}{(k+i)!} \begin{vmatrix} \frac{1}{(n-k)!} & \frac{1}{(n-k+1)!} & \cdots & \frac{1}{(n+m-2k-1)!} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{1}{(n-m+1)!} & \frac{1}{(n-m+2)!} & \cdots & \frac{1}{(n-k)!} \end{vmatrix}.$$

Next, we multiply the first column by $(n-k)!$, the second by $(n-k+1)!$, and so on, to arrive at the expression

$$\prod_{i=0}^{m-k-1} \frac{(n+i)!}{(k+i)!(n-k+i)!} \begin{vmatrix} 1 & 1 & \cdots \\ n-k & n-k+1 & \cdots \\ (n-k)(n-k-1) & (n-k+1)(n-k) & \cdots \\ \vdots & \vdots & \end{vmatrix}.$$

Finally, we can recognize the columns of this matrix as the series of monic polynomials 1, x, $x(x-1)$, $x(x-1)(x-2)$, $\ldots$, of degrees $0, 1, 2, \ldots, m-k-1$, applied to the integers $n-k, n-k+1, \ldots, n+m-2k-1$. It's determinant

is thus the van der Monde determinant

$$\begin{vmatrix} 1 & 1 & 1 & \dots \\ n-k & n-k+1 & n-k+2 & \dots \\ (n-k)^2 & (n-k+1)^2 & (n-k+2)^2 & \dots \\ \vdots & \vdots & \vdots & \end{vmatrix},$$

which has detereminant $\prod_{i=0}^{m-k-1} i!$. Our conclusion at last is the

Proposition 14.4. *The degree, in the projective space $\mathbb{P}^{mn-1}$ of $m \times n$ matrices, of the variety M_k of matrices of rank k or less is*

$$\deg(M_k) = \prod_{i=0}^{m-k-1} \frac{i!(n+i)!}{(k+i)!(n-k+i)!}$$

Exercise 14.5. Verify this formula in case

(a) $m = n$ and $k = m - 1$; and

(b) $k = 1$.

Exercise 14.6. In Exercise **??** of the first chapter, we introduced the variety of triples of collinear points, that is,

$$\Psi = \{(p, q, r) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n \mid p, q \text{ and } r \text{ are collinear in } \mathbb{P}^n\}.$$

Calculate the class $\psi = [\Psi] \in A^{n-1}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$ by applying Porteous to the evaluation map $E \to F$, where E is the trivial bundle on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ with fiber $H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ and

$$F = \pi_1^*\mathcal{O}_{\mathbb{P}^n}(1) \oplus \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1) \oplus \pi_3^*\mathcal{O}_{\mathbb{P}^n}(1),$$

with $\pi_i : \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ projection on the i^{th} factor.

14.4.2 Pinch points of surfaces

Let $C \subset \mathbb{P}^n$ be a smooth curve, and $\pi = \pi_\Lambda : C \to \mathbb{P}^2$ the projection from a general $(n-3)$-plane. It's a standard exercise that the map π will be an immersion, and that the image will have only ordinary nodes as singularities.

Is there a similar description for surfaces? In other words, if $\pi : S \to \mathbb{P}^3$ is a general projection of a smooth surface $S \subset \mathbb{P}^n$ to $\mathbb{P}^3$, can we say what singularities occur? The answer is yes (see for example [GH] for a fuller discussion): what we find is that there will be a curve in the image where the map is 2-1, and at a general such point the image surface $\overline{S}$ is simply the union of two smooth sheets crossing transversely; similarly, there'll be a finite number of points where the map is 3-1, and at each such point

the image surface will be the union of three smooth sheets intersecting transversely.

But there's one additional type of singularity. At a finite number of points of S, the map π will fail to be an immersion. (The geometry of the map, and of the image, is beautiful at these points: basically, their images are points of the double curve of $\overline{S}$ where the two sheets wind around each other. The local equation of the image will be

$$z^2 = xy^2$$

so that the double curve will be the x-axis, and the two sheets at the point $(x_0, 0, 0)$ will have tangent planes $z = \pm\sqrt{x_0} \cdot y$.) These points are called *pinch points* of the image surface $\overline{S}$.

We now ask the obvious enumerative question: in terms of the standard invariants of the surface $S \subset \mathbb{P}^n$, *how many pinch points will* $\overline{S}$ *have?* This is again a very straightforward application of Porteous' formula: the differential of the map π is a vector bundle map

$$d\pi : T_S \to \pi^* T_{\mathbb{P}^3}$$

and Porteous tells us that the number of points where this map fails to be injective will be simply the degree 2 piece of the quotient

$$\frac{c(\pi^* T_{\mathbb{P}^3})}{c(T_S)}.$$

Denote by $\alpha = c_1(\mathcal{O}_S(1))$ the pullback to S of the hyperplane class on $\mathbb{P}^3$ (equivalently, the restriction to S of the hyperplane class on $\mathbb{P}^n$), and write= $\omega = c_1(T_S^*)$ and $\chi = c_2(T_S^*)$. We have then

$$\begin{aligned}\frac{c(\pi^* T_{\mathbb{P}^3})}{c(T_S)} &= \frac{1 + 4\alpha + 6\alpha^2}{1 - \omega + \chi} \\ &= (1 + 4\alpha + 6\alpha^2)(1 + \omega + (\omega^2 - \chi))\end{aligned}$$

and taking the degree 2 part of this expression we have the

Proposition 14.7. *The number of pinch points of a general projection of a smooth surface $S \subset \mathbb{P}^3$ is*

$$6\alpha^2 + 4\alpha\omega + \omega^2 - \chi.$$

Exercise 14.8. Verify this directly in case

(a) S is a smooth surface in $\mathbb{P}^3$ to begin with; and

(b) $S \subset \mathbb{P}^5$ is the quadratic Veronese surface.

There's another way to interpret this last formula. Recall that, for a smooth surface $S \subset \mathbb{P}^n$, the *tangential variety* X of S is defined to be the union of the tangent planes $\mathbb{T}_p(S) \subset \mathbb{P}^n$ to S in $\mathbb{P}^n$; if $n \geq 4$ this is

always a fourfold. Now, if $\pi = \pi_\Lambda : S \to \mathbb{P}^3$ is the projection from a general $(n-4)$-plane $\Lambda \subset \mathbb{P}^n$, the points where the map fails to be an immersion are just the points p such that $\mathbb{T}_p(S) \cap \Lambda \neq \varnothing$. If the tangent planes to S sweep out the tangential variety X just once—that is, if a general point on X lies on just one tangent plane to S—then the number of such points is just the degree of X. Thus, in this case we can view the formula above as a formula for the degree of the tangential variety of a surface $S \subset \mathbb{P}^n$.

Exercise 14.9. Let $S \subset \mathbb{P}^n$ be a smooth surface.

(a) Show that we have a map from the projective bundle $\mathbb{P}(T_S \oplus \mathcal{O}_S)$ to $\mathbb{P}^n$ with image the tangential variety X (specifically, carrying the fiber over p to the tangent plane $\mathbb{T}_p(S)$);

(b) Show that the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ under this map is the line bundle $\mathcal{O}_{\mathbb{P}(T_S \oplus \mathcal{O}_S)}(1) \otimes \mathcal{O}_S(1)$; and

(c) Use this and our description of the Chow ring of the projective bundle $\mathbb{P}(T_S \oplus \mathcal{O}_S)$ to rederive the formula above for the degree of X.

Exercise 14.10. Let $X \subset \mathbb{P}^n$ be a smooth 3-fold, and $\pi : X \to \mathbb{P}^5$ a general projection. Find the number of points where the map π fails to be an immersion.

Exercise 14.11. Let $X \subset \mathbb{P}^n$ be a smooth 6-fold, and $\pi : X \to \mathbb{P}^7$ a general projection. Find the number of points where the differential $d\pi$ has rank 4 or less.

One last remark is in order here. It's natural to ask whether there is an analogous description of the possible singularities of a general projection of a smooth variety $X \subset \mathbb{P}^n$ of dimension k to $\mathbb{P}^{k+1}$. The answer is a resounding "no:" in fact, for large values of k we don't even know what the maximum multiplicity of a point on the image $\overline{X} = \pi(X) \subset \mathbb{P}^{k+1}$ will be, beyond the fact that it grows exponentially with k. (One way to see this is to use the logic suggested by the last exercise: we expect that there will be places where the corank of the differential $d\pi$ is roughly $\sqrt{k}$, and over the image of such points the degree of the fiber of π will be at least on the order of $2^{\sqrt{k}}$.)

14.4.3 Quadrisecant lines to a rational space curve

We come finally to our keynote problem: counting the quadrisecant lines to a space curve. We start with a qualitative discussion. Consider a smooth, irreducible, nondegenerate space curve $C \subset \mathbb{P}^3$, and ask: how would we expect lines $L \subset \mathbb{P}^3$ to intersect C?

To begin with, a general line is disjoint from C, while the lines that do meet C form an irreducible, 3-dimensional family (that is, an irreducible divisor in the Grassmannian $\mathbb{G}(1,3)$). (We can see this by considering the incidence correspondence $\Sigma = \{(L,p) : p \in L \cap C\} \subset \mathbb{G}(1,3) \times C$.) The locus of secant lines—that is, lines such that $\deg(L \cap C) \geq 2$—is likewise an irreducible surface in $\mathbb{G}(1,3)$, being just the image of the product $C \times C$ under the regular map $(p,q) \mapsto \overline{pq}$ (this is at first glance defined only on the complement of the diagonal in $C \times C$; it extends to all of $C \times C$ by sending (p,p) to the tangent line $\mathbb{T}_p(C)$).

What about trisecant lines, which we'll similarly define as lines L such that $\deg(L \cap C) \geq 3$? Well, this may be somewhat less clear initially, but we do have the

Exercise 14.12. Show that the general secant line to C is not trisecant, and deduce that the locus of trisecant lines to C, if nonempty, has dimension 1. (For extra credit, show that it is empty only in case C is a twisted cubic or an elliptic quartic.)

Note that the locus of trisecant lines need not be irreducible; an example where it's not would be the intersection $C = Q \cap S$ of a smooth quadric Q and a cubic surface S.

Continuing in this way, we might expect that there will be a finite number of 4-secant lines to C, and no 5-secant lines. Neither of these statements is true for an arbitrary curve, or even a general curve (meaning a curve corresponding to a general point of a component of the Hilbert scheme): look, for example, at complete intersections of quadric with quintic surfaces. Nonetheless, we may ask: is there an enumerative formula for the number of 4-secant lines to C, say in terms of the degree d and genus g of C?

The answer is that there is such a formula, and we'll see in this subsection how to derive it in case C is rational by using Porteous' formula. In fact, we can derive the general case by Porteous as well; we'll discuss how to do this at the end of the subsection. For the remainder of this discussion, though, we'll assume that $C \cong \mathbb{P}^1$.

How do we set up the problem? As if often the case, the key is to turn the problem around: instead of looking at all lines in $\mathbb{P}^3$ and imposing the condition of meeting C four times, we'll look at 4-tuples of points on C and impose the condition that they span only a line. More precisely, we've characterized a 4-secant line to C as a line L that contains a subscheme of C of degree 4; we'll introduce a parameter space for such subschemes $\Gamma \subset C$, and try to describe the condition that the span $\overline{\Gamma}$ is only 1-dimensional.

Now, in general if C is a smooth curve the parameter space for subschemes of a given degree d on C (that is, the Hilbert scheme $\mathcal{H}_d(C)$) is

simply the d^{th} symmetric power $C^{(d)}$ of C, which is to say the quotient of the d-fold product C^d by the symmetric group S_d. (By contrast, if C is singular, the Hilbert scheme can be quite tricky—see for example ** and **). In case $C \cong \mathbb{P}^1$, this is particularly simple: any subscheme $\Gamma \subset \mathbb{P}^1$ of degree d is the zero locus of a homogeneous polynomial $F \in H^0(\mathcal{O}_{\mathbb{P}^1}(d))$, unique up to scalars; from which we see that the d-fold symmetric product of $\mathbb{P}^1$ is simply $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \cong \mathbb{P}^d$.

So, for the following, we will write $\mathbb{P}^4$ for the space of subschemes $\Gamma \subset \mathbb{P}^1$ of degree 4. Now, how do we express the condition that $\dim \overline{\Gamma} = 1$? This is also straightforward. We start by introducing a vector bundle F of rank 4 on $\mathbb{P}^4$ whose fiber at a point Γ is

$$F_\Gamma = H^0(\mathcal{O}_\Gamma(d)).$$

Officially, we let

$$\Phi = \{(\Gamma, p) : p \in \Gamma\} \subset \mathbb{P}^4 \times \mathbb{P}^1$$

be the universal divisor of degree 4, with $\alpha : \Phi \to \mathbb{P}^4$ and $\beta : \Phi \to \mathbb{P}^1$ the projections, and set

$$F = \alpha_* \left(\mathcal{O}_\Phi \otimes \beta^* \mathcal{O}_{\mathbb{P}^1}(d)\right).$$

Now, let E be the trivial bundle of rank 4 on $\mathbb{P}^4$, with fiber $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$. The restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \hookrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \longrightarrow H^0(\mathcal{O}_\Gamma(d))$$

gives us a vector bundle map $\varphi : E \to F$, and we see that *the locus of subschemes $\Gamma \subset \mathbb{P}^1$ of degree 4 with $\dim \overline{\Gamma} = 1$ is simply the locus where the map φ has rank 2.* (Note that φ can never have rank < 2.) Porteous will then give us a formula for the class of this locus, in case it has the expected dimension

$$4 - 2 \times 2 = 0.$$

But first we have to find the Chern classes of the bundle F. Fortunately, we can do this directly from the definition, since F has a lot of sections. For example, suppose A is a general homogeneous polynomial of degree d on $\mathbb{P}^1$, viewed as a global section of $\mathcal{O}_{\mathbb{P}^1}(d)$. By restriction to each subscheme $\Gamma \subset \mathbb{P}^1$ in turn, A gives a section σ_A of F. When is $\sigma_A(\Gamma) = 0$? Exactly when Γ is contained in the zero locus of F on $\mathbb{P}^1$. Since the zero locus of A consists of d reduced points, the number of subschemes of degree 4 it contains is simply $\binom{d}{4}$; thus the fourth Chern class $c_4(F)$ is just $\binom{d}{4}$ times the class of a point in $\mathbb{P}^4$.

The other Chern classes may be calculated similarly. By way of notation, let $\omega \in A^1(\mathbb{P}^4)$ be the hyperplane class; we can write for each k

$$c_k(F) = m_k \omega^k$$

and we need just to determine the coefficient m_k for each k. We do this as usual by intersection with a cycle of complementary dimension, in this case a k-plane.

To carry this out, let $A_1, \ldots, A_{5-k}$ be general polynomials of degree d on $\mathbb{P}^1$ and $\sigma_1, \ldots, \sigma_{5-k}$ the corresponding sections of F on $\mathbb{P}^4$. The locus $\Psi \subset \mathbb{P}^4$ where the sections σ_i are linearly dependent is simply the locus of divisors $\Gamma \subset \mathbb{P}^1$ of degree 4 contained in the zero locus of some linear combination of the A_i. Now choose general points $p_1, \ldots, p_{4-k} \in \mathbb{P}^1$, and let $\Lambda \subset \mathbb{P}^4$ be the set of divisors of degree 4 on $\mathbb{P}^1$ containing $p_1, \ldots, p_{4-k}$. The intersection $\Psi \cap \Lambda$ is the locus of divisors of degree 4 on $\mathbb{P}^1$ contained in the zero locus of some linear combination of the A_i and containing $p_1, \ldots, p_{4-k}$. But there is a unique (up to scalars) linear combination A of the A_i vanishing at $p_1, \ldots, p_{4-k}$, and it will vanish simply at $d-4+k$ other points of $\mathbb{P}^1$; the intersection $\Psi \cap \Lambda$ will consist of those divisors equal to $p_1 + \cdots + p_{4-k}$, plus k of the $d-4+k$ remaining zeroes of A. We have thus

$$m_k = \#(\Psi \cap \Lambda) = \binom{d-4+k}{k}.$$

Exercise 14.13. Complete this calculation by verifying that, as defined above, the cycles Ψ and $\Lambda \subset \mathbb{P}^4$ do in fact intersect transversely.

Now we just apply Porteous' formula, which tells us that the class of the locus of subschemes $\Gamma \subset \mathbb{P}^1$ contained in a line is

$$\begin{vmatrix} \binom{d-2}{2}\omega^2 & \binom{d-1}{3}\omega^3 \\ \binom{d-3}{1}\omega & \binom{d-2}{2}\omega^2 \end{vmatrix} = \begin{vmatrix} \binom{d-2}{2} & \binom{d-1}{3} \\ \binom{d-3}{1} & \binom{d-2}{2} \end{vmatrix} \omega^4$$

$$= \left(\frac{(d-2)^2(d-3)^2}{4} - \frac{(d-1)(d-2)(d-3)^2}{6} \right) \omega^4$$

$$= \frac{(d-3)^2(d-2)}{12} \left(3(d-2) - 2(d-1)\right) \omega^4$$

$$= \frac{(d-2)(d-3)^2(d-4)}{12} \omega^4.$$

In other words, we expect a rational space curve of degree d to have $\frac{(d-2)(d-3)^2(d-4)}{12}$ 4-secant lines. As usual, this is subject to the condition that C has in fact only finitely many, and that we count each of those with the appropriate multiplicity. Note also as a check that this number is 0 in case $d = 2$, 3 or 4, as it should be.

Exercise 14.14. Check the answer in cases $d = 5$ and 6 by independently counting the number of quadrisecant lines to a general rational quintic and sextic. (Hint: such a curve will lie on a smooth cubic surface.)

Exercise 14.15. Let $C \subset \mathbb{P}^4$ now be a smooth, nondegenerate rational curve of degree d. Use similar methods to find the expected number of

(a) trisecant lines to C; and

(b) 6-secant 2-planes to C.

Exercise 14.16. Let $C \subset \mathbb{P}^3$ be a smooth curve, $L \subset \mathbb{P}^3$ a line meeting C in exactly 4 points $p_1, \ldots, p_4$ and not tangent to C at any of them. Suppose that the tangent lines $\mathbb{T}_{p_i}(C)$ to C at the p_i are independent mod L (that is, they span distinct planes with L), and that the cross-ratio of the four points $p_1, \ldots, p_4 \in L$ is *not* equal to the cross-ratio of the four planes $\mathbb{T}_{p_1}(C) + L, \ldots, \mathbb{T}_{p_4}(C) + L$. Show that $\Gamma = p_1 + \cdots + p_4 \in \mathbb{P}^4$ counts as a quadrisecant line with multiplicity 1 (that is, the 3×3 minors of a matrix representative of φ near Γ generate the maximal ideal $m_\Gamma \subset \mathcal{O}_{\mathbb{P}^4,\Gamma}$).

Exercise 14.17. Let C now be a general rational curve of degree d in $\mathbb{P}^3$. (Note that the family of rational curves $C \subset \mathbb{P}^3$ of degree d is irreducible, so this makes sense.)

(a) Show that C has no 5-secant lines.

(b) Show that if $L \subset \mathbb{P}^3$ is any quadrisecant line to C, then L meets C in four distinct points.

(c) Finally, show that every quadrisecant line to C satisfies the conditions of the preceding exercise, and deduce that the number of quadrisecant lines to C is exactly $\frac{(d-2)(d-3)^2(d-4)}{12}$. (Note: the first two items are straightforward dimension counts; this one is a little more subtle.)

Other proofs. A natural question to ask here is, Is there a more general formula, that applies to curves of arbitrary genus? The answer is yes, and we can derive it in a very similar fashion. The one issue that arises is that, where the Hilbert scheme parametrizing subschemes of degree 4 in $\mathbb{P}^1$ is simply $\mathbb{P}^4$, whose Chow ring we know, the space of subschemes of degree 4 of a smooth curve C of higher genus (again, the fourth symmetric power $C^{(4)}$ of C) is more complex; in particular, we don't know its Chow ring explicitly. It is possible, however, to say enough about the ring $A^*(C^{(4)})$ to carry out the same calculation there, and in fact we'll derive the relevant information in the final Chapter of this book. The most general formula, for the number of d-secant $(d - r - 1)$-planes to a curve of degree n and genus g in $\mathbb{P}^s$, is derived in this way in Chapter 8 of [ACGH]; the formula is on page 350 and the specialization to 4-secant lines on page 351.

There is also another, completely different way to approach the problem of counting quadrisecant lines to a space curve: via the classical theory of correspondences. This is described in Chapter 2 of [GH]; the formula is the same (once we specialize to $g = 0$), though it's far from clear that the multiplicities are equal!

14.5 Miscellaneous

There are a few items kicking around that I think belong in this Chapter, if we want to do them at all

14.5.1 Porteous for symmetric and skew-symmetric bundle maps

14.5.2 Positive vector bundles; Statement of Fulton-Lazarsfeld theorem

14.5.3 Webs of quadrics

A random exercise, but one that might be the lead-in to a larger discussion of webs of quadrics:

Exercise 14.18. Let $\{Q_\mu \subset \mathbb{P}^3\}_{\mu \in \mathbb{P}^3}$ be a general web of quadrics in $\mathbb{P}^3$.

(a) Find the number of 2-planes $\Lambda \subset \mathbb{P}^3$ that are contained in some quadric of the net.

(b) A line $L \subset \mathbb{P}^3$ is said to be a *special line* for the web if it lies on a pencil of quadrics in the web. Find the class of the locus $\Sigma \subset \mathbb{G}(1,3)$ of special lines.

****This was in Ch. 9 because the second half is an application of the notion of Segre class. But there's a lot more to be said about the geometry of webs of quadrics (it connects to the degree of the locus of rank 2 quadrics in the $\mathbb{P}^9$ of quadrics in $\mathbb{P}^3$, which is a case of the symmetric Porteous formula; there's the relation between the discriminant hypersurface in the $\mathbb{P}^3$ parametrizing the quadrics and the locus in the original $\mathbb{P}^3$ of singular points of quadrics in the web, etc.)****

****add special case of Giambelli via Porteous (referred to in Chapter on General Grasmannians; add reference there)****

15

The Chow Ring of a Blowup

****Various things:

(a) First example should be blow up of an arbitrary (smooth, projective) variety at a point.

(b) Possible second example (or series of exercises) would be the graph of the birational isomorphism between a smooth quadric $Q \subset \mathbb{P}^{n+1}$ and $\mathbb{P}^n$ given by projection from a point $p \in Q$: the graph is also the blow-up $Bl_{Q'}\mathbb{P}^n$, where $Q' \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ is a quadric hypersurface in a hyperplane in $\mathbb{P}^n$; the isomorphism $Bl_p Q \cong Bl_{Q'}\mathbb{P}^n$ should give us a way of determining $A^*(Q)$.

(c) Possible keynote: If $C \subset \mathbb{P}^3$ is a smooth curve of degree d and genus g, and $S, T \subset \mathbb{P}^3$ smooth surfaces of degrees e and f containing C, at how many point of C are S and T tangent? Note we can also do this in case $e = f$ by viewing the defining equations of S and T and sections of the twist $N^*(e)$ of the conormal bundle of C; we can also do it via the calculations in the three-surfaces-containing-a-curve section: if we write $S \cap T = C \cup D$, the answer is just the intersection number $(C \cdot D)$ in $A^*(S)$, which we calculate there.

****

Keynote Questions:

(a) Suppose that S, T and $U \subset \mathbb{P}^3$ are surfaces whose intersection consists of a smooth curve and some reduced points. How many points will there be?

15.1 The Chow Ring of a Blowup

In this Chapter we will give a full description of the Chow ring of the blowup of a smooth subvariety Z of a smooth projective variety X, following the treatment in Fulton [1984]. The description will be given in terms of the Chow ring of Z, the pushforward map from this ring into the Chow ring of X, and the Chern classes of the normal bundle $N := N_{Z/X}$ of Z. The reader will find a more general case in Fulton [1984] Section 6.7.

We will use the following notation throughout this section. Let $\pi : W \to X$ be the blowup of X along Z, and let $\pi_E : E = \pi^{-1}(Z) \to Z$ be the restriction of π to the exceptional divisor. Let $i : Z \to X$ and $j : E \to W$ be the inclusions, as in the following diagram:

$$(*) \qquad \begin{array}{ccc} W = Bl_Z X & \xleftarrow{j} & E \cong \mathbb{P}N \\ \pi \downarrow & & \downarrow \pi_E \\ X & \xleftarrow{\quad i \quad} & Z \end{array}$$

The exceptional divisor E, with its projection π_E, is the projectivization $\mathbb{P}(N)$ of the normal bundle $N = N_{Z/X}$. We write

$$0 \to \mathcal{O}_{\mathbb{P}N}(-1) \to \pi_E^*(N) \to Q \to 0$$

for the tautological exact sequence on $E = \mathbb{P}N$, so $\operatorname{rank} Q = \operatorname{codim} Z - 1$. We set $m = \operatorname{codim} Z = \operatorname{rank} N$.

As usual we write

$$\zeta = c_1(\mathcal{O}_{\mathbb{P}(N)}(1)) \in A^1(E)$$

for the first Chern class of the dual of the tautological subbundle on $\mathbb{P}N$. From the theory of Chapter 11 we know the ring structure of $A(E)$ completely in terms of the ring structure of $A(Z)$ and the Chern classes of N:

$$A(E) = A(Z)[\zeta]/(\zeta^m + c_1(N)\zeta^{m-1} + \cdots + c_m(N)).$$

Theorem 15.1. *If $\pi : W \to X$ is the blowup of a smooth projective variety along a smooth subvariety Z of codimension m then, with notation as above, there is a split exact sequence of additive groups, preserving the grading by*

dimension:

$$0 \longrightarrow A(Z) \xrightarrow{\begin{pmatrix} i_* \\ h \end{pmatrix}} A(X) \oplus A(E) \xrightarrow{\begin{pmatrix} j_* & \pi^* \end{pmatrix}} A(W) \longrightarrow 0$$

where $h : A(Z) \to A(E)$ *is defined by* $h(\alpha) = -c_{m-1}(Q)\pi_E^*(\alpha)$.

Perhaps the most mysterious element in this statement is the presence of the factor $c_{m-1}(Q)$ in the definition of h. We will see in the next chapter that this is an example of a general phenomenon: from that point of view, the bundle Q should be regarded as the the pull-back of the normal bundle of Z in X modulo the normal bundle $\mathcal{O}_{\mathbb{P}E}(-1)$ of E in W, and $m-1$ is the difference $\operatorname{codim}_X Z - \operatorname{codim}_W E$. The excess intersection theorem (or at least the special case proven in this chapter) is what is necessary to show that the two maps in the exact sequence compose to zero. We begin by proving the other parts of the Theorem. We will prove that the composition is actually zero in Section 15.3 after introducing the important technique of deformation to the normal cone. ******** not yet complete.********

Beginning of the Proof of Theorem 15.1. To prove that the right hand map is a surjection we will show that, for any $\alpha \in A(W)$, the difference $\alpha - \pi^*\pi_*\alpha$ is represented by a cycle supported on E. It suffices to prove this when $\alpha = [A]$ for some subvariety $A \subset W$ that is not contained in E. Let U be the open set $U = W \setminus E$, which we identify, via π, with $X \setminus Z$.

We have $\pi_*[A] = \pi A$. By Theorem 5.10, pullback commutes with restriction to an open set in the sense that $\pi|_U^*([\pi A \cap U]) = [A \cap U]$ is equal to the image of $\pi^*[\pi A]$ in $A(U)$. Thus the difference $[A] - \pi^*\pi_*[A]$ goes to zero in $A(U)$. From the "excision" right exact sequence of Proposition **??** it follows that this difference is in the image of $A(E) \to A(W)$, completing the argument.

To prove that the left-hand map is a split monomorphism, it is enough to prove that h is a split monomorphism. We will show that $\pi_E \circ h = 1$ on $A(Z)$. To this end, we compute the Chern class of Q in terms of ζ and the Chern class of N:

$$c(Q) = \frac{c(N)}{1-\zeta}; \quad \text{so} \quad c_{m-1}(Q) = \zeta^{m-1} + c_1(N)\zeta^{m-2} + \cdots + c_{m-1}(N),$$

where we are regarding the $c_i(N)$ as elements of $A(E)$ via the ring homomorphism π_E^*. Since $c_{m-1}(Q)$ is monic in ζ, Lemma 11.5 shows that

$$\pi_{E*}(h(\alpha)) = \alpha$$

as required.

********maybe postpone the following until after proving that the composition is generally zero? We use the weaker "key formula" here, in any

case.**** Now suppose that

$$(\chi,\epsilon)\in \mathrm{Ker}\left(A(X)\oplus A(E)\xrightarrow{(\pi^*,j_*)} A(W)\right) \text{ with } \epsilon=\sum_{i=0}^{m-1}\epsilon_i\zeta^i.$$

By the Moving Lemma χ can be represented by a cycle generically transverse to Z, from which it follows that $\pi_*\pi^*\chi=\chi$ (this would be true for any birational map.) Also, for $i<m-1$ the cycle ζ^i can be represented by a subvariety with positive fiber dimension over Z, so $\pi_{E*}\epsilon_i\zeta^i=\epsilon_i\pi_{E*}\zeta^i=0$, while $\pi_{E*}\zeta^{m-1}=1\in A(Z)$. Thus

$$0=\pi_*(\pi^*\chi+j_*\epsilon)=\chi+i_*\pi_{E*}\epsilon=\chi+i_*\epsilon_{m-1};$$

thus $\chi=-i_*\epsilon_{m-1}$. We want to show that $\epsilon=\epsilon_{m-1}c_{m-1}(Q)$.

Using Theorem 15.10 we get

$$0=j^*(\pi^*\chi+j_*\epsilon)=(\pi_E^*i^*\chi)+\epsilon\zeta=-\epsilon_{m-1}\pi_E^*(c_m(N))+\epsilon\zeta\in A(E).$$

Thus as elements of $A(Z)[\zeta]$ the polynomial

$$-\epsilon_{m-1}c_m(N)+\epsilon\zeta$$

is divisible by the monic polynomial

$$c_m(N)+c_{m-1}\zeta+\cdots+\zeta^m=c_m(N)+(c_{m-1}+\cdots+\zeta^{m-1})\zeta.$$

It follows that

$$\epsilon=\epsilon_{m-1}(c_{m-1}+\cdots+\zeta^{m-1})=\epsilon_{m-1}c_{m-1}(Q)\in A(E)$$

as required. □

Now that we know that $A(W)$ is generated by $j_*(A(E))$ and $\pi^*(A(X)$, the following result describes the multiplicative structure:

Proposition 15.2. *With hypotheses and notation as above, the rules for multiplication in $A(W)$ are:*

$$\begin{aligned}(\pi^*\alpha)(\pi^*\beta)&=\pi^*(\alpha\beta),\quad \textit{for } \alpha,\beta\in A^*(X);\\ (\pi^*(\alpha))(j_*\gamma)&=j_*(\pi_E^*i^*(\alpha)\gamma),\quad \textit{for } \alpha\in A^*(X) \textit{ and } \gamma\in A^*(E);\ \textit{and}\\ (j_*\gamma)(j_*\delta)&=-j_*(\gamma\delta\zeta),\quad \textit{for } \gamma,\delta\in A^*(E)\end{aligned}$$

Proof. The first formula says that the pullback $\pi^*:A^*(X)\to A^*(W)$ is a ring homomorphism, which is part of Theorem 5.10.To prove the second formula, choose a cycle A representing α whose components A_i are generically transverse to Z, and then choose a cycle $G\subset E$ representing γ whose components G_j are generically transverse to the preimages, in E, of the components of $A\cap Z$. The intersections $\pi_E^{-1}(A_i\cap Z)\cap G_j=\pi^{-1}A_i\cap G_j$ are then reduced. Computing the dimensions we see that these intersections are generically transverse either as subvarieties of E or of W. Thus, with

the multiplicites with which they appear in α and γ they represent both the left and right hand sides of the desired equality.

The third formula is a special case of Theorem 15.10, below, using the fact that the normal bundle $N_{E/W} = \mathcal{O}_{\mathbb{P}N}(-1)$. □

15.2 The blow-up of $\mathbb{P}^3$ along a curve

We pause to consider the first interesting case of a blow-up of a positive-dimensional variety. Let $X = \mathbb{P}^3$ and let Z be a smooth Zurve of degree d and genus g, and let $\pi : W \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along Z. Let E, i and j be as in the diagram (*) above.

Let $h \in A^1(\mathbb{P}^3)$ be the class of a plane, and write $l = h^2 \in A^2(\mathbb{P}^3)$ the class of a line. Set $\tilde{h} = \pi^* h \in A^1(W)$ and $\tilde{l} = \pi^* l \in A^2(W)$, their pullbacks to W. For $D \in Z^1(Z)$ any divisor, we denote by $F_D = \pi_E - 1D \in Z^1(E)$ the corresponding divisor on E, and for a divisor class $\delta \in \mathrm{Pic}(Z)$ we let $f_\delta = j_*\big(\pi_E^* \delta\big) \in A(W)$.

From the Adjunction formula we can determine the Chern classes of the normal bundle of Z. Indeed, we have

$$\omega_Z = \wedge^2 N_{Z/\mathbb{P}^3} \otimes \omega_{\mathbb{P}^3}|_Z = \wedge^2 N_{Z/\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(-4)|_Z$$

so, since $A^i(Z) = 0$ for $i > 1$ we get

$$c(N_{Z/\mathbb{P}^3}) = \frac{1 + c_1(\omega_Z)}{1 - 4h_Z} = 1 + c_1(\omega_Z) + 4h_Z$$

The formula for $c_{m-1}(Q)$ in the previous section now becomes

$$c_1(Q) = \big[\frac{1 + c_1(\omega_Z) + 4h_Z}{1 - \zeta}\big]_1 = \zeta + c_1(\omega_Z) + 4h_Z.$$

Theorem 15.1 now gives a complete description of $A(W)$, most of which can be derived directly. To begin with, $A^0(W)$ and $A^3(W) \cong \mathbb{Z}$, generated by the fundamental class of W and the class of a point, respectively.

The group $A^1(W)$ is also not hard to determine, even without the Theorem: the restriction of any line bundle on W to the complement $U = W \setminus E \cong \mathbb{P}^3 \setminus Z$ of E is isomorphic to $\mathcal{O}(n)$ for a unique n, and any line bundle on W trivial on U must be $\mathcal{O}_W(kE)$ for some k. So $A^1(W)$ is generated by the classes $\tilde{h}$ and e, which are independent since π_* sends $\tilde{h}$ to h and e to 0.

Lastly, by Theorem 15.1, the group $A^2(W)$ is generated by the classes $\tilde{l}$ (the pullback of $A^2(\mathbb{P}^3)$), and the classes $j_*\zeta$ and $j_* f_\delta$ for $\delta \in A^1(Z)$, and the only relation among these is that

$\pi^*([Z]) = j_*(c_1(Q)) = j_*(\zeta + c_1(\omega_Z) + 4h_Z)$.

****The following argument seems incomplete to me**** We can derive this relation concretely in this case. Applying a one-parameter subgroup of PGL_4 to Z we get a deformation of Z: that is, a family $\mathcal{Z} \subset U \times \mathbb{P}^3$ over an open subset $U \subset \mathbb{A}^1$ containing the origin $0 \in \mathbb{A}^1$, flat over U, with fiber $Z_0 = Z \subset \mathbb{P}^3$. If the subgroup is chosen generally, then for $t \neq 0 \in U$ the curve Z_t is disjoint from Z, and moreover the section σ of the normal bundle N of Z corresponding to the restriction of $\mathcal{Z}$ to the first-order neighborhood of $0 \in \mathbb{A}^1$ is nowhere zero. For $t \neq 0 \in U$, let $\tilde{Z}_t \subset W$ be the preimage of Z_t in W.

We claim that the flat limit $\tilde{Z}_0$ of $\tilde{Z}_t$ as $t \to 0$ is the section of $E = \mathbb{P}N$ corresponding to the trivial sub-line bundle $\mathcal{O}_Z \subset N$ spanned by σ. ****This needs a proof****.

Given the claim, Proposition 12.22 ****of Segre chapter 9.5**** then tells us the class of $\tilde{Z}_0$: we have

$$[\tilde{Z}_0] = \zeta + f_{c_1(N)} = \zeta + 4f_h + f_\omega \in A^*(E).$$

On the other hand, for $t \neq 0 \in U$ the class of $\tilde{Z}_t$ is visibly $d\tilde{l}$; and so the rational equivalence $\tilde{Z}_t \sim \tilde{Z}_0$ gives us the relation

$$d\tilde{l} = j_*\zeta + j_* f_{c_1(N)}$$

15.2.1 The intersection of three surfaces in $\mathbb{P}^3$ containing a curve

As an application, we can compute the number of "residual points" of intersection of three surfaces that meet in a smooth curve together with some additional points, answering Keynote Question a

Proposition 15.3. *Let S, T and $U \subset \mathbb{P}^3$ be surfaces of degrees s, t, u. If the intersection $S \cap T \cap U$ consists of the disjoint union of a smooth curve of degree d and genus g, and a zero-dimensional scheme Γ, then*

$$\deg \Gamma = stu - d(s + t + u) + (2g - 2 + 4d).$$

For example, if Z is a line then Γ will have degree $stu - (s+t+u) + 2$. It is easy to see this formula directly when $d = 1$, so that S is a plane: then the curves residual curves $(S \cap T) \setminus Z$ and $(S \cap U) \setminus Z$, begin plane curves of degrees $t - 1$ and $u - 1$, meet in $(t-1)(u-1) = stu - (s + t + u) + 2$ points.

As a second example, three quadric surfaces meet in the twisted cubic and no further points, and indeed

$$2^3 - 3(2 + 2 + 2) + (0 - 2 + 12) = 0.$$

As noted by Fulton [1984] Example 9.1.2, the Proposition can also be used to show that some curves in $\mathbb{P}^3$, such as smooth curve of genus 1 and degree 5, cannot be written as the scheme-theoretic intersection of three surfaces. The equation $stu-5(s+t+u)+20=0$ has the solution $(s,t,u)=(2,3,5)$ but such a curve cannot lie on a quadric, and the equation cannot be solved for $s,t,u\geq 3$. See Exercise 15.4 for another such application.

Exercise 15.4. (Fulton) Show that a nonsingular curve of degree 5 and genus 1 in $\mathbb{P}^3$ cannot be written as the scheme-theoretic intersection of three surfaces, and that a smooth rational curve of degree 5 in $\mathbb{P}^3$ can only be written as the scheme-theoretic intersection of 3 surfaces if it lies on a quadric. Show that the general rational curve of degree 5 in $\mathbb{P}^3$ does not lie on a quadric. This last example, first observed by Oscar Perron in 1941, settled a long-running confusion between scheme-theoretic and set-theoretic complete intersections. (Every algebraic set in $\mathbb{P}^n$, over an infinite field, is an intersection of n hypersurfaces; see Eisenbud and Evans [1973] for this and a brief history of the subject.)

Exercise 15.5. Let S, T, U and $V\subset\mathbb{P}^4$ be hypersurfaces of degrees d, e, f and g respectively, and suppose that the intersection

$$S\cap T\cap U\cap V=Z\cup\Gamma$$

with Z a smooth curve of degree d and genus g, and Γ a collection of reduced points disjoint from Z. What is the degree of Γ?

Proof of Proposition 15.3. Write $S\cap T\cap U=Z\cup\Gamma$, where Z is the smooth curve of degree d and genus g. If we let $W=Bl_Z(\mathbb{P}^3)\to\mathbb{P}^3$ be the blow up, with exceptional divisor $E\subset W$, and let $\tilde{S}$, $\tilde{T}$ and $\tilde{U}$ be the proper transforms of the three surfaces we started with, then the fact that the intersection $S\cap T\cap U$ is reduced along Z says that the proper transforms $\tilde{S}$, $\tilde{T}$ and $\tilde{U}$ have no common intersection along E. In other words,

$$\tilde{S}\cap\tilde{T}\cap\tilde{U}=\pi^{-1}(\Gamma)$$

and in particular, the degree of Γ is the degree of the product $[\tilde{S}][\tilde{T}][\tilde{U}]\in A^3(W)$.

If one of S,T,U, say S were double along Z, then our hypothesis implies that $T\cap U=Z\cup D$ for some curve D that meets Z in only finitely many points. Moreover, a complete intersection is necessarily connected (for example by Hartshorne's Theorem Eisenbud [1995] Theorem ****), so D is either empty or meets Z. In the first case $S\cap T\cap U=T\cap U$, and Γ is empty. In the second case the intersection $T\cap U$ is singular at the points of $Z\cap D$, and thus has Zariski tangent spaces of dimension ≥ 2 at these points. It follows that $S\cap T\cap U$ is not reduced at the points of $Z\cap D$, contradicting our hypothesis. Thus we may assume that each of S,T,U is generically reduced along Z.

This has the consequence that each of $\tilde{S}, \tilde{T}$ and $\tilde{U}$ contains E with multiplicity just 1, so we get

$$[\tilde{S}] = s\tilde{h} + [E], [\tilde{T}] = t\tilde{h} + [E], [\tilde{U}] = u\tilde{h} + [E].$$

in $A^1(W)$. Thus

$$\begin{aligned}\deg(\Gamma) &= \deg\Big((s\tilde{h} - [E])(t\tilde{h} - [E])(u\tilde{h} - [E])\Big) \\ &= \deg\Big((stu)\tilde{h}^3 - (st + su + tu)\tilde{h}^2[E] + (s + t + u)\tilde{h}[E]^2 - [E]^3\Big)\end{aligned}$$

It remains to identify the degrees of the monomials $\tilde{h}^3, \tilde{h}^2[E], \tilde{h}[E]^2$ and $[E]^3$. The rules for multiplication given in Proposition 15.2 do this for us.

First of all, $\deg \tilde{h}^3 = \pi^*(h^3) = 1$, while $\tilde{h}^2[E] = \pi_E^*(h^2[Z]) = 0$ since h^2 is the class of a general line. On the other hand, $\tilde{h}[E]^2 = \tilde{h}(-j_*\zeta) = -j_*(\pi_E^*(i^*h)\zeta)$. Since ζ restricts to a point (the "hyperplane section") on each fiber, this has the same degree as i^*h, that is,

$$\deg \tilde{h}[E]^2 = \deg Z = d.$$

Finally, $[E]^3 = j_*([E](-\zeta)) = j_*(\zeta^2)$, and from the relation

$$0 = \zeta^2 + c_1(N)\zeta + c_2(N) = \zeta^2 + (2g - 2 + 4d)\zeta,$$

in the Chow ring of E we deduce that

$$\deg[E]^3 = -(2g - 2 + 4d).$$

Inserting these into the formula for $\deg \Gamma$ above, we get

$$\deg \Gamma = stu - d(s + t + u) + (2g - 2 + 4d)$$

□

Exercises 15.6–15.9 given some further applications of what we have developed.

Exercise 15.6. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree e containing Z, $\tilde{S} \subset W$ its proper transform and $\Sigma_S = \tilde{S} \cap E$ the curve on E consisting of normal vectors to Z contained in the tangent space to S. Find the class $[\Sigma_S] \in A^*(W)$ of Σ_S in the blow-up

(a) by applying Proposition **??** of Chapter 11; and

(b) by multiplying the class $[\Sigma_S]$ by the class $e = [E]$ in $A^*(W)$.

Exercise 15.7. Let $q \in \mathbb{P}^3$ by any point not on the tangential surface of Z, and let $\Gamma \subset E \subset W$ be the curve of intersections with E of the proper transforms of lines $\overline{pq}$ for $p \in Z$. Find the class of Γ in $A^*(W)$.

Exercise 15.8. Let $B \subset \mathbb{P}^3$ be another curve, of degree m, and suppose that B meets Z in the points of a divisor D. Show that the class of the proper transform $\tilde{B} \subset W$ of B is

$$[\tilde{B}] = ml - F_D.$$

Exercise 15.9. In Section 11.2.1 of Chapter 11, we observed that the blow-up $X = Bl_{\mathbb{P}^k}\mathbb{P}^n$ of $\mathbb{P}^n$ along a k-plane was a $\mathbb{P}^{k+1}$-bundle over $\mathbb{P}^{n-k-1}$, and used this to describe the Chow ring of X. We now have another description of the Chow ring of X. Compare the two, and in particular

(a) Express the generators of $A^*(X)$ given in this chapter in terms of the generators given in Section 11.2.1; and thus

(b) Find the relations among the generators given here.

15.3 Intersections in a Subvariety

The last formula in Proposition 15.2 expresses the product, in $A(X)$, of two cycles that happen to be supported the subvariety E in terms of their product in $A(E)$. This is a special case of a more general assertion about any smooth subvariety (it remains true for locally complete intersection subvarieties, with essentially the same proof) :

Theorem 15.10. *Let $i : Z \to X$ is an inclusion of smooth projective varieties of codimension m, and let $N = N_{Z/X}$ be the normal bundle of Z in X. then for any class $\alpha \in A^*(Z)$ we have*

$$\textit{"Key Formula"}: \quad i^*\big(i_*\alpha\big) = \alpha \cdot c_m(N) \in A^{a+m}(Z).$$

Further, if $\alpha \in A^a(Z)$ and $\beta \in A^b(Z)$ then

$$i_*\alpha \cdot i_*\beta = i_*\big(\alpha \cdot \beta \cdot c_m(N_{Z/X})\big) \in A^{a+b+2m}(X).$$

As we shall see, the second formula of Theorem 15.10 follows easily from the first, but the first one requires a new idea. Recall that we have already proven the formula for the case when X is the total space of the compactification $\mathbb{P}(N_{Z/X} \oplus \mathcal{O}_X)$ of the normal bundle of Z (Proposition 12.26). The intuition behind the Key Formula in general is that a neighborhood of of Z in X looks "like" a neighborhood of Z in its normal bundle. Indeed, if we were working with C^∞ manifolds a neighborhood in the classical topology would be C^∞ equivalent to a neighborhood of Z in its normal bundle. This C^∞ equivalence is, at least, enough to make i^*i_* behave the same in homology.

This idea doesn't give a proof of Theorem 15.10, even in the complex analytic case: rational equivalence is more subtle than homological equivalence,

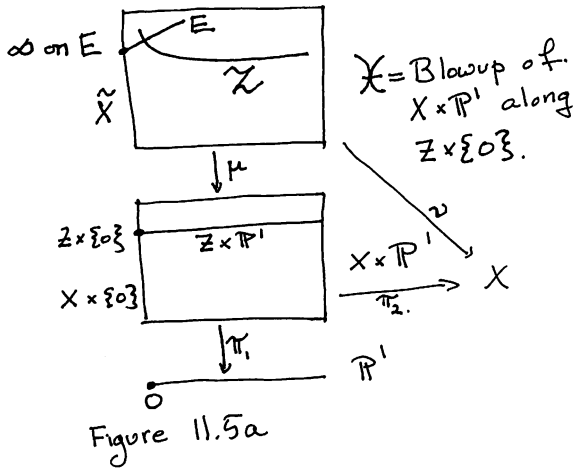


FIGURE 15.1. Deformation to the Normal Cone

and tubular neighborhood theorem that we used is false in the category of complex analytic or algebraic varieties (****simplest example?****). However, there is a way out: a construction called "deformation to the normal cone" provides a flat degeneration from the neighborhood of Z in X to the neighborhood of Z in its normal bundle. This is one of the beautiful and important techniques introduced into the subject by]. We first explain the construction.

15.3.1 Deformation to the normal cone

Suppose that X is a smooth projective variety of dimension n, and $Z \subset X$ a smooth subvariety of codimension m. Let

$$\mu : \mathcal{X} = Bl_{Z\times\{0\}}(X \times \mathbb{P}^1) \to X \times \mathbb{P}^1,$$

be the blowup of $X \times \mathbb{P}^1$ along the subvariety $Z \times \{0\}$, and write $E \subset \mathcal{X}$ for the exceptional divisor. As a variety, E is the projectivization of the normal bundle

$$N_{Z\times\{0\}/X\times\mathbb{P}^1} \cong N_{Z/X} \oplus \mathcal{O}_Z.$$

Thus E is the compactification of the total space of the normal bundle $N = N_{Z/X}$ described in Section **??** of Chapter 11.

We think of $\mathcal{X}$ as a family of projective varieties over $\mathbb{P}^1$ via the composition $\pi_2 \circ \mu : \mathcal{X} \to X \times \mathbb{P}^1 \to \mathbb{P}^1$. The fibers X_t of $\mathcal{X}$ over $t \neq 0$ are all isomorphic to X, while the fiber X_0 of $\mathcal{X}$ over $t = 0$ consists of two irreducible components: the proper transform $\tilde{X}$ of $X \times \{0\}$ in $\mathcal{X}$ (isomorphic

to the blow-up $Bl_Z(X)$), and the exceptional divisor $E \cong \mathbb{P}(N \oplus \mathcal{O}_Z)$, with the two intersecting along the "hyperplane at infinity" $\mathbb{P}N \subset \mathbb{P}(N \oplus \mathcal{O}_X)$ in E, which is the exceptional divisor in $\tilde{X} \cong Bl_Z(X)$.

Now consider the subvariety $Z \times \mathbb{P}^1 \subset X \times \mathbb{P}^1$, and let $\mathcal{Z}$ be its proper transform in $\mathcal{X}$. Because $Z \times \mathbb{P}^1$ intersects $X \times \{0\}$ transversely in $Z \times \{0\}$, the intersection $\mathcal{Z} \cap X_0$ consists exactly of the zero section $N_0 \subset |N| \subset \mathbb{P}(N \oplus \mathcal{O}_Z)$ corresponding to the sub-line bundle $\mathcal{O}_Z \subset N \oplus \mathcal{O}_Z$. In particular, all the fibers $\mathcal{Z}_t$ of $\mathcal{Z}$ are isomorphic to Z, and $\mathcal{Z}$ goes nowhere near the component $\tilde{X}$ of X_0.

Also, let

$$\nu = \pi_1 \circ \mu : \mathcal{X} \to X$$

be the composition of the blow-up map $\mu : \mathcal{X} \to X \times \mathbb{P}^1$ with the projection on the first factor. For $t \in \mathbb{P}^1$ other than 0, ν carries X_t isomorphically to X. As for the fiber X_0, on the component $\tilde{X} \subset X_0$ the map ν is the blow-down map $\tilde{X} = Bl_Z X \to X$, while on the component E the map ν is the composition $i \circ \pi$ of the bundle map $\pi : E \cong \mathbb{P}(N \oplus \mathcal{O}_Z) \to Z$ with the inclusion $i : Z \hookrightarrow X$. The situation is summarized in Figure 15.1.

The point of this construction is that as $t \in \mathbb{P}^1$ approaches 0, the neighborhood of Z in the fibers of the family $\mathcal{X} \to \mathbb{P}^1$ degenerates from the neighborhood of Z in X to the neighborhood of Z in the total space of its normal bundle in X. If we have a family $\{A_t\}_{t \in \mathbb{P}^1}$ of cycles in X that we'd like to intersect with Z, we can use this construction to transform the intersection of A_0 with Z into the intersection of the fiber of the proper transform of $\mathcal{A}$ in $\mathcal{X}$ with the zero section in the compactified normal bundle $E \cong \mathbb{P}(N \oplus \mathcal{O}_Z)$. We can use our knowledge of the Chow rings of projective bundles to analyze this intersection.

The idea of the following proof is that under the deformation to the normal cone the class $i^* i_* \alpha$ is deformed into the rationally equivalent class $i_N^* i_{N*} \alpha$, where

$$i_N : Z \to \mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z) = E$$

is the section sending Z to the zero locus $N_0 \subset |N| \subset \overline{N} := \mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z)$. By Proposition 12.26, $i_N^* i_{N*} \alpha = \alpha \cdot c_m(N)$

Proof of the "Key Formula" of Theorem 15.10. We will use the notation introduced in the construction of the deformation to the normal cone, above.

We begin with the first formula of the Theorem. We may assume that α is the class of an irreducible subvariety $A \subset Z$. Let $\mathcal{A}$ and $\mathcal{Z}$ be the proper transforms of the subvarieties $A \times \mathbb{P}^1$ and $Z \times \mathbb{P}^1$. Since $Z \times \mathbb{P}^1$ meets $X \times \{0\}$ transversely, $\mathcal{Z} \cong Z \times \mathbb{P}^1$ via the projection, and this isomorphism also induces an isomorphism $\mathcal{A} \cong A \times \mathbb{P}^1$. To simplify notation, we will

write $A_t \subset Z_t \subset X_t$ for the copies in the general fiber of $\mathcal{X}$ but we will write $A \subset Z \subset E$ instead of $A_0 \subset Z_0 \subset E$ for the fibers $A \subset Z$ contained in $E \subset X_0$.

By the moving lemma we can find a cycle $\mathcal{C}$ on $\mathcal{X}$ linearly equivalent to $\mathcal{A}$ and generically transverse to $\mathcal{Z}$, to Z, to E to X_t and to Z_t. The family $\mathcal{A}$ meets X_t and E generically transversely in A_t and A, respectively, so the equality $[\mathcal{C}] = [\mathcal{A}] \in A(\mathcal{X})$ restricts to equalities $i_*[A_t] = i_*[\mathcal{A} \cap X_t] = [\mathcal{C} \cap X_t] \in A(X_t)$, and $i_{N*}[A] = [\mathcal{C} \cap E] \in A(E)$. Since $\mathcal{C}$ meets Z_t and Z generically transversely as well, we have $i^* i_*[A_t] = [\mathcal{C} \cap Z_t]$ and $i_N^* i_{N*}[A] = [\mathcal{C} \cap E]$.

By generic transversality, neither Z_t nor Z can be contained in $\mathcal{C}$. It follows that, after removing any components that do not dominate $\mathbb{P}^1$, the cycle $\mathcal{C}$ in $\mathcal{Z} \cong Z \times \mathbb{P}^1$ is a rational equivalence between $i^* i_*[A]$ and $i_N^* i_{N*}[A]$. By Proposition 12.26, $i_N^* i_{N*}[A] = [A] \cdot c_m(N)$ as required.

Now it is easy to deduce the second formula of the Theorem. If $\beta = [Z]$, the formula follows from the "Key Formula" by taking the pushforward i_* of both sides. On the other hand, if $\beta \in A^*(Z)$ is arbitrary, we can use the push-pull formula: since

$$i^*(i_*\alpha) = \alpha \cdot c_m(N),$$

we can write

$$\begin{aligned} i_*\big(\alpha \cdot \beta \cdot c_m(N)\big) &= i_*\big(\beta \cdot i^*(i_*\alpha)\big) \\ &= i_*\alpha \cdot i_*\beta. \end{aligned}$$

□

Exercise 15.11. Let $X \subset \mathbb{P}^5$ be a smooth hypersurface of degree d, and $\Lambda \cong \mathbb{P}^2 \subset X$ a 2-plane contained in X. What is (the degree of) the self-intersection $[\Lambda]^2 \in A^4(X)$ of Λ in X? Check this in case $d = 1$ and 2.

Conclusion of the Proof of Theorem 15.1. We must show that the composition of the maps in the sequence shown in the Theorem is zero, that is, $j_*(c_{m-1}(Q)\pi_E^*\alpha) = \pi^* i_*\alpha$ for any $\alpha \in A(Z)$.

********sketch of a possible argument******** Blow up the subvariety $\mathcal{Z}$ in the "deformation to the normal cone" argument, and argue that the proper transform of $\mathcal{C}$ in the proof of the "key formula" connects the one side of the equality we want to prove with the corresponding side of the equality in the case where $X = \overline{N}$, the compactified normal bundle, while the other side is essentially constant in the family. This reduces to the case of the compactified normal bundle. But in that case ...

By Theorem 5.10, $\pi^* i_*\alpha$ goes to zero in the Chow group of $W \setminus E = X \setminus Z$, and thus is in the image of j_*. ********The following was based on an error!

Still needs a proof... **** Consequently, it is enough to prove the equality after applying j^* to both sides.

By Theorem 15.10 and the push-pull formula we have

$$\begin{aligned} j^* j_*(c_{m-1}(Q)\pi_E^*\alpha) &= c_1(N_{E/W})c_{m-1}(Q)\pi_E^*\alpha \\ &= c_1(\mathcal{O}_{\mathbb{P}N_{Z/W}}(-1))c_{m-1}(Q)\pi_E^*\alpha. \end{aligned}$$

From the tautological exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}N_{Z/W}}(-1) \to \pi_E^* N_{Z/W} \to Q \to 0$$

we get

$$c_1(\mathcal{O}_{\mathbb{P}N_{Z/W}}(-1))c_{m-1}(Q) = c_m(\pi_E^* N_{Z/W}).$$

On the other hand, by functoriality $j^*\pi^* = \pi_E^* i^*$ ****we need a reference for this!****, so $j^*\pi^* i_*\alpha = \pi_E^* i^* i_*\alpha = \pi_E^*(c_m(N_{Z/X})\alpha)$, proving the desired equality.

□

16

Excess Intersection Formulas

Keynote Questions:

(a) Suppose two surfaces in $\mathbb{P}^4$ meet in a smooth curve and some reduced points. How many points?

(b) How many nonsingular conics in $\mathbb{P}^2$ are simultaneously tangent to five given lines?

(c) Let $X \subset \mathbb{P}^5$ be the complete intersection of 3 general quadrics, and let Y be a general projection of X to $\mathbb{P}^4$. How many singular points does Y contain?

******** 12.3 is not ideal, since it's so easy another way. On the other hand doing the Chasles problem would involve assuming or proving that the sextics cut out the second inf nbhd of the Vero scheme-theoretically, not to mention worrying about the transversality elsewhere. Even Fulton doesn't do this – he just says what happens in the blowup.********

The keynote questions of this Chapter are examples of what is called *excess intersection*: subvarieties whose intersection contains a component with larger than expected dimension. This situation arises frequently in geometric problems when the evident equations defining the solution to the problem are also satisfied in some degenerate case. For example, the set of plane conics tangent to a given line is a quadric hypersurface in $\mathbb{P}^5$, so one might expect the number of plane conics simultaneously tangent to five general lines would be $2^5 = 32$. However each of these quadrics contains the set of double lines (in characteristic $\neq 2$ this is the Veronese surface in

$\mathbb{P}^5$) and these form a component of the intersection. This is the situation of Keynote Question b ****we should perhaps insert the solution in char 2 as well when we come to it!)****

In these circumstances, knowledge of the Chow ring of the ambient space seems useless; intersection theory, on the face of it, tells us something about the actual intersection of the cycles only under the hypothesis that the intersection has the expected dimension. Can we say something about the class of the components of an intersection that do happen to have the expected dimension even when some other components have too large a dimension?

One way to think about this question is to imagine that the intersection we wish to treat is the limit of intersections of varieties that intersect transversely. Then we could rephrase the question by asking: how many of the points of the nearby transverse intersection have limits in the component of too large dimension? Is this number determined by some aspect of the local geometry near the non-transverse intersection?

The answer is yes, in a remarkably strong sense: the *excess intersection formula* explains how the intersection class comes from classes in the chow groups of the connected components, whatever their dimension, in terms of the Chern classes of the normal bundles of these components, under hypotheses guaranteeing that these bundles exist (and even a little more generally). In the case where the intersection is supposed to be zero-dimensional we can get the degree of the part that is really zero-dimensional by subtracting the degrees of the cycles defined in this way on the components of the intersection that are too large.

In its most general form, this result is the centerpiece of Fulton's development of *refined* intersection products, and is in fact a different, and technically superior (though considerably less intuitive) foundation for the whole theory of the Chow ring of a smooth variety. For a proof of the main result we refer the reader to Fulton [1984].

In the next section we will give an introduction to this result in a special case that is not too technical. We then give an argument that indicates why the formula is reasonable; this argument could be made into a proof of a weak form of the Theorem. In the third and fourth sections we explain more general versions of the result, and in the last section we give a famous application, a formula that can be used to count the double points of certain projections of smooth varieties.

16.1 Introduction to the Excess Intersection Formula

Here is an elementary example that turns out to be typical:

Example 16.1 (Two Plane Curves)**.** Consider two plane curves $A_1, A_2 \subset \mathbb{P}^2$ of degrees d_1, d_2, so that $[A_1][A_2] = d_1 d_2[p] \in A(\mathbb{P}^2)$, where $[p]$ is the class of a point. Suppose now that A_1 and A_2 have a component Z of degree e in common, but are otherwise general.

In this case we can "subtract" Z from each of the A_i (by dividing their equations by the equation of Z, say) to get curves $A_1' := A_1 - Z$ and $A_2' = A_2 - Z$ of degrees $d_1 - e, d_2 - e$. Since we have assumed that A_1' and A_2' are general, they will meet in $(d_1 - e)(d_2 - e) = d_1 d_2 - (d_1 + d_2)e + e^2$ reduced points away from Z. In this sense the intersection along Z "counts for" precisely $(d_1 + d_2)e - e^2$ points.

In particular, for any flat families of curves $A_{1,t}, A_{2,t}$ such that $A_{1,0} = A_1$ and $A_{2,0} = A_2$, while $A_{1,t}$ and $A_{2,t}$ are transverse for $t \neq 0$, exactly $(d_1 + d_2)e - e^2$ points of the intersection $A_{1,t} \cap A_{2,t}$ will approach Z.

Exercise 16.2. Suppose that n hypersurfaces $D_i \subset \mathbb{P}^n$ have intersection $\cap_i D_i$ equal to a set of reduced points and a smooth hypersurface Z. Find the number of isolated points in terms of the degrees of the D_i and the degree of Z.

At least if we assume that resolution of singularities is possible (as we know it to be in characteristic zero), we can see that this is part of a general phenomenon. Suppose, for example, that A_1, A_2 are subvarieties of a smooth projective variety X whose codimensions add up to the dimension of X, and that they meet transversely except along a subscheme Z that is a common component of their intersection. We can blow up repeatedly along centers in Z and in its preimage to arrive at a smooth projective variety X' dominating X where the strict transforms A_i' of the A_i do not meet along the preimage of Z. Of course we have not changed their intersection away from Z, so we see that Z "accounts for" a number of intersection points that is the difference between $\deg[A_1][A_2] \in A(X)$ and $\deg[A_1'][A_2'] \in A(X')$—a number that must be entirely determined by the local geometry of the intersection near Z. We wish to compute this number. The result will say in particular that under favorable circumstances there is a formula that depends only on the normal bundles of the varieties involved.

To express the formula, we will again use the notation

$$[W]_m \in A_m(Z)$$

to denote the component of dimension m of an element $W \in A(Z)$. For example, if F is a bundle on Z then

$$[c(F)]_m = c_{\dim Z - m} \in A_m(Z).$$

****note that Fulton uses $\{[A]\}_m$ instead of $[A]_m$. Do we want to change? Also, the notation $[]_m$ was used to denote something of codim m later in this chapter, and I've changed it. OK?****

To understand the terms that enter the formula, note that if we have inclusions $Z \subset A_i \subset X$ of smooth varieties, then there are bundle inclusions of the normal bundles $N_{Z/X} \to N_{A_i/X}|_Z$. Putting these together we have an inclusion

$$N_{Z/X} \to N_{A_1/X}|_Z \oplus N_{A_2/X}|_Z.$$

The cokernel of this map is a bundle F on Z. If we write $m := \dim X - \operatorname{codim} A_1 - \operatorname{codim} A_2$ for the expected dimension of the intersection, then the rank of the bundle F is the excess dimension $\dim Z - m$.

The varieties A_i meet transversely in Z if and only if $F = 0$, so the nontriviality of F is a measure of the failure of transversality. The Excess Intersection Theorem, in its simplest case, says that the top Chern class

$$c_{\operatorname{rank} F}(F) = [c(F)]_m$$

represents the "amount of transverse intersection that Z has swallowed up." For example, if $\dim Z = m$, then $F = 0$ and $c_0(F) = [Z]$, as it should. Here is a preliminary form of the main result (see Theorem 16.10 for a more general form.)

Theorem 16.3 (Excess Intersection Theorem—Preliminary Version)**.** *Let A_1, A_2 be smooth subvarieties of a smooth projective variety X, and suppose that $A_1 \cap A_2 = \cup_j Z_j$ is a disjoint union of smooth varieties Z_j. Write $\iota : Z_j \to X$ for the inclusion morphism, and set*

$$F_j = \operatorname{coker} N_{Z_j/X} \to N_{A_1/X}|_{Z_j} \oplus N_{A_2/X}|_{Z_j}.$$

If $m = \dim X - \operatorname{codim} A_1 - \operatorname{codim} A_2$ is the "expected dimension" of the intersection, then

$$[A_1][A_2] = \sum_j \iota_* c_{\operatorname{rank} F_j}(F_j) \in A(X).$$

In applications it is often convenient to write the classes $c_{\operatorname{rank} F_j}(F_j)$ directly in terms of the normal bundles of A_1, A_2 and Z. We can do this through the the Whitney sum formula, which yields

$$c_{\operatorname{rank} F_j}(F_j) = \left[\frac{c(N_{A_1/X}|_{Z_j}) c(N_{A_2/X}|_{Z_j})}{c(N_{Z_j/X})} \right]_m.$$

An important point to notice is that the intersection class $[A_1][A_2]$ is obtained by pushing forward the well-defined rational equivalence classes

$c_{\operatorname{rank} F_j}(F_j)$ on the components of the set-theoretic intersection $\cap A_i$. These *refined intersection classes* are a sharper invariant of the intersection even when the intersection does have the expected dimension. It is a key to many applications of intersection theory, and a central point of the Fulton-MacPherson development of intersection theory.

We will spend the rest of this section working out a collection of simple examples.

16.1.1 First Examples

Example 16.4. $A_1 \cap A_2$ is transverse along a component Z_j we have $\dim Z = m$ so $[c(F)]_m = [Z]$, no matter what bundle F is (in this case it is the zero bundle.) Thus transverse intersections are counted correctly.

Example 16.5 (Two Plane Curves, continued)**.** In the case of two plane curves $A_1, A_2 \subset X := \mathbb{P}^2$ treated above, the normal bundle of A_i is $\mathcal{O}_{\mathbb{P}^2}(d_i)|_{A_i}$, and the normal bundle of Z is $\mathcal{O}_{\mathbb{P}^2}(e)|_Z$ Thus $c(N_{A_i/X}) = 1 + d_i\zeta$, where ζ is the class of a line. If we write ζ_Z for the restriction of the class of ζ to the curve Z, then $c(N_{A_i/X}|_Z) = 1 + d_i\zeta_Z$. The assertion of Theorem 16.3 is:

$$d_1d_2 = (\#\text{ isolated points of intersection}) + \iota_*\left(\left[\frac{(1+d_1\zeta_Z)(1+d_2\zeta_z)}{(1+e\zeta_Z)}\right]_0\right),$$

To parse the right hand side, note that $1 \in A(Z)$, the fundamental class, has dimension 1, while ζ_Z has dimension 0—it is the class of e points, the intersection of Z with a line. Thus $1/(1+e\zeta_Z) = 1 - e\zeta_Z$ and

$$\left[\frac{(1+d_1\zeta_Z)(1+d_2\zeta_Z)}{(1+e\zeta_Z)}\right]_0 = (d_1 + d_2 - e)\zeta_Z$$

is the class in Z of the intersection of Z with $d_1 + d_2 - e$ lines. It follows that

$$\iota_*\left(\left[\frac{(1+d_1\zeta_Z)(1+d_2\zeta_z)}{(1+e\zeta_Z)}\right]_0\right)$$

is the class of $(d_1 + d_2 - e)e = (d_1 + d_2)e - e^2$ points in $\mathbb{P}^2$, as required.

Example 16.6. Another instructive special case is the self intersection of a smooth subvariety A in a smooth projective variety X. Here we take $A_1 = A_2 = A$, so $Z = A_1 \cap A_2 = A$, and each inclusion $N_{Z/X} \to N_{A_i/X}|_Z$ is the identity. Thus

$$F := \operatorname{coker} N_{Z/X} \to N_{A_1/X}|_Z \oplus N_{A_2/X}|_Z = N_{A/X},$$

and the assertion becomes

$$[A]^2 = \iota_*\big([c(N_{A/X}|_Z)]_m\Big),$$

where $m = \operatorname{codim} A + \operatorname{codim} A - \operatorname{codim} A = \operatorname{codim} A = \operatorname{rank} N_{A/X}$. Thus

$$[c(N_{A/X}|_Z)]_m$$

is the top Chern class of the normal bundle. This is part of the assertion that we proved in Theorem 15.10.

16.1.2 Intersection of two surfaces in $\mathbb{P}^4$ containing a curve

We come now to Keynote Question a: given two smooth surfaces S and $T \subset \mathbb{P}^4$ of degrees s and t whose intersection consists of a reduced curve C and a collection Γ of reduced points, we ask for the degree of Γ.

By Theorem 16.3,

$$\begin{aligned}\deg \Gamma &= st - \deg\left(\left[\frac{c(N_{S/\mathbb{P}^4}|_C)c(N_{T/\mathbb{P}^4}|_C)}{c(N_{C/\mathbb{P}^4})}\right]_0\right)\\ &= st - c_1(N_{S/\mathbb{P}^4}|_C) + c_1(N_{T/\mathbb{P}^4}|_C) - c_1(N_{C/\mathbb{P}^4}).\end{aligned}$$

It is interesting to express this in terms of the geometry of the curve on the surfaces. From the sequence

$$0 \to N_{C/S} \to N_{C/\mathbb{P}^4} \to (N_{S/\mathbb{P}^4})|_C \to 0$$

we have

$$c_1(N_{S/\mathbb{P}^4}|_C) = c_1(N_{C/\mathbb{P}^4}) - c_1(N_{C/S})$$

and similarly for T; so we can rewrite the result as

$$\deg(\Gamma) = st + \deg(c_1(N_{C/S}) + c_1(N_{C/T}) - c_1(N_{C/\mathbb{P}^4})).$$

Of course the degree of $c_1(N_{C/S})$ is the self-intersection number of C on S, which we will write $(C \cdot C)_S$, and similarly for T. On the other hand, The adjunction formula

$$\omega_C = \wedge^3 N_{C/\mathbb{P}^4} \otimes_{\mathcal{O}_C} \omega_{\mathbb{P}^4}|_C,$$

so if C has genus g and degree d then $\deg c_1(N_{C/\mathbb{P}^4} = 2g - 2 + 5d$. Putting this together, we get

$$\deg(\Gamma) = st + (C \cdot C)_S + (C \cdot C)_T - 5d - 2g + 2.$$

In particular, if we take $C = L$ a line in $\mathbb{P}^4$, we arrive at the answer to our second keynote question:

$$\deg(\Gamma) = st + (L \cdot L)_S + (L \cdot L)_T - 3.$$

Note that, in contrast to the answer to our first keynote question, this does not depend only on the degrees of S and T (that is, their classes in $A(\mathbb{P}^4)$), but on their geometry; the simplest example of this is described in Exercise 16.8 below.

Exercise 16.7. Check the answer to our second keynote question in the following cases:

(a) S and T are both 2-planes;

(b) More generally, S is a smooth surface of degree d in a hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$ containing a line L, and T is a general 2-plane in $\mathbb{P}^4$ containing L; and

(c) S and T are smooth quadric surfaces.

Exercise 16.8. Let $S \subset \mathbb{P}^4$ be a cubic scroll **ref**. Show directly that a general 2-plane $T \subset \mathbb{P}^4$ containing a line of the ruling of S meets S in one more point, but a general 2-plane containing the directrix of S does not meet S anywhere else.

16.2 Justification of the Formula

In this section we explain the idea behind Theorem 16.3, allowing ourselves to assume any plausible general position statement. This does not constitute a proof, but may give the reader some intuition for why the result is true.

Heuristic Argument for Theorem 16.3. We place ourselves in the situation of Theorem 16.3, and assume that we can deform A_1 inside X to a variety that meets A_2 transversely. More formally, we suppose that there exists a flat family over $\mathbb{A}^1$ of subvarieties,

$$\mathcal{A}_1 \quad \subset \quad X \times \mathbb{A}^1$$

$$\mathbb{A}^1,$$

****why don't the arrows from the first row to the second row of the diagram show??**** with fiber $\mathcal{A}_{1,0} = A_1$ over $0 \in \mathbb{A}^1$, such that for $0 \neq t \in \mathbb{A}^1$ the subvariety $\mathcal{A}_{1,t}$ meets $A_2 = A_2 \times \{t\}$ transversely inside $X \times \{t\}$. This will be the case, for example, when $X = \mathbb{P}^n$, since then we can define $\mathcal{A}_t$ to be $g_t(A_1)$, where g_t is a general path in $PGL(n+1)$ starting from the origin. To simplify the notation we set $\mathcal{X} = X \times \mathbb{A}^1$ and $\mathcal{A}_2 = A_2 \times \mathbb{A}^1$, and write $\mathcal{A}_{i,t}$ and $\mathcal{X}_t \cong X$ for the fibers over $t \in \mathbb{A}^1$.

Suppose that $\dim X = n$, while $\dim A_i = n - c_i$, so $m = n - c_1 - c_2$ is the "expected dimension" of the intersection. In the open set $\mathcal{X} \setminus \mathcal{X}_0$ the intersection $\mathcal{A}_1 \cap \mathcal{A}_2$ will be transverse and of dimension $1 + n - c_1 - c_2$, and, for $t \neq 0$, the intersection $\mathcal{A}_{1,t} \cap \mathcal{A}_{2,t} \subset \mathcal{X}_t \cong X$ has class $[\mathcal{A}_{1,t}][\mathcal{A}_{2,t}] =$

$[A_1][A_2]$. Since the closure

$$\Phi := \overline{\mathcal{A}_1 \cap \mathcal{A}_2 \cap (\mathcal{X} \setminus \mathcal{X}_0)} \subset X \times \mathbb{A}^4$$

is flat over $\mathbb{A}^1$, it meets $X = \mathcal{X}_0$ in the class $[A_1][A_2]$. It follows that $[A_1][A_2] = [\Phi \cap X_0] = \sum_i [\Phi \cap Z_i]$.

Thus to prove the Theorem, we must show that that $[\Phi \cap Z_i] = [c(\mathcal{F}_i)]_m$ for each i. If Z_i has the expected dimension m then $\Phi \cap Z_i = Z_i$, the bundle $\mathcal{F}_i$ is zero, and $[c(\mathcal{F}_i)]_m = [Z_i]$, as we noted before stating the Theorem.

If, on the other hand, a component $Z := Z_i$ of $A_1 \cap A_2$ has dimension $d+m > m$ then Z is also component of $\mathcal{A}_1 \cap \mathcal{A}_2$, so the set $\Phi \cap Z$ is the locus along Z where $\mathcal{I}_{Z/\mathcal{X}}$ is not the sum of $\mathcal{I}_{\mathcal{A}_1/\mathcal{X}}$ and $\mathcal{I}_{\mathcal{A}_2/\mathcal{X}}$; that is where the map of restrictions of conormal bundles

$$\mathcal{I}_{\mathcal{A}_1/\mathcal{X}}|_Z \oplus \mathcal{I}_{\mathcal{A}_2/\mathcal{X}}|_Z \to \mathcal{I}_{Z/\mathcal{X}}|_Z$$

or, dually, the map of restricted normal bundles

$$\alpha : N_{Z/\mathcal{X}} \to N_{\mathcal{A}_1/\mathcal{X}}|_Z \oplus N_{\mathcal{A}_2/\mathcal{X}}|_Z.$$

is not injective.

We know a priori that the degeneracy locus of α is (at least set-theoretically) the intersection of Z and Φ, which has dimension m (or is empty). On the other hand, α is a map from a vector bundle of rank $1 + c_1 + c_2 - d$ to a vector bundle of rank $c_1 + c_2$, so expected codimension of the degeneracy locus of α in the $m+d$-dimensional variety Z is d. Thus we can use Porteous formula to compute the class of the degeneracy locus.

We now make the "general position" assumption that the intersection of Φ and Z is, at least generically, the degeneracy locus of α, and it follows from Porteous' formula that

$$[\Phi \cap Z] = \left[\frac{c(N_{\mathcal{A}_1/\mathcal{X}}|_Z \oplus N_{\mathcal{A}_2/\mathcal{X}}|_Z)}{c(N_{Z/\mathcal{X}})} \right]_m .$$

Since $N_{Z/\mathcal{X}} = N_{Z/X} \oplus \mathcal{O}_Z$ we have $c(N_{Z/\mathcal{X}}) = c(N_{Z/X})$. Also, since $\mathcal{A}_i \cap \mathcal{X}_0 = A_i$ we have $N_{\mathcal{A}_i/\mathcal{X}}|_Z = N_{A_i/X}|_Z$, so we get

$$[\Phi \cap Z] = \left[\frac{c(N_{A_1/X}|_Z \oplus N_{A_2/X}|_Z)}{c(N_{Z/X})} \right]_m ,$$

as required. □

As in the Theorem, let

$$F = \operatorname{coker}(N_{Z/X} \to N_{A_1/X}|_Z \oplus N_{A_2/X}|_Z),$$

a vector bundle of rank $d = \dim Z - m$. A feature of the proof just given is that we computed $[c(F)]_m = c_d(F)$ as the degeneracy locus of the map

$$N_{Z/X} \oplus \mathcal{O}_Z \to N_{A_1/X}|_Z \oplus N_{A_2/X}|_Z$$

To see why this is the same, recall that $N_{Z/X}$ is a summand of $N_{A_1/X}|_Z \oplus N_{A_2/X}|_Z$. Thus the degeneracy locus is the locus where the section of $N_{A_1/X}|_Z \oplus N_{A_2/X}|_Z$ that is the image of $1 \in \mathcal{O}_Z$ is contained in the image of $N_{Z/X}$. Equivalently, this is the zero locus of the image of this section in F—which is $c_d(F)$. This same argument is part of the justfication for the general position hypothesis imposed near the end of the proof: it shows that the map α at least never drops rank by more than 1.

16.3 Avoiding Smoothness Hypotheses

A serious problem in applying the version of the Excess Intersection Theorem given above is that it requires both the A_i and their intersection to be smooth. This hypothesis was weakened in a very useful way by Fulton and MacPherson, ****is this accurate? Give reference!**** and although the statements go beyond the context of this book, we digress to explain them and to indicate some of their importance.

The first way in which we have apparently used the smoothness hypothesis is to form the normal bundles of the A_i. Recall that for a smooth subvariety $A \subset X$ the normal bundle is defined from the tangent bundle T_A by the short exact sequence

$$0 \longrightarrow T_A \longrightarrow T_X|_A \longrightarrow N_{A/X} \longrightarrow 0.$$

This is the dual over $\mathcal{O}_A$ of a right exact sequence of modules of differentials:

$$\mathcal{I}_A/\mathcal{I}_A^2 \longrightarrow \Omega_X|_A \longrightarrow \Omega_A \longrightarrow 0,$$

defined for *any* subscheme A of a scheme X. Whenever A is locally a complete intersection in X, the left hand term is a vector bundle and the the sequence is exact (Eisenbud [1995], Theorem ****.) Thus when A is locally a complete intersection (for example, a hypersurface in $\mathbb{P}^n$, no matter how singular), we can define

$$N_{A/X} = \mathrm{Hom}_{\mathcal{O}_A}(\mathcal{I}_A/\mathcal{I}_A^2, \mathcal{O}_A) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_A, \mathcal{O}_A),$$

a vector bundle of rank equal to the codimension of A in X.

A second way in which smoothness is apparently necessary in the Theorem is to define the multiplication of classes; on a singular variety, the Chow groups do not form a ring. However, we are not multiplying arbitrary classes, but only the Chern classes of vector bundles, and it turns out that this is always possible. Indeed, by the Whitney formula (Theorem 7.3), the Chern class of the direct sum of two vector bundles is the Chern class of their product; so it is enough that Chern classes be well defined. The definition we gave can be adapted to the case of vector bundles on an

arbitrary projective scheme, and if we take the formula of Theorem 11.4as the definition, they can be defined for vector bundles on any scheme.

Using these ideas we can make sense of the formula in Theorem 16.3 whenever the A_i and Z are locally complete intersections. But even when the A_i satisfy this, there is usually no reason for their intersection Z to do so, so it is worthwhile weakening the hypothesis on Z. To do so, we single out one of the A_i, say $A := A_2$, and consider the fraction $c(N_{A/X}|_Z)/c(N_{Z/X})$. If we assume again, for a moment, that Z and A are both smooth, then from the exact sequence

$$0 \longrightarrow \mathcal{I}_{A/X} \longrightarrow \mathcal{I}_{Z/X} \longrightarrow \mathcal{I}_{Z/A} \longrightarrow 0$$

we get an exact sequence of vector bundles on Z,

$$0 \longrightarrow N_{Z/A} \longrightarrow N_{Z/X} \longrightarrow N_{A/X}|_Z \longrightarrow 0,$$

whence $c(N_{A/X}|_Z)/c(N_{Z/X} = 1/c(N_{Z/A})$, and we recall that this latter is the Segre class $s(N_{Z/A})$. Thus we could modify the formula of Theorem 16.3 using

$$\frac{c(N_{A_1/X}|_{Z_j})c(N_{A_2/X}|_{Z_j})}{c(N_{Z_j/X})} = c(N_{A_1/X}|_{Z_j})s(N_{Z_j/A_2}).$$

This might be a useless exercise except for the fact the *the definition of $s(N_{Z/A})$ can be generalized to any pair of schemes $Z \subset A$*. The case where A is a projective variety will suffice for our purposes:

Definition 16.9. Let $Z \subset A$ be a closed subscheme of a projective variety A, and let E be the exceptional divisor of the blowup of Z in A, so that $E = \operatorname{Proj} \oplus_{i\geq 0}\mathcal{I}^i/\mathcal{I}^{i+1}$, with tautological line bundle $\mathcal{O}_E(1)$. Let $\pi : E \to Z$ be the projection. The segre class $s(Z, A)$ is defined to be

$$s(Z, A) = \sum_j \pi_*\big((c_1\mathcal{O}_E(1))^j\big) \in A(Z).$$

Putting this together, and allowing an arbitrary number of varieties A_i, we can give something close to the form of the Excess Intersection Theorem given by Fulton and MacPherson ****check the citation, and give reference****:

Theorem 16.10 (Excess Intersection Theorem—Definitive Version)**.** *Suppose $A_1, \dots A_{u-1}$ are locally complete intersection subvarieties of a projective variety X, and let A_u be an arbitrary subvariety. Suppose that $\cap_{i=1}^u A_i = \cup_j Z_j$ is a disjoint union of subschemes Z_j. Write $\iota : Z_j \to X$ for the inclusion morphism. If m is the "expected dimension" of the intersection, $m = \dim X - \sum_{i=1}^u \operatorname{codim} A_i$ then*

$$\prod_{i=1}^u [A_i] = \sum_j \iota_*\left(\left[(\prod_{i=1}^{u-1} c(N_{A_i/X}|_{Z_j}))s(Z_j, A_u)\right]_m\right) \in A(X).$$

As noted above, the formula in Theorem 16.10 gives the intersection class $\prod[A_i]$ in terms of a class defined on the actual intersection $Z = \cap A_i$. This is a substantial improvement even when X is smooth and the dimension of Z is the expected dimension of the intersection. But it requires all but one of the A_i to be a complete intersection, so it is not directly applicable to general intersections, even in a smooth variety.

Nevertheless, this idea can be used to define intersection products of any cycles on a smooth variety in terms of Chern classes and their direct images, and thus without the Moving Lemma, and one gets refined intersection products as well. This is the approach taken in Fulton [1984], Chapter ****. To understand how this works, note first that the intersection of subvarieties A_1, A_2 in X is the same as the intersection of the direct product $A_1 \times A_2$ in $X \times X$ with the diagonal $X \cong \Delta \subset X \times X$. Next observe that when X is smooth, Δ is locally a complete intersection: for if $\dim X = n$, then the completion of the local ring of X at a point p is isomorphic to $K[[x_1, \ldots, x_n]]$, so the completion of the local ring of $X \times X$ at (p, p) is isomorphic to $K[[x_1, \ldots, x_n, y_1, \ldots y_n]]$ and the ideal of Δ at this point is generated by $x_1 - y_1, \ldots, x_n - y_n$, a regular sequence. Putting this together, we get the promised refined intersections:

Corollary 16.11. *Suppose that X is a smooth projective variety, and A_1, A_2 are subvarieties. Set $m = \dim X - \operatorname{codim} A_1 - \operatorname{codim} A_2$. The intersection product $[A_1][A_2]$ is the pushforward into $A_m(X)$ of the class*

$$\left[c(N_{\Delta/X\times X}|_{A_1\cap A_2})s(A_1 \cap A_2, A_1 \times A_2)\right]_m \in A_m(A_1 \cap A_2),$$

Corollary 16.11 is essentially the definition of intersection products in Fulton [1984].

16.3.1 Computing the Segre class of a Cone

Exercise 16.12. Prove that if $Z \subset A$ is a locally complete intersection subscheme inside the projective variety A, then $s(Z, A) = s(N_{Z/A})$.

Example 16.13. Here is an example of how to compute the Segre class of a cone. Let $A = \mathbb{P}^3$ and suppose that Z is the scheme defined by the ideal $(x_1x_2, x_1x_3, x_2x_3) \in K[x_0, \ldots, x_3]$, which is the union of three lines meeting in a point. We will compute $s(Z, A)$.

The blowup of Z is

$$\operatorname{Proj} \mathcal{O}_{\mathbb{P}^3}[a, b, c]/(x_3a - x_2b, x_1c - x_3a)$$

where a, b, c are sections of $\mathcal{O}_E(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^3}(2)$. Thus E is a reducible, 2-dimensional variety, with fibers over the point $p : x_1 = x_2 = x_3 = 0$ consisting of a projective plane, and fibers over each other point of Z consisting of a projective line. The scheme $a = b = 0$ exists in the (relative)

affine chart with $c \neq 0$, on which the equations are x_1x_2, x_1x_3, x_2x_3 and $x_1 = 0$. It thus consists of two reduced points q_1 q_2. It follows that

$$[q_1 + q_2] = \big(c_1(\mathcal{O}_E(1)) + c_1(\pi^*\mathcal{O}_Z(2))\big)^2.$$

Because Z is 1-dimensional we have $\big(c_1(\mathcal{O}_Z(2|_Z))\big)^i = 0$ for $i > 1$, so

$$\big(c_1(\mathcal{O}_E(1))\big)^2 = [q_1 + q_2] - 2\big(c_1(\mathcal{O}_E(1))c_1(\pi^*\mathcal{O}_Z(2))\big).$$

Since $\big(c_1(\mathcal{O}_E(1))\big)$ has degree 1 on every fiber, and $c_1(\pi^*\mathcal{O}_Z(2))$ is represented by $2\zeta_Z$, where ζ_Z is the restriction of the hyperplane class to Z, this shows that

$$\pi_*\big(c_1(\mathcal{O}_E(1))\big)^2 = [p_1 + p_2] - 2\zeta_Z,$$

where $p_i = \pi(q_i)$. It follows that

$$s(Z, A) = 1 - ([p_1 + p_2] - 2\zeta_Z) \in A(Z).$$

Theorem 16.10 now gives

$$[x_1x_2 = 0][x_1x_3 = 0][x_2x_3 = 0] = [(1 + 2\zeta_Z)(1 + 2\zeta_Z)(1 + [p_1 + p_2] - 2\zeta_Z)]_0$$

the class of 8 points, as one would expect!

16.3.2 How Many Conics are Tangent to Five Lines?

****we're working only with the intersection of two subvarieties, and for the five lines problem we need more.****

****need to change the notation here, change the discussion into an application of the formula.****

As an example of the general excess intersection formula, let's consider a simpler version of the five-conic problem of Chapter **: given 5 general lines $L_1, \ldots, L_5 \subset \mathbb{P}^2$ in the plane, how many smooth conics are tangent to all 5? The answer is 1—by passing to the dual plane, the problem translates into the question, how many plane conics pass thought 5 general points?—but we can also make the computation using the the excess intersection formula .

The situation is like the one we encountered in analyzing Chasles' five-conic problem in Chapter 10: plane conics are parametrized by $\mathbb{P}^5$, and in that $\mathbb{P}^5$ the closure of the locus Z_L of conics tangent to a given line L is a hypersurface Z_i of degree 2. But the hypersurfaces Z_i all contain the variety $S \subset \mathbb{P}^5$ of double lines, which is (in characteristic not 2) a copy of the Veronese surface . The intersection $\bigcap Z_i$ will consist of the union of a finite set Γ—the actual solutions—with the surface S.

In Chapter 10 we dealt with this by introducing a new parameter space for conics; we mentioned also that we could solve the problem by blowing

up the locus $S \subset \mathbb{P}^5$. But now we can apply the excess intersection formula. Here the hypersurfaces Z_i are singular, so the hypotheses of Proposition 16.3 are not satisfied; but since the Z_i and S are local complete intersections in $\mathbb{P}^5$, so that the normal bundles $N_{Z_i/\mathbb{P}^5} \cong \mathcal{O}_{\mathbb{P}^5}(2)$ and $N_{S/\mathbb{P}^5}$ are locally free, we can apply the more general form of the excess intersection formula.

We first compute the Chern class of the normal bundle of S in $\mathbb{P}^5$. To find this, let $\zeta \in A^1(S)$ be the hyperplane class in $S \cong \mathbb{P}^2$, and let $\eta \in A^1(\mathbb{P}^5)$ be the hyperplane class on $\mathbb{P}^5$; note that the restriction of η to S is 2ζ. Now we know

$$c(T_S) = (1+\zeta)^3 = 1 + 3\zeta + 3\zeta^2,$$

and

$$\begin{aligned} c(T_{\mathbb{P}^5}|_S) &= (1+\eta)^6|_S \\ &= 1 + 12\zeta + 60\zeta^2. \end{aligned}$$

Applying the Whitney formula to the sequence

$$0 \to T_S \to T_{\mathbb{P}^5}|_S \to N_{S/\mathbb{P}^5} \to 0,$$

we conclude that

$$\begin{aligned} c(N_{S/\mathbb{P}^5}) &= \frac{1 + 12\zeta + 60\zeta^2}{1 + 3\zeta + 3\zeta^2} \\ &= 1 + 9\zeta + 30\zeta^2 \end{aligned}$$

and inverting this we have

$$c(N_{S/\mathbb{P}^5})^{-1} = 1 + 9\zeta + 51\zeta^2.$$

Applying Theorem 16.10 we see that the contribution of S to the class $\prod[Z_i]$ is

$$\begin{aligned} \left[\frac{\prod_i c(N_{Z_i/\mathbb{P}^5}|_S)}{c(N_{S/\mathbb{P}^5})}\right]_2 &= \left[(1+2\eta)^5|_S(1 - 9\zeta + 51\zeta^2)\right]_2 \\ &= \left[(1 + 20\zeta + 160\zeta^2)(1 - 9\zeta + 51\zeta^2)\right]_2 \\ &= 51 - 180 + 160 \\ &= 31 \end{aligned}$$

and we conclude that the degree of Γ—that is, the number of actual solutions to the problem—is 1.

Exercise 16.14. Let $M \cong \mathbb{P}^8$ be the space of 3×3 matrices mod scalars, and let $\Sigma \cong \mathbb{P}^2 \times \mathbb{P}^2$ be the Segre variety, that is, the locus of matrices of rank 1. Using the excess intersection formula, find the degree of the intersection of Σ with the linear subspace $N \cong \mathbb{P}^5 \subset M$ of symmetric matrices (a.k.a. the Veronese surface).

Exercise 16.15. Let $\Lambda \cong \mathbb{P}^{n-2} \subset \mathbb{P}^n$ be a codimension 2 linear subspace of $\mathbb{P}^n$, and let $Q_1, \ldots, Q_{n-k}$ be general quadrics containing Λ. Write the intersection of the Q_i as

$$\bigcap_{i=1}^{n-k} Q_i = \Lambda \cup X$$

with X of dimension k.

(a) Use the excess intersection formula to show that the degree of X is $n-k+1$, and hence that X is a rational normal k-fold scroll.

(b) Verify this conclusion directly by writing the defining equations of the Q_i as 2×2 minors of a $2 \times (n-k+1)$ matrix M of linear forms on $\mathbb{P}^n$, and realizing X as the rank 1 locus of this matrix.

16.4 Pullbacks

Suppose we have a morphism $f : X \to Y$ of smooth projective varieties, and $A \subset Y$ a subvariety of codimension a. As we've seen, if the preimage $f^{-1}(A) \subset X$ is generically reduced of codimension a in X, then

$$[f^{-1}(A)] = f^*([A]);$$

that is, the class of $f^{-1}(A)$ is the pullback of the class $\alpha = [A] \in A(X)$. But what can be said about $f^*[A]$ when $f^{-1}(A)$ has codimension strictly less than a? In the case where f is a closed immersion, this question is addressed by the excess intersection formula, and we'll now describe how to extend the formula to the general case. Although weaker hypotheses suffice, we'll suppose throughout that A and $f^{-1}(A)$ are smooth.

In Chapter 5 we defined the pullback $f^*[A]$ by considering the cycles $X \times A$ and $\Gamma_f \subset X \times Y$, where Γ_f is the graph of f; we defined $f^*[A]$ to be the pushforward of under the projection map $X \times Y \to X$ on the first factor of the product $[X \times A][\Gamma_f]$. If we assume that A and $f^{-1}(A)$ are locally complete intersections, we can apply the excess intersection formula to compute this product rather than using the moving lemma.

To derive an explicit formula, suppose that B is a connected component $f^{-1}(A)$. Via the identification of X with Γ_f we think of B as a connected component of $(X \times A) \cap \Gamma_f$. We associate to B the class

$$\begin{aligned} \gamma_B &= \left[\frac{c(N_{X \times A / X \times Y})}{c(N_{B/\Gamma_f})} \right]_{\dim X - a} \\ &= \left[\frac{c(f^* N_{A/Y})}{c(N_{B/X})} \right]_{\dim X - a} \in A(B). \end{aligned}$$

Writing $j_B : B \to X \times Y$ for the inclusion, Theorem 16.3 shows that the desired product $[\Gamma_f][X \times A]$ is the sum of the classes $j_*\gamma_B$. Observing that the inclusion $i_B : B \to X$ can be factored as $i_B = \pi \circ j$, we get the pullback formula:

Corollary 16.16 (Excess intersection formula for pullbacks)**.** *Let $f : X \to Y$ be a morphism of smooth projective varieties. Let $A \subset Y$ be a smooth subvariety of codimension a. If $f^{-1}(A) \subset X$ is smooth, then*

$$f^*[A] = \sum_B i_{B*}\left[\frac{c(f^*N_{A/Y})}{c(N_{B/X})}\right]_{\dim X - a} \in A(X)$$

where the sum is taken over all connected components B of $f^{-1}(A)$, and $i_B : B \to X$ is the inclusion. □

Suppose, for example, that we want to calculate the degree of a generically finite map $f : X \to Y$. If we can find a point $q \in Y$ such that f is étale over a neighborhood of q—that is, such that the differential df_p is an isomorphism for all $p \in f^{-1}(q)$—then all we have to do is to count the number of points in the fiber $f^{-1}(q)$. More generally, if we can find a point $q \in Y$ over which f is finite, we can count the points of $f^{-1}(q)$ with multiplicity. But with Corollary 16.16 we can express the degree of f in terms of the local geometry of f around a fiber $f^{-1}(q)$, even when that fiber is positive-dimensional. Since the normal bundle of a point is trivial, the formula becomes quite simple:

Corollary 16.17. *Let $f : X \to Y$ be a generically finite surjective map of smooth projective varieties, $q \in Y$ any point such that $f^{-1}(q)$ is smooth. Then*

$$\deg(f) = \sum_B \deg s_{\dim B}(N_{B/X}),$$

where the sum is over connected components of $f^{-1}(q)$ and s denotes the Segre class.

For example, suppose $f : X = Bl_qY \to Y$ is the blow-up of an n-dimensional variety Y at a point q. Since the normal bundle to the exceptional divisor $E = f^{-1}(q) \cong \mathbb{P}^{n-1}$ is $\mathcal{O}_E(-1)$, we have

$$s(N_{E/X}) = \frac{1}{1-\zeta} = 1 + \zeta + \cdots + \zeta^{n-1};$$

we conclude the degree of f is 1.

It may seem unnecessary to use Proposition 16.17 in this case: after all, if we were truly ignorant of the fact that a blow-up map has degree 1, we could discover it by looking at any point at all on the target other than the one point we blew up. But there are many cases where we know more about the geometry of a map in a neighborhood of a special fiber

than a general one. A beautiful example is the calculation in Donagi and Smith [1980] of the degree of the Prym map in genus 6. This is a map from the space R_6 of unramified covers of curves of genus 6 to the space A_5 of abelian varieties of dimension 5 (both of dimension 15) defined by the Prym construction. While it does not seem possible to enumerate the points of a general fiber, Donagi and Smith are able to calculate the degree of the map by looking at a special point of the target (the Prym variety of a double cover of a smooth plane quintic), over which the fiber has three connected components: a point, a curve and a surface.

Exercise 16.18. Let $Q \subset \mathbb{P}^3$ be a quadric cone with vertex p, and $X = Bl_pQ \cong \mathbb{F}_2$ its blow-up at p. Let $\pi : Q \to \mathbb{P}^2$ be a general projection, and $f : X \to \mathbb{P}^2$ the composition of the blow-up map with the projection. Find the degree of f by looking at the fiber over $q = \pi(p)$.

16.5 The double point formula

A famous result of Whitney says that a generic map from a smooth compact real manifold X of dimension m to $\mathbb{R}^{2m+1}$ is an embedding; for maps to R^{2m} one should expect some double points in the image, that is, pairs of points in the source that are mapped to the same point in $\mathbb{R}^{2m}$. The same sort of thing happens in algebraic geometry. In this section we will introduce a simple version of the *double-point formula*, which counts these pairs of points, or, in fancier language, computes the degree of the double point locus.

To describe the formula, suppose that $f : X \to Y$ is a map of smooth projective varieties, with $\dim Y = 2 \dim X$. Let $\mathcal{D}(f)$ be the *double point locus of* f, that is, the locus of of pairs $(p, q) \in X \times X$ with $p \neq q$ but $f(p) = f(q)$. If we write Δ_X and Δ_Y for the the diagonal subvarieties of $X \times X$ and $Y \times Y$, respectively, then

$$\mathcal{D}(f) = (f \times f)^{-1}(\Delta_Y) \setminus \Delta_X.$$

Since $\mathcal{D}(f)$ is an open set of the preimage $(f \times f)^{-1}(\Delta_Y)$, and Δ_Y has codimension $\dim Y$ in $Y \times Y$, we expect $\mathcal{D}(f)$ to have dimension 0, and we would like to know its degree under the hypothesis that it really is finite. However, the set Δ_X will also be a component of $(f \times f)^{-1}(\Delta_Y)$, so we need an excess intersection formula to compute its contribution to the class $(f \times f)^*[\Delta_Y]$. To apply Corollary 16.16 directly, we want $(f \times f)^{-1}(\Delta_Y)$ to be smooth. As we shall see, this is equivalent to the condition that f be *immersive* in the sense that the differential of f is everwhere injective.

Corollary 16.19. *Let $f : X \to Y$ be a map of smooth projective varieties with $\dim Y = 2 \dim X$. If f is immersive and the double-point locus $\mathcal{D}(f)$*

is finite then

$$\deg \mathcal{D} = \deg(f \times f)^*[\Delta_Y] - \deg\left\{ \frac{f^* c(T_Y)}{c(T_X)} \right\}_0 .$$

Proof. The normal bundle of the diagonal $\Delta_Y \subset Y \times Y$ is the tangent bundle to $\Delta_Y \cong Y$, and likewise for X. ****I think we said why earlier in the book; refer to this.**** Moreover, for $p \in X$ the tangent map $df : T_{X,p} \to T_{Y,f(p)}$ is the same as the map of fibers of the normal bundle

$$N_{\Delta_X / X \times X,(p,p)} \to N_{\Delta_Y / Y \times Y,(f(p),f(p))}$$

This map is dual to the map of conormal bundles induces by the pullback $\mathcal{I}_{\Delta_Y / Y \times Y} \mathcal{I}_{\Delta_X / X \times X}$, and we conclude from Nakayama's Lemma that f is immersive if and only if the ideal of Δ_X is generated at each point (p, p) by the pullback of the ideal of Δ_Y; that is, Δ_X is a connected component of Δ_Y.

The desired formula is now a special case of Corollary 16.16 □

In the situation of the Corollary, if we assume $\dim X = 1$ and that $Y = \mathbb{P}^2$ we are computing the number of double points of a plane curve with no cusps (see exercise 16.20.)

Exercise 16.20. Let $C \subset \mathbb{P}^2$ be a plane curve of degree d whose singularities are only ordinary nodes, and let $\overline{C}$ be the normalization of C. Apply the double point formula to the map $\overline{C} \to \mathbb{P}^2$ to deduce the familiar fact that the number of nodes of C is $\binom{d-1}{2} - g$, the difference between the arithmetic genus and the geometric genus of C.

16.5.1 Apparent Double Points of a Surface

For a less familiar case, we consider the situation where $\dim X = 2$ and $Y = \mathbb{P}^4$:

Example 16.21 (The degree of the secant variety and the number of "apparent double points" of a surface). Let $X \subset \mathbb{P}^n$ be a smooth surface embedded in projective space. If we project from a point of $\mathbb{P}^n$ that does not lie on the secant locus $S_2(X)$ of X, then X will be projected isomorphically onto it's image. Since $\dim S_2(X) = 2 \dim X + 1 = 5$, this shows that any smooth surface can be embedded in $\mathbb{P}^5$.

Now suppose that $n = 5$ so that $X \subset \mathbb{P}^5$, and suppose that X is not contained in any hyperplane. Since the union of the tangent planes to X can have dimension at most $2 \times 2 = 4$, a generic projection to $\mathbb{P}^4$ will automatically be immersive. Furthermore, the double point locus corresponds to the set of secant lines that pass through the point of projection, and for

a generic point of projection this set must be finite (if the secant locus fills $\mathbb{P}^5$) or empty, so we can use Corollary 16.19 to compute the class of the double point locus.

In case the double point locus is reduced, its degree has a charming interpretation: it is the *number of apparent double points* of the surface—that is, the number of pairs of points that would line up with the eye of a generically placed observer, or again the number of double points of the image, counted appropriately.

Corollary 16.22. *Let X is a smooth nondegenerate surface in $\mathbb{P}^n$ of degree d. If the secant locus $S_2(X)$ has dimension 5, then the degree of the secant locus of X, and (in case $n = 5$) the number of apparent double points of X, is*

$$\frac{d^2 - 10d + 5c_1(T_X)\zeta - \deg c_1(T_X)^2 + \deg c_2(T_X)}{2}$$

where ζ is the class of a hyperplane section of X.

For example, when $X \cong \mathbb{P}^2$ is the Veronese surface in $\mathbb{P}^5$ we have $d = 4$ and $c(T_X) = (1+\eta)^3$, where η is the class of the image of a line (a conic in $\mathbb{P}^4$), so the number of apparent double points is

$$\frac{4^2 - 10 \cdot 4 + 5 \cdot 3 \cdot 2 - 3^2 + 3}{2} = 0,$$

which is why the Veronese surface can be projected isomorphically into $\mathbb{P}^4$.

A famous result (partially) proven by Severi in 1901 says that the only non-degenerate surface in $\mathbb{P}^n$, for $n \geq 5$ that can be projected isomorphically into $\mathbb{P}^4$ is the Veronese surface (see Dale [1985]for history and a modern proof). Equivalently, the Veronese surface is the only surface for which the expression in the double point formula is zero!

For another example, suppose that X is the complete intersection of three forms of degrees d_1, d_2, d_3 in $\mathbb{P}^5$. From the exact sequence sequence

$$0 \to T_X \to T_{\mathbb{P}^5}|_X \to N_{X/\mathbb{P}^5} \to 0$$

and the formula $N_{X/\mathbb{P}^5} = \oplus_{i=1}^3 \mathcal{O}_X(d_i)$ we get

$$c(T_X) = \frac{(1+\zeta)^6}{\prod_{i=1}^3 (1+d_i\zeta)} = 15\zeta^2 + (6 - d_1 - d_2 - d_3)\zeta + 1 \in A(X).$$

The degree of such a surface is $d := d_1 d_2 d_3$. Writing $e := d_1 + d_2 + d_3$ for the degree of the normal bundle and doing a little arithmetic, we see that the value of the expression in the double point formula is

$$\begin{aligned} &\frac{d^2 - 10(d + 5d(6-e) - d(6-e)^2 + 15d}{2} \\ &= d\frac{-e^2 + d + 62e - 331}{2}. \end{aligned}$$

In particular, counting double points with appropriate multiplicities, the projection of a general complete intersection of quadrics will have 52 double points, answering Keynote Question c. ****any way to see that they are simple?****

Proof. Since the generic hyperplane section of X consists of a set of points in linearly general position, the generic secant line to X meets X in just 2 reduced points. It follows that the dimension of the union of the locus of lines that meet X in a scheme of length 3 or more, or are tangent to X, is at most 4, and a generic $n-5$-plane in $\mathbb{P}^n$ will not lie on any of them. Thus the degree of intersection of such a plane with $S_2(X)$, which is the degree of $S_2(X)$, is the number of double points of the generic projection of X into $\mathbb{P}^4$.

By Corollary 16.19 this number of pairs of points with the same image is

$$\deg(f\times f)^*[\Delta_{\mathbb{P}^4}] - \deg\Big\{\frac{f^*c(T_{\mathbb{P}^4})}{c(T_X)}\Big\}_0,$$

so the number of double points of the image will be half this,

$$\frac{1}{2}\deg(f\times f)^*[\Delta_{\mathbb{P}^4}] - \deg\Big\{\frac{f^*c(T_{\mathbb{P}^4})}{c(T_X)}\Big\}_0.$$

Writing η for the hyperplane class of $\mathbb{P}^4$ we know from *** that

$$\begin{aligned}[\Delta_{\mathbb{P}^4}] &= (\eta^4,1)+(\eta^3,\eta)+\cdots+(1,\eta^4)\\ &\in A(\mathbb{P}^4\times\mathbb{P}^4) = \mathbb{Z}[\eta]/(\eta^4)\otimes\mathbb{Z}[\eta]/(\eta^4).\end{aligned}$$

Since the hyperplane class of $\mathbb{P}^4$ pulls back to that of X, and $\zeta^3[X]=0$, we get

$$\deg(f\times f)^*[\Delta_{\mathbb{P}^4}] = d^2.$$

On the other hand, $f^*c(T_{\mathbb{P}^4}) = 1+\zeta^5$, while by definition $c(T_X) = 1 + c_1(T_X)+c_2(T_X)$. Putting this together, we get

$$\begin{aligned}\frac{f^*c(T_{\mathbb{P}^4})}{c(T_X)} &= \frac{1+\zeta^5}{1+c_1(T_X)+c_2(T_X)}\\ &= (1+5\zeta+10\zeta^2)(1-c_1(T_X)-c_2(T_X)+c_1(T_X)^2)\end{aligned}$$

Taking the dimension 0 part we get

$$10\zeta^2 - 5\zeta c_1(T_X) - c_2(T_X) + c_1(T_X)^2,$$

Since $\deg\zeta^2 = \deg X = d$, this gives the Corollary. □

16.5.2 Generalizations

There are versions of the double point formula much more general than the one we present (see Fulton [1984], Section 9.3), which compute the class of the double point locus even in situations where its expected dimension is not zero or the map f is not immersive; and also *multiple point formula* counting (under strong hypotheses) the number of d-tuples of points that have a common image (see Kleiman et al. [1996], Kleiman [1990]).

For example, without the hypothesis of immersiveness we can define a *double point scheme* of f that reflects the failure of f to be an embedding and has class given by the formula of Corollary 16.19, assuming it has the expected codimension: We let $\pi : Z = Bl_{\Delta_X}(X \times X) \to X \times X$ be the blow-up along the diagonal and write $E \subset Z$ for the inverse image of the diagonal. The preimage W of Δ_Y under the composition $\pi \circ f$ contains the cartier divisor E, and we let $\tilde{D}_f$ be the scheme defined by the ideal

$$\mathcal{I}_{\tilde{D}_f} = (\mathcal{I}_W : \mathcal{I}_E).$$

The *double point scheme* D_f of f is defined to be the image of $\tilde{D}_f$ under π. See Exercise 16.23 for the relation of this to f being an embedding, and see Fulton [1984] Section **** for the computation of $[\tilde{D}_f]$.

Exercise 16.23. Show that f is an embedding if and only if the double point scheme D_f is empty.

17

The Grothendieck-Riemann-Roch Theorem

Convention: To simplify notation in this chapter we often identify a class in $A_0(X)$ with its degree when X is a projective algebraic variety.

17.1 The Riemann-Roch Formula for Curves and Surfaces

17.1.1 Nineteenth Century Riemann-Roch

Riemann-Roch formulas relate the dimension of the space of solutions of an analytic or algebraic problem—typically realized as the space of global sections of a coherent sheaf on a compact analytic or projective algebraic variety—to topological invariants or, more generally, discrete numerical invariants such as the Chern classes of the sheaf. Our goal in this chapter is to state, explain and apply a version of this Theorem proved by Grothendieck. To clarify its context, we start this chapter with a review of some classical versions. Although these were first proven (and still are important!) in an analytic context, we will stick with the category of projective algebraic varieties.

The original Riemann-Roch formula deals with a smooth projective curve C. It says in particular that the dimension $h^0(K_C)$ of the space of regular 1-forms on C, an algebraic invariant, is equal to the topological genus

$g(C) = 1 - \chi_{\text{top}}(C)/2$. To express this in modern language and suggest the generalizations to come, we invoke Serre duality, which says that

$$h^0(K_C) = h^1(\mathcal{O}_C),$$

and the Hopf index theorem for the topological Euler characteristic $\chi_{\text{top}}(C) = c_1(T_C)$, and state the Riemann-Roch Theorem as the formula

$$\chi(\mathcal{O}_C) := h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) = \frac{c_1(T_C)}{2}.$$

Once we know this, the usual Riemann-Roch formula for line bundles on a curve follows: if $L = \mathcal{O}_C(D)$ for some effective divisor D of degree $c_1(L) = d$, then from the sequence

$$0 \to \mathcal{O}_C \to L \to L|_D \to 0$$

we see that

$$\begin{aligned}\chi(L) &= c_1(L) + \frac{c_1(T_C)}{2} \\ &= d + 1 - g.\end{aligned}$$

It's not hard to go from this to the version for an arbitrary sheaf $\mathcal{F}$ on C,

$$\chi(\mathcal{F}) = c_1(\mathcal{F}) + \text{rank}(\mathcal{F})\frac{c_1(T_C)}{2} :$$

any sheaf $\mathcal{F}$ of positive rank admits a sub-line-bundle $\mathcal{L} \subset \mathcal{F}$, and from the relation

$$\chi(\mathcal{F}) = \chi(\mathcal{L}) + \chi(\mathcal{F}/\mathcal{L})$$

and an induction on $\text{rank}(\mathcal{F})$, we are reduced to the case of rank 1. Here, the cokernel $\mathcal{F}/\mathcal{L}$ is supported on a finite set, and the relation follows.

To state a Riemann-Roch Theorem for a smooth projective surface S, we start again from a special case,

$$\chi(\mathcal{O}_S) = \frac{c_1(T_S)^2 + c_2(T_S)}{12},$$

usually referred to as *Noether's formula*. From this, the prior Riemann-Roch for curves, and sequences of the form

$$0 \to L \to L(D) \to L(D)|_D \to 0$$

for smooth effective divisors $D \subset S$, we can deduce the version for line bundles:

$$\chi(L) = \frac{c_1(L)^2 + c_1(L)c_1(T_C)}{2} + \frac{c_1(T_S)^2 + c_2(T_S)}{12}.$$

For example, from the sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(D) \to \mathcal{O}_S(D)|_D \to 0$$

we have

$$\begin{aligned}\chi(\mathcal{O}_S(D)) &= \chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(D)|_D) \\ &= \frac{c_1(T_S)^2 + c_2(T_S)}{12} + \chi(\mathcal{O}_S(D)|_D).\end{aligned}$$

To evaluate the last term, observe that $\mathcal{O}_S(D)|_D$ is a line bundle of degree $D \cdot D$ on the curve D, which by adjunction has genus

$$g(D) = \frac{D \cdot D + D \cdot K_S}{2} + 1;$$

by the Riemann-Roch for curves, then, we have

$$\begin{aligned}\chi(\mathcal{O}_S(D)|_D) &= D \cdot D - \frac{D \cdot D + D \cdot K_S}{2} \\ &= \frac{D \cdot D + D \cdot c_1(T_S)}{2}\end{aligned}$$

and the Riemann-Roch formula above follows for $L = \mathcal{O}_S(D)$.

More generally if $\mathcal{F}$ is any coherent sheaf on S then

$$\chi(\mathcal{F}) = \frac{c_1(\mathcal{F})^2 - 2c_2(\mathcal{F}) + c_1(\mathcal{F})c_1(T_C)}{2} + \operatorname{rank}(\mathcal{F})\frac{c_1(T_S)^2 + c_2(T_S)}{12}.$$

An important step in the proof is to show that both sides of the formula are additive on exact sequences. For the left hand side, this follows from the long exact sequence in cohomology, and for the right hand side it comes from the properties of Chern characters and Todd classes, which we'll describe next.

Using this additivity, and resolving a sheaf by vector bundles, we reduce the proof to the case when $\mathcal{F}$ is a vector bundle. By the push-pull formula for direct images from Proposition17.9, the Euler characteristic doesn't change when we pull a sheaf back to a projective bundle, so (as with the "splitting principle" we used to compute Chern classes) we can use additivity again to reduce to the case of a line bundle.

17.2 The Chern character

Our next construction, the Chern character, does not represent a new invariant of a vector bundle; it's a repackaging of the data in the Chern class (in $A^*_{\mathbb{Q}}(X) := \mathbb{Q} \otimes_Z A^*(X)$) in a way that is much more convenient for some purposes. To define it, suppose E is a vector bundle/locally free sheaf of rank n on a smooth variety X, and formally factor its Chern class:

$$c(E) = \prod_{i=1}^{n}(1 + \alpha_i)$$

We define the *Chern character* to be

$$\mathrm{Ch}(E) = \sum_{i=1}^{n} e^{\alpha_i} \in A^*(X) \otimes \mathbb{Q}.$$

In other words, the k^{th} graded piece $\sum \frac{\alpha_i^k}{k!}$ of the power series $\sum e^{\alpha_i}$ is symmetric in the variables $\alpha_1, \ldots, \alpha_n$, and hence expressible uniquely as a polynomial in the elementary symmetric polynomials in the α_i; we take the k^{th} graded piece $\mathrm{Ch}_k(E)$ of the Chern character to be this polynomial, applied to the Chern classes of E. We have immediately

$$\mathrm{Ch}_0(E) = \mathrm{rank}(E)[X] \quad \text{and} \quad \mathrm{Ch}_1(E) = c_1(E).$$

Further, it is not hard to compute that

$$\begin{aligned} \mathrm{Ch}_2(E) &= \frac{1}{2} \sum \alpha_i^2 \\ &= \frac{1}{2} \left((\sum \alpha_i)^2 - 2 \sum \alpha_i \alpha_j \right) \\ &= \frac{c_1(E)^2 - 2c_2(E)}{2}. \end{aligned}$$

Since the Chern roots of E^* are the negatives of the Chern roots of E the Chern character of E^* satisfies a similar formula to the Chern class,

$$\mathrm{Ch}_i(E^*) = (-1)^i \, \mathrm{Ch}_i(E).$$

Because of the denominators in the power series for e^α, the Chern character necessarily takes values in the tensor product $A^*(X) \otimes \mathbb{Q}$ rather than in $A^*(X)$. In fact, since the power sums $\sum \alpha_i^k$ generate the ring of symmetric polynomials over $\mathbb{Q}$, the Chern character encodes exactly the same information as the rational Chern class—that is, the image $c(E)_{\mathbb{Q}}$ of $c(E)$ under the map $A^*(X) \to A^*(X) \otimes \mathbb{Q}$—but possibly ****this is ambiguous**** slightly less information than the Chern class itself.

Suppose that

$$0 \to F \to G \to H \to 0$$

is an exact sequence of sheaves. If we formally factor the Chern classes of F and H:

$$c(F) = \prod_{i=1}^{m} (1 + \alpha_i) \quad \text{and} \quad c(H) = \prod_{j=1}^{n} (1 + \beta_j)$$

then the Chern class of G will be given by

$$c(G) = \prod_{i=1}^{m} (1 + \alpha_i) \prod_{j=1}^{n} (1 + \beta_j).$$

Thus

$$\mathrm{Ch}(G) = \sum_{i=1}^{m} e^{\alpha_i} + \sum_{j=1}^{n} e^{\beta_j}.$$

It follows that the Chern characters satisfy the Whitney Sum Formula

$$\mathrm{Ch}(G) = \mathrm{Ch}(F) + \mathrm{Ch}(H)$$

(now the name is unambiguous!)

We can use this to extend the definition of Chern character to all coherent sheaves on a smooth projective variety X: if $\mathcal{F}$ is any coherent sheaf on X, and

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

a resolution in which all the sheaves $\mathcal{E}_i$ are locally free, then we simply set

$$\mathrm{Ch}(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \,\mathrm{Ch}(\mathcal{E}_i)$$

Unlike Chern classes, Chern characters behave in a simple way with respect to tensor products of vector bundles: if the Chern classes of the vector bundles F and H are formally factored as above then, by the splitting principle,

$$c(F \otimes H) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + \alpha_i + \beta_j).$$

Thus

$$\mathrm{Ch}(F \otimes H) = \sum_{i,j} e^{\alpha_i + \beta_j};$$

that is, the Chern character satisfies the identity

$$\mathrm{Ch}(F \otimes H) = \mathrm{Ch}(F)\,\mathrm{Ch}(H).$$

This is what we mean when we say that the Chern character is formally better behaved than the Chern class.

Since the Chern character is equivalent data to the rational Chern class, this yields a formula for the rational Chern class of a tensor product. The result is quite convenient for machine computation, but the conversion of polynomials in the power sums to polynomials in the elementary symmetric polynomials is complicated enough that it is not so useful for computation by hand.

Exercise 17.1. (a) Find the Chern characters of the universal bundles S and Q on the Grassmannian $\mathbb{G} = \mathbb{G}(1, 3)$.

(b) Use this to find the Chern character of the tangent bundle $T_{\mathbb{G}} = S^* \otimes Q$.

(c) Use this in turn to find the Chern class of $T_{\mathbb{G}}$.

17.2.1 K-theory

Let X be any variety. The *Grothendieck ring* of finite rank vector bundles on X, written $K^*(X)$, is the group of formal finite linear combinations of vector bundles of finite rank on X, modulo the relations $E - F + G = 0$ whenever there exists an exact sequence

$$0 \to E \to F \to G \to 0.$$

The product in this ring is defined by setting the product of the classes of bundles E and F equal to the class of the tensor product bundle $E \otimes F$, and extending by linearity. This operation preserves the equivalence relation because tensoring with a vector bundle preserves exact sequences.

We define a the *Grothendieck group of coherent sheaves* on X, written $K_*(X)$, by taking formal combinations of coherent sheaves in place of vector bundles. The tensor product with a a vector bundle preserves short exact sequences, so the group $K_*(X)$ is a module over the ring $K^*(X)$. Since a vector bundle of finite rank may be regarded as a coherent sheaf, there is a natural map $K^*(X) \to K_*(X)$. If X is smooth and projective then any coherent sheaf has a finite resolution by vector bundles, so this map is an isomorphism.

In these terms, we can express the Whitney sum formula and the analogous relation for the Chern character of a tensor product by saying that *the Chern character is a ring homomorphism from the K-ring $K^*(X)$ to the rational Chow ring $A^*(X)$*. In fact, we can say much more:

Theorem 17.2 (Grothendieck)**.** *If X is a smooth projective variety, then the Chern character is an isomorphism of rings*

$$K^*(X) \otimes \mathbb{Q} \to A^*(X) \otimes \mathbb{Q}; \qquad E \mapsto Ch(E)$$

For the proof see].

Strikingly, there is an analogous statement in the category of differentiable manifolds: If we define the topological K-group of a manifold M to be the group of formal linear combinations of $\mathcal{C}^\infty$ vector bundles, with ring structure given as above by tensor products, then the Chern character gives an isomorphism

$$\text{gr } K(M) \otimes \mathbb{Q} \cong H^{2*}(X, \mathbb{Q})$$

where the term on the right is the ring of even-degree rational cohomology classes. ****again, we need a reference****

17.2.2 Chern classes of coherent sheaves

Let X be a smooth projective variety, and $\mathcal{F}$ a coherent sheaf of X. By the Hilbert Syzygy Theorem, we can resolve $\mathcal{F}$ by locally free sheaves: that is,

we can find an exact sequence

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

in which all the sheaves $\mathcal{E}_i$ are locally free, that is, sheaves of sections of vector bundles E_i. We can in turn use this to extend the definition of Chern classes to all coherent sheaves. If $\mathcal{F}$ were locally free, by Whitney sum its Chern class would be the alternating product of the Chern classes of the other bundles E_i in the sequence, and we adopt this as our definition in general: we set

$$c(\mathcal{F}) = \prod_{i=0}^{n} c(E_i)^{(-1)^i}.$$

Exercise 17.3. Verify that this is well-defined; that is, $c(\mathcal{F})$ does not depend on the choice of resolution. ****this should have a hint.****

Caution: If $Y \subset X$ is a subvariety of a smooth variety, then 1 plus the cycle class $[Y]$ is in general *not* equal to the Chern class of $\mathcal{O}_Y$; in general the Chern class of $\mathcal{O}_Y$ may have components of codimensions $> \operatorname{codim} Y$, and even the component in $A^{\operatorname{codim} Y}$ differs from $[Y]$ by a factor of $(-1)^{\operatorname{codim} Y - 1}(codim Y)!$. This is a consequence of the Grothendieck Riemann-Roch Theorem that we will discuss in Chapter 13 ****check this; and make sure it's in Ch 13****. For now, we invite the reader to verify a special case:

Exercise 17.4. Let $p \in \mathbb{P}^n$ be a point. Using the Koszul complex, show that the Chern class of the structure sheaf $\mathcal{O}_p$, viewed as a coherent sheaf on $\mathbb{P}^n$, is

$$c(\mathcal{O}_p) = 1 + (-1)^{n-1}(n-1)![p].$$

Here are some examples where the geometry of a subvariety is reflected in the Chern class of its structure sheaf:

Exercise 17.5. Let $C \subset \mathbb{P}^3$ be a smooth curve. Find the Chern class of the structure sheaf $\mathcal{O}_C$ in case

(a) C is a twisted cubic;

(b) C is an elliptic quartic curve, and

(c) C is a rational quartic curve.

17.3 Hirzebruch-Riemann-Roch

Much of the content of the formulas above was known to 19^{th} century algebraic geometers, although the formulas were expressed without cohomology, and only for line bundles (represented by divisors). For curves, any

cohomology group is either an H^0, or dual to one; for surfaces, what we would write as h^2 was expressed in terms of the dual h^0, and h^1 was left as an "error term," yielding an inequality in place of the equality. (The term $h^1(\mathcal{F})$ was called the *superabundance* of $\mathcal{F}$, in view of the fact that $h^0(\mathcal{F})$ could be that much greater than "expected.")

In the 20^{th} century these formulas were extended to sheaves on varieties of arbitrary dimension. Before we describe this extension, we need to introduce the *Todd class* of a coherent sheaf, which is a certain polynomial in the Chern classes of the sheaf that has already made itself felt in the previous formulas

$$\chi(\mathcal{O}_X) = \frac{c_1(T_X)}{2} \text{ when } C \text{ is a smooth projective curve,}$$
$$\chi(\mathcal{O}_S) = \frac{c_1(T_S)^2 + c_2(T_S)}{12} \text{ when } S \text{ is a smooth projective surface.}$$

Todd conjectured that there should "universal" polynomials giving the Euler characteristic of the structure sheaf of a smooth projective variety X of any dimension n as a polynomial $\mathrm{Td}_n(c_1, c_2, \dots)$ in the Chern classes $c_i = c_i(T_X)$ of the tangent bundle of X, and he wrote down a formula for it would have to be using generating functions.

The resulting polynomial is now called the n-th Todd polynomial, $\mathrm{Td}_n(c) = \mathrm{Td}_n(c_1, c_2, \dots)$. For example, from the Riemann-Roch formula for curves and Noether's formula for surfaces we must have

$$\mathrm{Td}_1(c) = \frac{c_1}{2}$$
$$\mathrm{Td}_2(c) = \frac{c_1^2 + c_2}{12}.$$

Todd's conjecture was verified by Hirzebruch. Since it is a polynomial in the Chern classes, we can define the Todd class $\mathrm{Td}_n(\mathcal{F}) = \mathrm{Td}_n(c_1(\mathcal{F}), c_2(\mathcal{F}), \dots)$ for any coherent sheaf $\mathcal{F}$.

Exercise 17.6. Consider the three varieties $X_1 = \mathbb{P}^3$, $X_2 = \mathbb{P}^1 \times \mathbb{P}^2$ and $X_3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

(a) In each case, calculate the degrees of the classes $c_3(T_{X_i})$, $c_1(T_{X_i})c_2(T_{X_i})$ and $c_1(T_{X_i})^3$.

(b) Show that the resulting 3×3 matrix is nonsingular.

(c) Show that the Euler characteristic $\chi(\mathcal{O}_{X_i}) = 1$ for each i.

(d) Given that the Euler characteristic of the structure sheaf of a smooth projective 3-fold X is expressible as a polynomial of degree 3 in the Chern classes of its tangent bundle, show from the above examples that the polynomial must be

$$\mathrm{Td}_3(c_1, c_2, c_3) = \frac{c_1 c_2}{24}.$$

Though we do not know a proof, it seems reasonable to hope that the formula for Td_n is determined in the same way for all n by the requirement that it be homogeneous of degree n (thinking of c_i as having degree i and give the right answer for products of projective spaces $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_j}$ with $\sum a_j = n$.

Following Todd and Hirzebruch, we give the definition for all the Todd polynomials at once, via their generating function, which we write as a symmetric power series in variables a_i of degree 1. The degree n part of this power series, $\mathrm{td}_n(a) = \mathrm{td}_n(a_1, \cdots)$, being a symmetric function of variables a_j, is a polynomial in the symmetric functions $c_j = c_j(a_1, \dots)$ defined by the formula $\sum_i c_i = \prod_j (1 - a_j)$. Thus we can write $\mathrm{td}_n(a) = \mathrm{Td}_n(c_1, \dots)$, the Todd polynomial, and $\mathrm{Td}(c) = \sum_{d=0}^{\infty} \mathrm{Td}_n(c)$. For any sheaf coherent sheaf $\mathcal{F}$ we define $\mathrm{Td}(\mathcal{F}) := \mathrm{Td}(c(\mathcal{F}))$.

To carry this out, let $a := a_1, a_2, \dots$ be variables, and set

$$\mathrm{td}(a) = \prod_{i=1}^{n} \frac{\alpha_i}{1 - e^{-\alpha_i}}.$$

To see what this is, write

$$1 - e^{-\alpha} = \alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{6} - \frac{\alpha^4}{24} + \dots$$

so

$$\frac{1 - e^{-\alpha}}{\alpha} = 1 - \frac{\alpha}{2} + \frac{\alpha^2}{6} - \frac{\alpha^3}{24} + \frac{\alpha^4}{120} - \dots;$$

inverting this, we get

$$\frac{\alpha}{1 - e^{-\alpha}} = 1 + \frac{\alpha}{2} + \frac{\alpha^2}{12} - \frac{\alpha^4}{720} + \dots$$

so

$$\mathrm{td}(a) = \prod_{i=1}^{n} (1 + \frac{\alpha_i}{2} + \frac{\alpha_i^2}{12} - \frac{\alpha_i^4}{720} + \dots).$$

Rewriting the first few of these in terms of the symmetric polynomials $c_i = c_i(a_1, \dots)$ we get

$$\mathrm{td}_0(a) = 1 = \mathrm{Td}_0(c),$$
$$\mathrm{td}_1(a) = \sum \frac{\alpha_i}{2} = \frac{c_1}{2} = \mathrm{Td}_1(c)$$

and

$$\mathrm{td}_2(a) = \frac{1}{12} \sum \alpha_i^2 + \frac{1}{4} \sum_{i<j} \alpha_i \alpha_j$$
$$= \frac{c_1^2 + c_2}{12} = \mathrm{Td}_2(c),$$

as they should be.

Now we can state the Hirzebruch Riemann-Roch formula: first, by construction, we have a formula for the Euler characteristic of the structure sheaf of a smooth projective variety X of dimension n:

$$\chi(\mathcal{O}_X) = \mathrm{Td}_n(T_X).$$

From this the formula for an arbitrary coherent sheaf $\mathcal{F}$ on X follows:

Theorem 17.7 (**?**])**.** *If X is a smooth projective variety of dimension n and $\mathcal{F}$ a coherent sheaf on X, then*

$$\chi(\mathcal{F}) = \deg\big(\mathrm{Ch}(\mathcal{F})\,\mathrm{Td}(T_X)\big).$$

To prepare for what is to come, recall that the deg map is defined as the pushforward map to the Chow group of a point, which is $\mathbb{Z}$.

This formula was first stated and proved in the setting of algebraic varieties by Hirzebruch **ref**; it was then generalized to the differentiable setting by Atiyah and Singer **ref**.

Exercise 17.8. Verify that this gives the classical Riemann-Roch formula in case $n = 1$ or 2, and write down the analogous formula in case $n = 3$.

********The next two subsections were just dumped here from Ch 7. Also, they repeat some of what's now in the Overture.********

17.4 Higher Direct Images of sheaves

******** Coordinate this with the "interlude" chapter!********

The direct image functor is a generalization of the functor sending a sheaf to its space of global sections. The *higher direct images*, which have the same relation to the other cohomology functors. Just as the higher cohomology groups of a sheaf are often used to derive information about global sections, we will use the higher direct image sheaves to shed light on the direct image itself. The reader may find it useful to review Section 6.4 before reading this material.

We will deal both with the cohomology of sheaves and with the homology of complexes. To keep from confusion, we will use H^i to denote the i-th Zariski cohomology functor, while H^i denotes the i-th homology of a complex.

Suppose that $\pi : X \to B$ is a projective morphism. This means that π can be factored as a closed immersion $X \subset \mathbb{P}^n_B = \mathbb{P}^n_K \times B$, for some n, followed by the projection to B. To simplify notation, we will simply write $\mathbb{P}$ for $\mathbb{P}^n_B$ in what follows.

By definition, $\mathbb{P} = Proj(S)$, where S is the sheaf of graded algebras $\mathcal{O}_B[x_0, \ldots, x_n]$. Let U_i be the open subscheme $x_i \neq 0$, and let

$$\mathcal{C}^\bullet : \quad \bigoplus_i \mathcal{O}_{\mathbb{P}}|_{U_i} \longrightarrow \bigoplus_{i,j} \mathcal{O}_{\mathbb{P}}|_{U_i \cap U_j} \longrightarrow \cdots$$

be the Čech complex on $\mathbb{P}$. If $\mathcal{F}$ is a quasicoherent sheaf on X we define $R^i\pi_*\mathcal{F}$ to be the degree zero part of the homology of the complex $\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{C}^\bullet$, that is,

$$R^i\pi_*\mathcal{F} := \big(\mathrm{H}^i(\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{C}^\bullet)\big)_0.$$

What makes this somewhat technical definition useful is that

$$(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}|_{U_i})_0$$

is the sheaf of modules over

$$\mathcal{O}_B[x_0, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n]$$

corresponding to the restriction of the sheaf $\mathcal{F}$ to the open set $U_i = \mathbb{A}^n_B$. This and the assumption that $\mathcal{F}$ is a sheaf shows that $R^0\pi_*\mathcal{F} = \pi_*\mathcal{F}$. Also, when $U \subset B$ is affine, so that each $U_i \cap \pi^{-1}(U)$ is affine, we get

$$R^i\pi_*\mathcal{F}|_{\pi^{-1}(U)} = H^i(\mathcal{F}|_{\pi^{-1}(U)}).$$

Since any sheaf is determined by its restriction to affine open subsets, this property characterizes $R^i\pi_*\mathcal{F}$ and shows that the definition is independent of the embedding $X \subset \mathbb{P}^n_B$ that we chose.

If $b \in B$ is a point (closed or not) then

$$H^i(\mathcal{F}|_b) = \mathrm{H}^i\big(\kappa(b) \otimes \mathcal{F} \otimes \mathcal{C}^\bullet\big),$$

but this is generally not equal to the fiber

$$(R^i\pi_*\mathcal{F})_b = \kappa(b) \otimes \mathrm{H}^i(\mathcal{F} \otimes \mathcal{C}^\bullet)$$

of the higher direct image. However, if z is a cycle or boundary in $\mathcal{F} \otimes \mathcal{C}^\bullet$ then $1 \otimes z$ is a cycle or boundary in $\kappa(b) \otimes \mathcal{F} \otimes \mathcal{C}^\bullet$ so we get maps $(R^i\pi_*\mathcal{F}) \to H^i(\mathcal{F}|_{X_b})$ which in turn induce comparison maps

$$\varphi^i_b : (R^i\pi_*\mathcal{F})|_b \to H^i(\mathcal{F}|_{X_b}).$$

We ask under what circumstances these are isomorphisms. We will give a partial answer to this question in the next section, but first we prove some basic properties of the $R^i\pi_*$.

Proposition 17.9. *Let $\pi : X \to B$ be a projective morphism.*

(a) **Restriction to open sets** *Let $U \subset B$ be an open subset, and let $\pi' : \pi^{-1}(U) \to U$ be the restriction of π. If $\mathcal{F}$ is any quasicoherent sheaf on X, then $(R^i\pi_*\mathcal{F})|_U = R^i\pi'_*(\mathcal{F}|_{\pi^{-1}(U)})$.*

(b) **Long Exact Sequence:** *The functor π_* is left exact, and the functors $\mathbb{R}^i\pi_*$ are the right derived functors of π_*. In particular, if*

$$\epsilon : 0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$$

is a short exact sequence of quasicoherent sheaves on X, then there are natural "connecting homomorphisms" η_i making the sequence

$$\cdots \longrightarrow R^i\pi_*\mathcal{F} \xrightarrow{R^i\pi_*\alpha} R^i\pi_*\mathcal{G} \xrightarrow{R^i\pi_*\beta} R^i\pi_*\mathcal{H} \xrightarrow{\eta_i} R^{i+1}\pi_*\mathcal{F} \longrightarrow \cdots$$

exact.

(c) **Push-Pull Formula:** *If $\mathcal{E}$ is a vector bundle on B and $\mathcal{F}$ is a quasicoherent sheaf on X, then*

$$R^i\pi_*(\pi^*\mathcal{E} \otimes \mathcal{F}) \cong \mathcal{E} \otimes R^i\pi_*\mathcal{F}.$$

Proof. (a): The since $\mathcal{O}_U$ is flat over $\mathcal{O}_B$, the restriction to U commutes with taking homology.

(b): The terms of the complex $\mathcal{C}^\bullet$ are flat, so when we tensor $\mathcal{C}^\bullet$ with the short exact sequence ϵ we get a short exact sequence of complexes, and thus a long exact sequence of homology sheaves. Taking the degree zero part preserves exactness. Since the long exact sequence begins with

$$0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{G} \to \cdots$$

we see that π_* is left exact.

To show that $R^i\pi_*$ is the i-th right derived functor of π_* it now suffices to show that $R^i\pi_*\mathcal{F} = 0$ when $\mathcal{F}$ is injective (Eisenbud [1995] Appedix A.3.9), or more generally flasque. It suffices to prove this after restricting to an affine subset $U \subset B$. Since the restriction of a flasque sheaf to an open subset is flasque, the result follows from the corresponding result for cohomology.

(c): The sheaf $\pi^*\mathcal{E}$ is also a vector bundle, and thus flat, so tensoring with $\pi^*\mathcal{E}$ commutes with taking homology,

$$\mathrm{H}^i(\pi^*\mathcal{E} \otimes \mathcal{F} \otimes \mathcal{C}^\bullet) = \pi^*\mathcal{E} \otimes \mathrm{H}^i(\mathcal{F} \otimes \mathcal{C}^\bullet).$$

Taking the degree zero part yields the desired formula. □

The other foundational result we will use is Grothendieck's Coherence Theorem:

Theorem 17.10. *If $\pi : X \to B$ is a projective morphism, and $\mathcal{F}$ is a coherent sheaf on X, then $R^i\pi_*\mathcal{F}$ is coherent for each i.*

The proof involves some useful ideas from homological commutative algebra.

Proof. Since the formation of $R^i\pi_*\mathcal{F}$ commutes with the restriction to an open set in the base, it suffices to treat the case where $B = \operatorname{Spec} A$ is affine. The Čech complex $\mathbb{C}$ is the direct limit of the duals of the Koszul complexes that are S-free resolutions of the of the ideals $(x_0^m, \dots x_n^m) \subset S$, and direct limits commute with taking homology. Thus if M is any finitely generated graded $S = A[x_0, \dots, x_n]$-module representing the sheaf $\mathcal{F}$, the homology of $\mathcal{F} \otimes \mathcal{C}^\bullet$ is

$$R^i\pi_*\mathcal{F} = \lim_m Ext^i_S((x_0^m, \dots, x_n^m), M)_0.$$

Write $\mathfrak{m}$ for the "irrelevant" ideal $(x_0, \dots, x_n) \subset S$. For each m there is an integer $N(m)$ such that $\mathfrak{m}^{N(m)} \subset (x_0^m, \dots, x_n^m) \subset \mathfrak{m}^m$. It follows that

$$\lim_m Ext^i_S((x_0^m, \dots, x_n^m), M) = \lim_m Ext^i_S(\mathfrak{m}^m, M).$$

Each term $Ext^i_S(\mathfrak{m}^m, M)$ of this limit is a finitely generated S-module, so its degree 0 part is a finitely generated A-module, and it suffices to show that the natural map

$$Ext^i_S(\mathfrak{m}^m, M)_0 \to Ext^i_S(\mathfrak{m}^{m+1}, M)_0$$

is an isomorphism for large m. From the long exact sequence in Ext_S, and the fact that $\mathfrak{m}^m/\mathfrak{m}^{m+1}$ is a direct sum of copies of $A(-m)$ (the free A-module of rank 1 with generator in degree m) we see that it is enough to prove that $Ext^i_S(A(-m), M)_0 = 0$ when m is large. Disentangling the degree shifts, we see that

$$Ext^i_S(A(-m), M)_0 = Ext^i_S(A, M)(m)_0 = Ext^i_S(A, M)_m.$$

However, A is annihilated (as an S-module) by $\mathfrak{m}$, so $Ext^i_S(A, M)$ is annihilated by $\mathfrak{m}$. Since it is a finitely generated S-module, it can only be nonzero in finitely many degrees, whence, indeed, $Ext^i_S(A, M)_m = 0$ when m is large. □

We remark that, using the notion of Castelnuovo-Mumford regularity it is possible to bound the degree m for which $Ext^i_S(A, M)_m = 0$ in terms of the data in a free resolution of M (similar to Smith [2000]), so the proof just given allows effective computation of the functors $R^i\pi_*(\mathcal{F})$.

17.5 Grothendieck-Riemann-Roch

****this first para was moved from elsewhere – reconcile.**** Grothendieck's version of the Riemann-Roch Theorem extends Hirzebruch's formula to a statement about a family of schemes and a family of sheaves—in other words, a morphism $\pi : X \to B$ and a sheaf $\mathcal{F}$ on X. Hirzebruch's Riemann-Roch formula is the special case when B is a single point. The general result

says something about the Chern classes of the direct image $\pi_*\mathcal{F}$ introduced in Chapter6.

The Grothendieck-Riemann-Roch formula tells us something about the Chern characters of the sheaves $\pi_*(\mathcal{F})$ and $R^i\pi_*(\mathcal{F})$. As we saw in the previous Section the ranks of these sheaves are the dimensions of the vector spaces $H^0(\mathcal{F}|_{X_b})$ for a generic point b. The additional information on the Chern classes tells us something aboutbut something about how the spaces $H^0(\mathcal{F}|_{X_b})$ fit together as b varies.

While the alternating sum of the ranks is the content of the Hirzebruch Riemann-Roch Theorem applied to the restriction of $\mathcal{F}$ to a general fiber of π, the information on Chern classes is new.

Theorem 17.11 (Grothendieck's Riemann-Roch Formula)**.** *If $\pi : X \to B$ is a projective morphism and $\mathcal{F}$ is a coherent sheaf on X then*

$$\sum_{i=0}^{n}(-1)^i \operatorname{Ch}(R^i\pi_*\mathcal{F}) = \pi_*\Big[\frac{\operatorname{Ch}(\mathcal{F})\cdot \operatorname{Td}(T_X)}{\pi^*\operatorname{Td}(T_B)}\Big].$$

A very readable proof can be found in Borel and Serre [1958].

In case the map π is a submersion—that is, the differential $d\pi$ is surjective everywhere—the formula simplifies a little. In this case we write $T_{X/B}$ for the *relative tangent bundle*, defined by the short exact sequence

$$0 \to T^v_{X/B} \to T_X \to \pi^*T_B \to 0.$$

Corollary 17.12. *If $\pi : X \to B$ is a projective morphism and a submersion, and $\mathcal{F}$ is a coherent sheaf on X, then*

$$\sum_{i=0}^{n}(-1)^i \operatorname{Ch}(R^i\pi_*\mathcal{F}) = \pi_*\Big[\operatorname{Ch}(\mathcal{F})\cdot \operatorname{Td}(T^v_{X/B})\Big].$$

Proof. This follows at once from Theorem17.11 and the multiplicativity of the Todd class. □

When applying these formulas it is crucial to know that the sheaves $R^i\pi_*(\mathcal{F})$ have fibers at an arbitrary point b equal to $H^i(\mathcal{F}|_{\pi^{-1}(b)})$. The Theorem on Cohomology and Base Change proved in the next section, gives conditions under which this happens. The remainder of the Chapter is concerned with examples using these techniques.

A major application of the Grothendieck Riemann Roch Theorem to the geometry of the moduli space of curves is described in Chapter **??**. Here we'll treat a more elementary application to vector bundles on projective space.

17.6 Application: Jumping lines

In this section, we'll describe the notion on *jumping lines*, an invariant used to study the geometry of vector bundles on projective space. To keep the notation relatively simple, we'll deal here just with the case of vector bundles of rank 2 on $\mathbb{P}^2$; but as indicated in Exercises **-**, the generalizations to bundles of arbitrary rank on $\mathbb{P}^n$ is straightforward.

We start with a familiar fact: that every vector bundle E on $\mathbb{P}^1$ is a direct sum

$$E = \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i)$$

of line bundles, with

$$\sum a_i = c_1(E).$$

The analogous statement is clearly false in general for $\mathbb{P}^n$ with $n \geq 2$: for example, we have the

Exercise 17.13. Use Chern classes to prove that the following bundles are not direct sums:

(a) The tangent bundle to $\mathbb{P}^n$

(b) The universal subbundle on the Grassmannian $G(k, n)$

Now, while the classification of bundles on $\mathbb{P}^1$ doesn't generalize to $\mathbb{P}^n$, it does provide a useful tool for analyzing bundles on $\mathbb{P}^n$, based on the fact that while a bundle E on $\mathbb{P}^n$ may not split, its restriction to each line $L \subset \mathbb{P}^n$ does. For every collection of integers a_i with $\sum a_i = c_1(E)$, then, we may define a subset $\Gamma_a \subset \mathbb{G}(1, n)$ of the Grassmannian of lines in $\mathbb{P}^n$ by

$$\Gamma_a = \{L \in \mathbb{G}(1, n) : E|_L \cong \bigoplus \mathcal{O}_L(a_i)\}.$$

This will be a locally closed subset of $\mathbb{G}(1, n)$, and together these give a stratification of $\mathbb{G}(1, n)$ whose geometry is an important invariant of E; the closures of the strata Γ_a are called loci of jumping lines.

****At this point, we need a discussion of (in effect) the deformation theory of vector bundles/ruled surfaces over $\mathbb{P}^1$. This should lead up to the notion of versality—though we don't have to call it that—and a description of the direct image and higher direct image of a family in the versal case****

To keep the notation relatively simple, we'll deal here just with the case of vector bundles of rank 2 on $\mathbb{P}^2$; but as indicated in Exercises **-**, the generalizations to bundles of arbitrary rank on $\mathbb{P}^n$ is straightforward.

So: let E be a vector bundle of rank 2 on $\mathbb{P}^2$. As we'll see, the loci of jumping lines depends very much on the parity of $c_1(E) \in A^1(\mathbb{P}^2) \cong \mathbb{Z}$; we'll consider the case of $c_1(E)$ even and odd in turn.

17.6.1 Vector bundles with even first Chern class

Assume first that $c_1(E) = 2k\alpha$, where $\alpha \in A^1(\mathbb{P}^2)$ is the class of a line. We want to think of the restrictions of E to each line in $\mathbb{P}^2$ in turn as a family of vector bundles on $\mathbb{P}^1$, parametrized by the Grassmannian $\mathbb{G}(1,2)$ (otherwise known as the dual plane $\mathbb{P}^{2*}$). To do this, we introduce the by now familiar universal line

$$\Phi = \{(L,p) : p \in L\} \subset \mathbb{P}^{2*} \times \mathbb{P}^2;$$

let $\pi_1 : \Phi \to \mathbb{P}^{2*}$ and $\pi_2 : \Phi \to \mathbb{P}^2$ be the projections. We can view $\pi_1 : \Phi \to \mathbb{P}^{2*}$ as the projectivization of the universal subbundle S on $\mathbb{P}^{2*} = \mathbb{G}(1,2)$; if we denote by

$$\zeta = c_1(\mathcal{O}_{\mathbb{P}S}(1)) \in A^1(\Phi)$$

the tautological class, and by α both the hyperplane class on $\mathbb{P}^{2*}$ and its pullback to Φ, then the Chow ring of Φ is given by

$$A^*(\Phi) = \mathbb{Z}[\alpha,\zeta]/(\alpha^3, \zeta^2 - \alpha\zeta + \alpha^2).$$

Note that ζ can also be realized as the pullback of the hyperplane class from $A^1(\mathbb{P}^{2*})$; we can see either from this or the representation of $A^*(\Phi)$ above that $\zeta^3 = 0$. We see also that the pushforward map $(\pi_1)_*$ is given by

$$\begin{aligned} \alpha^2 &\mapsto 0 \\ \alpha\zeta &\mapsto \alpha \\ \zeta^2 &\mapsto \alpha \end{aligned}$$

Now, to realize the restrictions of E to the lines in $\mathbb{P}^2$ as a family, we want to consider simply the pullback bundle

$$F = \pi_2^*(E)$$

on Φ. We would expect that, for an open dense subset U of $L \in \mathbb{P}^{2*}$, the restriction of F to the fiber over L would split evenly, that is, as

$$E|_L = F|_{\pi_2^{-1}(L)} \cong \mathcal{O}_L(k) \oplus \mathcal{O}_L(k);$$

that there will be a curve $C \subset \mathbb{P}^{2*}$ of lines L such that

$$E|_L = F|_{\pi_2^{-1}(L)} \cong \mathcal{O}_L(k+1) \oplus \mathcal{O}_L(k-1),$$

and that these are the only two splitting types that will occur. The first question we might ask, then, is just: what is the degree of the curve C?

We'd like to answer this by applying the Grothendieck-Riemann-Roch formula to the direct image of the bundle F under the map π_1. It's not clear, though, what this will tell us: if $k \geq 0$, the direct image $(\pi_1)_* F$ will be the bundle on $\mathbb{P}^{2*}$ whose fiber at each point $L \in \mathbb{P}^{2*}$ is the space of sections of $E|_L$, and it's not clear how knowing the Chern classes of this bundle reflect the different possible splittings of $E|_L$.

The trick, in fact, is to use the theorem on cohomology and base change. If we replace E by $E' = E \otimes \mathcal{O}_{\mathbb{P}^2}(-k-1)$ and F by the corresponding $F' = \pi_2^* E'$, then the restriction of the bundle E' to L will split as

$$E'|_L \cong \begin{cases} \mathcal{O}_L(-1)^{\oplus 2} & \text{if } L \in U, \text{ and} \\ \mathcal{O}_L \oplus \mathcal{O}_L(-2) & \text{if } L \in C. \end{cases}$$

The point is, we now have

$$h^0(E'|_L) = h^1(E'|_L) = \begin{cases} 0 & \text{if } L \in U, \text{ and} \\ 1 & \text{if } L \in C. \end{cases}$$

What happens now when we take the direct image $(\pi_1)_*(F')$? First of all, the direct image itself will be 0: over the open subset $U \subset \mathbb{P}^{2*}$ there are no sections, and since $(\pi_1)_*(F')$ is torsion free, it follows that $(\pi_1)_*(F') = 0$. Where the jump in the cohomology groups $H^i(E'|_L)$ will be reflected is in the higher direct image $R^1(\pi_1)_*(F')$; this will be a sheaf supported on the curve C and having rank 1 there. It follows that

$$c_1(R^1(\pi_1)_*(F')) = [C]$$

and this is something we can calculate from Grothendieck-Riemann-Roch.

To set up, we first need the Todd class of the relative tangent bundle of $\pi_1 : \Phi \to \mathbb{P}^{2*}$. In fact, we've already calculated the Chern class of this bundle: we have, from **ref**

$$c_1(T^v_{\Phi/\mathbb{P}^{2*}}) = -\alpha + 2\zeta$$

and correspondingly

$$\mathrm{Td}(T^v_{\Phi/\mathbb{P}^{2*}}) = 1 + \frac{-\alpha + 2\zeta}{2} + \frac{(-\alpha + 2\zeta)^2}{12}$$

(note that since $T^v_{\Phi/\mathbb{P}^{2*}}$ is a line bundle, there's no cubic term). If we write the Chern class of the bundle F' as

$$c(F') = -2\zeta + e\zeta^2$$

then we have

$$\begin{aligned} \mathrm{Ch}(F') &= \mathrm{rank}(F') + c_1(F') + \frac{c_1^2(F') - 2c_2(F')}{2} \\ &= 2 - 2\zeta + (2-e)\zeta^2. \end{aligned}$$

Altogether, then, Grothendieck-Riemann-Roch tells us that

$$\begin{aligned}[C] &= \mathrm{Ch}_1\left(R^1(\pi_1)_*(F')\right)\\ &= -(\pi_1)_*\left\{\left(2-2\zeta+(2-e)\zeta^2\right)\left(1+\frac{-\alpha+2\zeta}{2}+\frac{(-\alpha+2\zeta)^2}{12}\right)\right\}_2\\ &= -(\pi_1)_*\left(\frac{(-\alpha+2\zeta)^2}{6}-\zeta(-\alpha+2\zeta)+(2-e)\zeta^2\right)\\ &= (e-1)\zeta;\end{aligned}$$

or, in other words, the degree of the curve of jumping lines is $e-1$. Given that $E' = E\otimes\mathcal{O}_{\mathbb{P}^2}(-k-1)$, we have

$$e = c_2(E') = c_2(E) - (k^2-1)$$

and so for our original bundle E, the statement is that *the degree of the curve of jumping lines is* $c_2(E) - k^2 = c_2(E) - c_1^2(E)/4$.

17.6.2 Vector bundles with odd first Chern class

Assume now that $c_1(E) = (2k+1)\alpha$. The basic set-up is the same as in the preceding case, but here our expectations are qualitatively different. As before, we would expect that, for an open dense subset U of $L\in\mathbb{P}^{2*}$, the restriction of F to the fiber over L would split "as evenly as possible," that is, as

$$E|_L = F|_{\pi_2^{-1}(L)} \cong \mathcal{O}_L(k+1)\oplus\mathcal{O}_L(k).$$

But in this case we'd expect that the locus where the splitting is more unbalanced—specifically, the set of lines L such that

$$E|_L = F|_{\pi_2^{-1}(L)} \cong \mathcal{O}_L(k+2)\oplus\mathcal{O}_L(k-1)—$$

will be not a curve but a finite set Γ of points. (Again, we'd expect that these are the only two splitting types that will occur.) In this case we ask: what is the degree of the finite set Γ?

Our approach to this problem starts out similarly: we begin by twisting the vector bundle E so that the jump in splitting type is reflected in the ranks of the cohomology groups of its restriction to lines. Specifically, we replace E by $E' = E\otimes\mathcal{O}_{\mathbb{P}^2}(-k-1)$, so that the restriction of the bundle E' to L will split as

$$E'|_L \cong \begin{cases}\mathcal{O}_L\oplus\mathcal{O}_L(-1) & \text{if } L\in U, \text{ and}\\ \mathcal{O}_L(1)\oplus\mathcal{O}_L(-2) & \text{if } L\in\Gamma.\end{cases}$$

The point is, we now have

$$h^0(E'|_L) = \begin{cases}1 & \text{if } L\in U, \text{ and}\\ 2 & \text{if } L\in\Gamma.\end{cases}$$

and

$$h^1(E'|_L) = \begin{cases} 0 & \text{if } L \in U, \text{ and} \\ 1 & \text{if } L \in \Gamma. \end{cases}$$

As before, this means that the sheaf $R^1\pi_*(F')$ is supported on the exceptional locus, in this case Γ; and in the transverse case the analysis of families of vector bundles shows that it is isomorphic to the structure sheaf $\mathcal{O}_\Gamma$.

The big difference is that the direct image $\pi_*(F')$ is nonzero; in fact, away from Γ it's locally free of rank 1. In the transverse case, moreover, we've seen that it's also locally free in a neighborhood of a point $p \in \Gamma \subset \mathbb{P}^{2*}$ corresponding to a line $L \subset \mathbb{P}^2$, though the comparison map

$$\pi_*(F')|_p \to H_0(F'|_L)$$

is zero: none of the sections of $F'|_L$ extend to a neighborhood of L. In other words, we have

$$\pi_*(F') \cong \mathcal{O}_(m)$$

for some m; the value of m will also emerge in the calculation.

We can now get to work and apply GRR. To begin with, write the Chern class of the bundle E' as

$$c(E') = 1 - \zeta + e\zeta^2;$$

the Chern character of E', and of its pullback F' to Φ, is then

$$\mathrm{Ch}(F') = 2 - \zeta + (\frac{1}{2} - e)\zeta^2.$$

The Grothendieck Riemann Roch formula then tells us that

$$\begin{aligned} &\mathrm{Ch}(\pi_*F') - \mathrm{Ch}(R^1\pi_*F') \\ &\qquad = \pi_*\left[\left(2 - \zeta + (\frac{1}{2} - e)\zeta^2\right)\left(1 + \frac{-\alpha + 2\zeta}{2} + \frac{(-\alpha + 2\zeta)^2}{12}\right)\right] \\ &\qquad = 1 - e\alpha + \frac{e}{2}\alpha^2. \end{aligned}$$

Now, from the isomorphism $\pi_*(F') \cong \mathcal{I}_\Gamma(m)$ we have

$$\mathrm{Ch}(\pi_*F') = 1 + m\alpha + \frac{m^2}{2}\alpha^2;$$

since

$$\mathrm{Ch}(R^1\pi_*F') = \mathrm{Ch}(\mathcal{O}_\Gamma) = [\Gamma] = \gamma\alpha^2,$$

all in all we have

$$1 - e\alpha + \frac{e}{2}\alpha^2 = 1 + m\alpha + (\frac{m^2}{2} - 2\gamma)\alpha^2.$$

From the degree 1 terms, we see that $m = -e$; and then equating the degree 2 terms we arrive at our answer: the degree of Γ is given by

$$\gamma = \frac{e^2 - e}{2}.$$

Finally, to express this in terms of the Chern classes of our original bundle E, we observe that by the splitting principle, for an arbitrary bundle E of rank 2 on $\mathbb{P}^2$ with first Chern class $c_1(E) = (2k+1)\zeta$ the degree of the second Chern class of the twist $E' = E(-k-1)$ is

$$\begin{aligned} e &= c_2(E) - (k+1)c_1(E) + (k+1)^2 \\ &= c_2(E) - k(k+1). \end{aligned}$$

17.6.3 Examples and Generalizations

Some examples would be in order. We start with the simplest examples of vector bundles of rank 2 on $\mathbb{P}^2$, the universal quotient bundle Q and the tangent bundle $T_{\mathbb{P}^2} \cong Q(1)$. In either case, when we twist the bundle to make the second Chern class equal to $-\zeta$, the bundle $E' \cong Q(-1) \cong T_{\mathbb{P}^2}(-2)$ will have Chern class

$$c(E') = 1 - \zeta + \zeta^2.$$

Our formula then says that these bundles will have no jumping lines, which is clear in any event from homogeneity: since the automorphism group of $\mathbb{P}^2$ carries each of these bundles to itself and acts transitively on lines, the splitting type of their restrictions to lines can't vary.

For another example, suppose now that F_0, F_1 and F_2 are general homogeneous polynomials of degree d on $\mathbb{P}^2$. We can then define a vector bundle of rank 2 on $\mathbb{P}^2$ by associating to each point p the quotient space

$$E_p = \mathbb{C}^3 / \langle (F_0(p), F_1(p), F_2(p)) \rangle;$$

in other words, this is the pullback of the universal quotient bundle Q by the map $\mathbb{P}^2 \to \mathbb{P}^2$ given by the triple $[F_0, F_1, F_2]$. Since $c(Q) = 1 + \zeta + \zeta^2$, we have

$$c(E) = 1 + d\zeta + d^2\zeta^2.$$

When the degree $d = 2k$ is even, the bundle $E' = E(-k-1)$ will have Chern class

$$c(E') = 1 - 2\zeta + (3k^2 + 1)\zeta^2,$$

and so the curve of jumping lines will have degree $3k^2$, or $\frac{3}{4}d^2$. When the degree $d = 2k+1$ is odd, we have

$$c(E') = 1 - \zeta + (3k^2 + 3k + 1)\zeta^2,$$

so the number of jumping lines will be the binomial coefficient $\binom{3k^2+3k+1}{2}$.

Exercise 17.14. In case $d = 3$—that is, E is the pullback of the universal quotient bundle under the map $\mathbb{P}^2 \to \mathbb{P}^2$ given by a general net of cubic curves—show that the jumping lines of E are exactly the 21 lines in $\mathbb{P}^2$ contained in some element of the net.

Finally, as we suggested at the outset of this section, there are natural analogs of the notion of jumping lines for vector bundles of larger rank on possibly larger projective spaces. We won't go into these; the following exercises give a couple simple cases.

Exercise 17.15. Let E be a vector bundle of rank 3 on $\mathbb{P}^2$, with first Chern class $c_1(E) = d\zeta$. Show that if $d \equiv 0 \pmod 3$, we expect a curve of jumping lines, while if $d \equiv 1$ or $2 \pmod 3$, we expect a finite number. Calculate the class of the locus of jumping lines in any case (note that the answer will be different in the cases $d \equiv 1$ and 2).

Exercise 17.16. Now consider a vector bundle E of rank 2 on $\mathbb{P}^3$, and the corresponding locus $\Sigma \subset \mathbb{G}(1,3)$ of jumping lines. Again, if $c_1(E)$ is even we expect Σ to be codimension 1 in $\mathbb{G}(1,3)$, while if $c_1(E)$ is odd we expect it to have codimension 2. In either case, calculate the class of Σ. Note that in both cases, we expect there to be more than 2 splitting types, with the most unbalanced occurring in codimension 3 or 4 in $\mathbb{G}(1,3)$. Can you find formulas for the classes of these loci?

Exercise 17.17. As an example, let $q : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$ be a nondegenerate skew-symmetric bilinear form. We can define a bundle of rank 2 E on $\mathbb{P}^3$ by setting

$$E_p = \langle p \rangle^{\perp} / \langle p \rangle.$$

Describe the locus of jumping lines for such a bundle.

18

Brill-Noether

18.1 What maps to projective space "should" a curve have?

Until the 20th century, varieties defined as subsets of projective space. In this respect, algebraic geometry was much like other fields in mathematics: for example, in the 19^{th} century a group was by definition a subset of GL_n or S_n closed under the operations of composition and inversion. But in about 1860 Riemann's work established a way of talking about curves that crystallized, over the next hundred years, into our notion of an abstract variety. Riemann also studied the family of such abstract curves, and gave a parametrization. These developments made it possible, soon after Riemann's work, for Brill and Noether to ask the questions: "What embeddings, or more generally, what sort of maps to projective space will every curve of genus g have? When will there be at least one map of given degree d to a given projective space $\mathbb{P}^r$?" They gave a correct answer to this question in **?**], but it took about 100 years before **?**] produced the first correct proof! Here is the result:

Theorem 18.1 (Brill and Noether]). *Every smooth projective curve of genus g over $\mathbb{C}$ admits a rational map to $\mathbb{P}^r$ of degree d if and only if*

$$d \geq g - (r+1)(d-g+r)$$

Conventions: In this chapter we will concern ourselves only with smooth complete curves C. To avoid trivialities we will often limit ourselves to *non-degenerate* rational maps—that is, to maps whose image is not contained in any hyperplane. We will generally identify two such maps if they differ by an automorphism group of the target projective space. Finally, we the *degree* of a map $\varphi : C \to \mathbb{P}^r$ we will mean the degree of the pullback $\varphi^* H$ of a general hyperplane $H \subset \mathbb{P}^r$ as a divisor; thus, the degree of φ is the degree of the image curve $\varphi(C)$ times the degree of the cover $C \to \varphi(C)$.

Given these conventions, maps $\varphi : C \to \mathbb{P}^r$ correspond one-to-one to pairs (L, V) where L is a line bundle of degree d on C and $V \subset H^0(L)$ an $(r+1)$-dimensional vector space of sections without common zeroes (base locus). If we drop the requirement that the sections of V have no common zeroes, such an object is called a *linear system of degree d and dimension r*; classically, it was referred to as a g^r_d. (Note that a g^r_d with base locus still gives rise to a map to $\mathbb{P}^r$; it just may have degree smaller than d). In these terms, the central problem of Brill Noether theory is this:

For a given curve C, describe the set of g^r_ds on C and the geometry of the corresponding maps $C \to \mathbb{P}^r$.

The starting point for any study of this question involves one essential construction and two theorems. The construction is that of the *Jacobian* $J(C)$ of the curve C; the theorems are the Abel Theorem and the Riemann-Roch Theorem. In the next page or two we'll summarize the content of these theorems.

One remark: for the remainder of this Chapter, we'll be working over the complex numbers and making use of the singular homology and cohomology of the varieties we deal with. The results we obtain are thus proved only in characteristic 0. In fact, all our conclusions (the relations among the numerical equivalence classes of various subvarieties of Jacobians and the Chern classes of certain bundles; in particular, the existence half of Brill-Noether) are true over an algebraically closed field of arbitrary characteristic, but the constructions and proof require different (and more technically demanding) methods.

The Jacobian. One of the early motivations for studying algebraic curves came from calculus, specifically the desire to make sense of integrals of algebraic functions. In modern terms, this means integrals

$$\int_{p_0}^{p} \omega$$

where ω is a holomorphic (or more generally meromorphic) differential on a complex curve C, p_0 and p are points of C and the integral is taken along a path from p_0 to p on C. One problem in trying to define such integrals is that if the curve in question has positive genus, they depend on the choice

of path. We can incorporate this by viewing the expression $\int_{p_0}^{p}$ as a linear functional on the space $H^0(K_C)$ of holomorphic differentials on C, defined *modulo the subgroup of those linear functions obtained by integration over closed loops on* C. Now, the integral of a holomorphic (and hence closed) forms over a closed loop γ depends only on the homology class of γ; we thus have a map

$$H_1(C, \mathbb{Z}) \to H^0(K_C)^*$$

which embeds $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ as a discrete, cocompact lattice in $H^1(K_C)^* \cong \mathbb{C}^g$. We may thus view the integral $\int_{p_0}^{p}$ as an element of the quotient

$$\int_{p_0}^{p} \in H^0(K_C)^*/H_1(C, \mathbb{Z})$$

We then define the *Jacobian* $J(C)$ of the curve C to be this quotient. By our construction, $J(C)$ is a complex torus, and in particular a compact complex manifold; in fact, it is a projective variety over $\mathbb{C}$, though that takes a good bit more work to prove. We will indicate below other approaches to the construction of $J(C)$ that apply much more generally (to smooth curves over an arbitrary base scheme B, yielding a projective B-scheme); but since we will need for our calculation the description of $J(C)$ as a complex torus we might as well start with this one.

Having thus defined the Jacobian, we see that integration defines a map $C \to J(C)$, and more generally maps

$$\mu_d : C_d \to J(C)$$

from the symmetric products of C to its Jacobian, defined by

$$D = \sum p_i \mapsto \sum \int_{p_0}^{p_i}.$$

These maps are called *Abel-Jacobi* maps.

Abel's Theorem. In this context, Abel's Theorem is easy to state: it says simply that *two divisors D and E ofn the same degree d are linearly equivalent if and only if $\mu_d(D) = \mu_d(E)$*. Note that one direction is relatively immediate: if D and E are linearly equivalent, there is a pencil of divisors, parameterized by $\mathbb{P}^1$, interpolating between them; and since any map from $\mathbb{P}^1$ to a torus is constant it follows that $\mu(D) = \mu(E)$. The hard part (which was in fact proved by Clebsch, not Abel) is the converse.

The import of Abel's Theorem is that we may, for each d, identify the set $\mathrm{Pic}^d(C)$ of linear equivalence classes of divisors of degree d on C with the Jacobian $J(C)$ (though the identification is not canonical; it depends on the choice of a base point $p_0 \in C$). In particular, it shows that the functor of families of line bundles of degree d on C is representable. We also see that the subset of line bundles L of degree d such that $h^0(L) \geq r + 1$ is

an algebraic subvariety of $\mathrm{Pic}^d(C)$ (it's the locus where fiber dimension of μ_d is r or greater). We will denote this subvariety $W^r_d(C)$, and interpret the problem "describe the set of g^r_ds on C" to mean describe the variety $W^r_d(C)$—its dimension, irreducible components, smooth and singular loci, etc. We should also mention that $W^r_d(C)$ has a natural scheme structure, with which it represents the functor of line bundles of degree d with $r+1$ or more sections; to see how this is done we refer the reader to [ACGH].

Riemann-Roch. The Riemann-Roch Theorem for curves, in its classical form, says that for a line bundle L on a smooth curve of genus g,

$$h^0(L) = d - g + 1 + h^0(K - D).$$

For line bundles L of degree $d > 2g-2$, the last term is zero, and so Riemann-Roch tells us the dimension precisely:

$$h^0(L) = d - g + 1.$$

For line bundles of degree close to $2g-2$ it gives us approximate information: for example, if $d = 2g-2$, Riemann-Roch says that

$$h^0(L) = \begin{cases} g, & \text{if } L = K; \\ g-1, & \text{otherwise} \end{cases} \tag{18.1}$$

and if $g > 0$ and $d = 2g-3$, it says that

$$h^0(L) = \begin{cases} g-1, & \text{if } L = K - p \text{ for some point } p \in C; \\ g-2, & \text{otherwise.} \end{cases} \tag{18.2}$$

It also tells us that for any line bundle,

$$h^0(L) \geq d - g + 1.$$

Beyond this, however, it's a pretty crude tool; as we'll see shortly, there are much better inequalities.

One thing the last inequality does tell us is that *any line bundle L of degree $d = g$ has $h^0(L) > 0$*. Thus the map

$$\mu_g : C_g \to \mathrm{Pic}^g(C)$$

is surjective, and in fact birational (this is the *Jacobi Inversion Theorem*). This in turn may be used to give a construction of the Jacobian that applies in much greater generality than the one given above: composing μ_g with translations, we see that $\mathrm{Pic}^g(C)$, and hence $J(C)$, may be covered by open sets isomorphic to open subsets of the symmetric product C_g, and these can be glued together to construct $J(C)$. (Historically, it was the desire to carry out this construction that led André Weil to the definition of an abstract variety in the first place!) In even greater generality, for $d \geq g$ we can apply Grothendieck's theory of étale equivalence relations to construct $\mathrm{Pic}^d(C)$ as the quotient of C_d by linear equivalence; see **Kleiman**.

18.1.1 Behavior on special curves: Clifford and Castelnuovo Theorems

We may summarize what Riemann-Roch tells us about the possible existence of g^r_ds on curves of genus g in the following diagram:

** parallelogram chart **

Any point in the shaded area of this chart represents a pair (d, r) such that Riemann-Roch allows for the existence of a line bundle L on a curve C of genus g with $h^0(L) = r + 1$.

As we indicated above, however, we can do better. To begin with, we have *Clifford's Theorem*, which says that if L is any line bundle such that $h^0(L)$ and $h^0(KL^{-1})$ are both nonzero, then

$$h^0(L) \leq \frac{d}{2} + 1.$$

Note that this inequality therefore applies to any line bundle L of degree $d \leq 2g$. An extension of Clifford also says that if moreover equality holds then either $L = \mathcal{O}$, $L = K$ or C is hyperelliptic and L is a multiple of the g^1_2 on C. If we exclude the cases of degree $d = 0$ and $d \geq 2g - 2$ (where after all Clifford is not telling us anything new), we may state this as

Theorem 18.2 (Clifford)**.** *If C is a curve of genus g and L a line bundle of degree d on C with $0 < d < 2g - 2$, then*

$$h^0(L) \leq \frac{d}{2} + 1,$$

with equality holding only if C is hyperelliptic and L a multiple of the g^1_2.

This then cuts the above chart of allowed values of d and r essentially in half:

** triangle chart **

Now, Clifford's Theorem is sharp: for every d, g and r allowed by Clifford, there do exist curves of genus g and g^r_ds on them. It does not, however, represent a satisfactory answer to our basic problem, for two reasons:

• Given our motivation for studying g^r_ds on curves—the classification of curves in projective space—you may say that our real object of interest is not g^r_ds in general but those whose associated maps give birational embeddings in $\mathbb{P}^r$. But the linear systems satisfying equality in Clifford's Theorem, and (as we'll see in just a moment) those that are even close to this, are not birationally very ample. Thus, we may refine our original question and ask: *what birationally very ample linear series exist on curves of genus g*—in other words, for which d, g and r do there exist irreducible, nondegenerate curves of degree d and geometric genus g in $\mathbb{P}^r$?

• The linear series that achieve equality in Clifford exist on hyperelliptic curves; and similarly (as we'll at least assert in the statement of the Brill-Noether Theorem below) those than are close likewise exist only on very special curves. We may ask, accordingly, *what linear series exist on* all *curves of genus g?*

The first of these two issues is dealt with in a classical theorem of Castelnuovo:

Theorem 18.3 (Castelnuovo)**.** *Let $C \subset \mathbb{P}^r$ be an irreducible, nondegenerate curve of degree d and geometric genus g. Then*

$$g \leq \binom{m}{2}(r-1) + m\epsilon$$

where $d = m(r-1) + \epsilon + 1$ and $0 \leq \epsilon \leq r-2$.

Castelnuovo also shows that this bound is sharp; and using his analysis it's not hard to see that curves of all geometric genera between 0 and the bound do occur. Note that for given r the bound on g is asymptotically $d^2/2(r-1)$, so this is very much stronger than Clifford.

The second question raised above—what linear series exist on *every* curve of genus g—is the one addressed by Brill-Noether theory; we'll take up this question now.

18.1.2 Behavior on general curves: naive dimension counts leading to statement of Brill-Noether

As we indicated, the Brill-Noether Theorem answers the question, "for which values of d, g and r is it the case that every curve of genus g has a g^r_d? Rather than leave the reader in suspense, here's the answer:

Theorem 18.4. *If C is a general curve of genus g, the dimension of $W^r_d(C)$ is*

$$\rho(g, r, d) = g - (r+1)(g-d+r).$$

In particular, it's the case that every curve has a g^r_d if and only if $\rho \geq 0$.

** Maybe another chart here, with the RR, Clifford, Castelnuovo and Brill-Noether areas all drawn in **

There are various extensions of this theorem for general curves C of genus g: Fulton and Lazarsfeld show that if $\rho > 0$ then $W^r_d(C)$ is irreducible; Gieseker proves that the singular locus of $W^r_d(C)$ is exactly $W^{r+1}_d(C)$; and Griffiths and Harris show that if $r \geq 3$ then the general point of $W^r_d(C)$ corresponds to a very ample line bundle (that is, we actually do get a biregular embedding of C in $\mathbb{P}^r$ as a curve of degree d when $\rho \geq 0$). But

here we will focus only on the dimension estimate in Theorem 18.4, and will prove only half of that: the statement that for any curve C of genus g,

$$\dim W^r_d(C) \geq g - (r+1)(g-d+r).$$

Note some special cases of Theorem 18.4 (and the last of the extensions following):

• Any curve C of genus g is expressible as a d-sheeted cover of $\mathbb{P}^1$ with $d \leq \lfloor (g+1)/2 \rfloor + 1$.

Thus, elliptic curves and curves of genus 2 are double covers of the sphere $\mathbb{P}^1$; curves of genus 3 and 4 are either 2- or 3-sheeted covers of $\mathbb{P}^1$, and so on.

• A general curve of genus g is birational to a plane curve of degree $d = \lfloor (2g+2)/3 \rfloor + 2$.

Note that this applies only to a general curve; by Theorem 18.4, every curve of genus g will have a g^2_d, but it need not be birationally very ample on every curve.

• A general curve of genus g may be embedded in $\mathbb{P}^3$ as a curve of degree $d = \lfloor (3g+3)/4 \rfloor + 3$.

How you might be led to the Brill-Noether Theorem. There are numerous naive dimension counts that suggested to 19^{th} geometers (Brill and Noether among them) the statement of the Brill-Noether Theorem; we will mention one here.

Let $D = p_1 + \cdots + p_d$ be a divisor of degree d on a curve C, viewed as a point on the symmetric product C_d; for simplicity, let's take $d \leq g$. It's not hard to see that a general such divisor does not move in a positive-dimensional linear series; that is, $h^0(\mathcal{O}_C(D)) = 1$. We'll ask: how many conditions is it on D that $h^0(\mathcal{O}_C(D)) \geq r+1$?

Happily, Riemann-Roch provides at least the beginning of an answer to this question: it says that $h^0(\mathcal{O}_C(D)) = r+1$ exactly when $h^0(K_C(-D)) = g-d+r$, that is, *exactly when the points $p_1, \ldots, p_d$ of d fail by r to impose independent conditions on the space $H^0(K_C)$ of regular differentials on C.*

To express this condition, choose a basis $\omega_1, \ldots, \omega_g$ for the space $H^0(K_C)$, and choose a local coordinate z_i centered around $p_i \in C$. In terms of the coordinate z_i, we can identify the cotangent space to C at p_i with $\mathbb{C}$, and write $\omega(p_i)$ for the value of a differential ω at p_i; in other words, if

$$\omega = (a_0 + a_1 z + \ldots) dz_i$$

we'll write $\omega(p_i)$ for a_0.

Now consider the matrix

$$\Omega(D) = \begin{pmatrix} \omega_1(p_1) & \omega_2(p_1) & \dots & \omega_g(p_1) \\ \omega_1(p_2) & \omega_2(p_2) & \dots & \omega_g(p_2) \\ \vdots & \vdots & & \vdots \\ \omega_1(p_d) & \omega_2(p_d) & \dots & \omega_g(p_d) \end{pmatrix}.$$

What Riemann-Roch is telling us is that the divisor D will move in an r-dimensional linear series if and only if the rank of the matrix $\Omega(D)$ has rank $d-r$ or less. Now, if $U \subset C_d$ is a suitably small analytic neighborhood of the point D in C_d (say, the product of coordinate neighborhoods of the p_i), we may view $\Omega(D)$ as a matrix valued function on U; and the locus in U of divisors moving in an r-dimensional series is then the rank $d-r$ determinantal variety associated to this matrix function. We would expect, then, that the locus of such divisors would have dimension

$$d - r(g - d + r);$$

and since this locus maps to $W^r_d(C)$ with r-dimensional fibers, we might guess that

$$\begin{aligned} \dim W^r_d(C) &= d - r(g - d + r) - r \\ &= g - (r+1)(g - d + r) \end{aligned}$$

which is the Brill-Noether estimate.

18.1.3 Description of various methods of proof

There are many other dimension counts that lead to the same conclusion, but rather than go through them we'll talk about possible proofs of the theorem.

To begin with, there are two cases, $r = 1$ and 2, where special techniques can be used (and where are least part of the statement of the theorem was proved before the general one). In both cases, we have an a priori estimate of the dimension of the locus $\mathcal{G}^1_d$ of pairs $(C, \mathcal{D})$ where C is a curve of genus g and $\mathcal{D}$ a g^r_d.

For example, in case $r = 1$ we can construct the *Hurwitz scheme* $\mathcal{H}$ parametrizing branched covers $C \to \mathbb{P}^1$ of degree d and genus g. Such a cover will have a branch divisor $B \subset \mathbb{P}^1$ of degree $b = 2d + 2g - 2$; and for a given divisor $B \subset \mathbb{P}^1$ of this degree there will be a finite number of such covers. The scheme $\mathcal{H}$ thus has dimension $2d + 2g - 2$; subtracting 3 for automorphisms of $\mathbb{P}^1$, we conclude that the dimension of $\mathcal{G}^1_d$ is $2d + 2g - 5$. Finally, since the moduli space of smooth curves of genus g has dimension $3g - 3$, we might expect that the family of g^1_ds on a general curve of genus

g will have dimension

$$2d + 2g - 5 - (3g - 3) = 2d - g - 2 = \rho(g, 1, d).$$

Does this prove anything? It does in one direction: it shows that for a general curve C, the dimension of W^1_d is at most $\rho(g, 1, d)$, and in particular that when $\rho(g, 1, d) < 0$ a general curve is not expressible as a d-sheeted cover of $\mathbb{P}^1$. To prove the other direction, we would have to show that the map $\mathcal{H} \to M_g$ is dimensionally proper, for example by exhibiting a curve C with $\dim W^1_d(C) = \rho(g, 1, d)$ or by evaluating the rank of the differential of this map. Neither of these approaches, however, is easy to carry out; the former can be done (and has been done; see **limit linear series**) if we compactify both $\mathcal{H}$ and M_g to allow certain singular curves.

A similar dimension count can be made for plane curves, with similar results, but after that, this method fails: no one has an a priori idea of the dimension of the family of curves of degree d and genus g in $\mathbb{P}^r$ for $r \geq 3$.

Interestingly, there was also a proof of the existence half of Brill-Noether in the $r = 1$ case using the double point formula! This is based on the observation that, if there are no g^1_ds on a curve C, then the Abel-Jacobi map $\mu_d : C_d \to J(C)$ is actually an embedding. When $d \geq g/2$, however, we can apply the double point formula to the map μ_d and, finding that the class of the double point locus is nonzero when $d \geq (g + 2)/2$, deduce the existence of a g^1_d on an arbitrary curve.

This approach requires that we know the Chern classes of the tangent bundle to the symmetric product C_d of C, so it's not one that we can carry out without a fair amount of preparatory work. By the end of this Chapter, however, we will know enough about the symmetric products of a curve and their Chow rings (mod numerical equivalence, at any rate) to do this; the necessary calculations and the ul;timate proof of the theorem is carried out in Exercises **-** below. See also **Gunning** for an account of this proof.

discussion of actual proofs

Finally, we should say that the proof of the nonexistence half of Brill-Noether—the statement that for a general curve C, the dimension $\dim W^r_d(C) \leq \rho(g, r, d)$, and in particular that C possesses no g^r_ds when $\rho < 0$—requires in general very different ideas. One could prove it simply by exhibiting, for each g, r and d a smooth curve C of genus g with $\dim W^r_d(C) = \rho(g, r, d)$ (or with $W^r_d(C) = \varnothing$ if $\rho < 0$); interestingly, though, no one has ever succeeded in doing this. All existing proofs involve some variational element, and so are beyond the scope of this book and its techniques. We hope to describe a proof of this half, and so complete the proof of Theorem 18.4, in a later volume on specialization and deformation methods, if we and the reader live that long.

18.2 Chow rings of Jacobians and Symmetric Products of Curves

The central objects in our proof are the symmetric products C_d of a curve C and its Jacobian J, and it would seem that a necessary first step in carrying out any enumerative calculation on them would be to describe their Chow rings.

Unfortunately, these are among the spaces whose Chow groups (beyond codimension 1) seem to be essentially unknowable. To give one example of this, it's not hard to see that a cycle $Z \subset J$ will not in general be rationally equivalent to its translate $Z_a = Z + a$ by an element $a \in J$: for example, the group law on J gives a map from the space of effective 0-cycles on J to J, and since any map from $\mathbb{P}^1$ to J is constant, rationally equivalent 0-cycles must "add up" to the same sum in J. But in fact this merely scratches the surface of our ignorance: for any two points $a, b \in J$ and any cycle Z on J, we can ask if the sum

$$Z - Z_a - Z_b + Z_{a+b}$$

is rationally equivalent to 0. It is true for cycles in codimension 1, but beyond this the answer is unknown, even for 0-cycles. And our ignorance of when linear combinations of translates of cycles are rationally equivalent to zero is very much an issue here: most of the cycles whose classes we might hope to determine are in fact only defined after a choice of base point—in effect, only up to translation.

To get around this difficulty, we'll do something we have avoided up to now: we'll work not with the Chow rings, but rather with the topological cohomology groups of these spaces, given the analytic topology. More accurately, since we'll be concerned principally with the classes of their algebraic subvarieties, we'll work with the *algebraic cohomology rings* H^*_{alg} of these spaces, that is, the image in H^* of the Chow ring, under the map $H^* \to A^*$ given by associating to any subvariety its fundamental class.

We start with the Jacobian. Topologically, this is straightforward: as the quotient of a complex vector space $H^0(K_C)^* \cong \mathbb{C}^g$ by a lattice $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, the first homology group of $J(C)$ is naturally identified with that of the curve C:

$$H_1(J(C), \mathbb{Z}) = H_1(C, \mathbb{Z}).$$

The first cohomology $H^1(J(C), \mathbb{Z})$ is similarly identified with $H^1(C, \mathbb{Z})$. Note moreover that the Abel-Jacobi map $\mu : C \to J(C)$ induces these identifications. For future calculations, we choose a basis $\alpha_1, \ldots \alpha_g, \beta_1, \ldots, b_g$ for $H^1(C, \mathbb{Z})$, normalized so that the cup product is skew-diagonal: that is,

$$(\alpha_i \cup b_j)[C] = \begin{cases} 1, & \text{if } i = j; \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\alpha_i \cup \alpha_j = \beta_i \cup \beta_j = 0 \quad \text{for all } i, j$$

By abuse of notation, we will also use the symbols α_i and β_i to denote the corresponding cohomology classes in $H^1(J(C), \mathbb{Z})$.

Now, since topologically $J(C) \cong (S^1)^{2g}$ is a product of S^1s, by the Künneth formula we have

$$H^*(J(C), \mathbb{Z}) = \bigwedge^* H^1(J(C), \mathbb{Z}) = \bigwedge^* H^1(C, \mathbb{Z}).$$

(Note in particular that $H^*(J(C), \mathbb{Z})$ has no torsion; in particular, the notion of numerical equivalence coincides with homological equivalence for cycles in $J(C)$.) For a multiindex $I = (i_1, \ldots, i_k)$ with $i_1, \ldots i_k \in \{1, \ldots, g\}$, we will write α_I for the class

$$\alpha_I = \alpha_{i_1} \cup \cdots \cup \alpha_{i_k} \in H^k(J(C), \mathbb{Z})$$

and similarly

$$\beta_I = \beta_{i_1} \cup \cdots \cup \beta_{i_k} \in H^k(J(C), \mathbb{Z});$$

the classes

$$\{\alpha_I \cup \beta_J : I, J \subset \{1, \ldots, g\}\}$$

then form a basis for $H^*(J(C), \mathbb{Z})$. Note that the cup product in complementary dimension is given, for $I, J, K, L \subset \{1, \ldots, g\}$, by

$$((\alpha_I \cup \beta_J) \cup (\alpha_K \cup \beta_L))[J(C)] = \begin{cases} \pm 1, & \text{if } K = I^\circ \text{ and } L = J^\circ; \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{18.3}$$

Note that

$$(\alpha_1 \cup \beta_1 \cup \alpha_2 \cup \beta_2 \cup \cdots \cup \alpha_g \cup b_g)[J(C)] = +1;$$

the sign of all other products above is determined accordingly. Of special interest are the classes $\eta_i \in H^2(J(C), \mathbb{Z})$ defined as

$$\eta_i = \alpha_i \cup \beta_i \quad \text{for } i = 1, \ldots, g.$$

For a multiindex $I = (i_1, \ldots, i_k)$ with $i_1, \ldots i_k \in \{1, \ldots, g\}$, we will write η_I for the class

$$\eta_I = \eta_{i_1} \cup \cdots \cup \eta_{i_k} \in H^{2k}(J(C), \mathbb{Z}).$$

Again, the cup product in complementary dimension is simple: we have for $I, J \subset \{1, \ldots, g\}$

$$(\eta_I \cup \eta_J)[J(C)] = \begin{cases} 1, & \text{if } J = I^\circ; \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{18.4}$$

18.2.1 Poincaré's formula

With the notation established (at least until we get to the symmetric products!), let's focus on the objects of interest to us: the classes of algebraic subvarieties. Most basic among these are the subvarieties $W_d \subset J(C)$ parametrizing divisor effective classes of degree d, that is, the image of the map $\mu_d : C_d \to J(C)$. (Note that the map μ depends on a choice of base point $p_0 \in C$, and so the subvarieties $W_d \subset J(C)$ are really only defined up to translation, but their classes in $H^*(J(C))$ are well-defined.) Our basic result is the

Proposition 18.5.

$$[W_d] = \sum_{\substack{I \subset \{1,\dots,g\} \\ |I| = g-d}} \eta_I \ \in \ H^{2g-2d}(J(C), \mathbb{Z}).$$

Note in particular that the divisor $W_{g-1} \subset J(C)$ has class

$$[W_{g-1}] = \eta_1 + \cdots + \eta_g.$$

This class occurs often, and we will denote it simply by θ. In these terms, we can write the formula above as

$$[W_d] = \frac{\theta^{g-d}}{(g-d)!}$$

or, if you like,

$$[W_d] = \frac{[W_{g-1}]^{g-d}}{(g-d)!}.$$

In this form, it is called *Poincaré's formula*. Note that this last formulation makes sense in the ring $N^*(J(C))$ of cycles on $J(C)$ mod numerical equivalence—we don't need to introduce the topological cohomology of $J(C)$ to state it—and indeed it is valid for curves and their Jacobians over arbitrary fields. We don't know if there is an analogous formula in a finer cycle theory. **ref, if there is one!**

Proof. In order to prove Proposition 18.5, we will take the product of both sides of the formula with an arbitrary element $\alpha_I \cup \beta_J$ of our basis for the cohomology $H^{2d}(J(C), \mathbb{Z})$ of $J(C)$ in complementary dimension, evaluate on the fundamental class of $J(C)$ and show that they are the same; by Poincaré duality, this will suffice to prove the formula. In view of the formulas 18.3 and 18.4 above, we need to prove that

(18.5)

$$(\mu^*(\alpha_I \cup \beta_J))[C_d] = \begin{cases} (-1)^{\binom{d}{2}}, & \text{if } I = J \text{ (in which case } \alpha_I \cup \beta_J = (-1)^{\binom{d}{2}}\eta_I\text{); and} \\ 0 & \text{otherwise} \end{cases}$$

Now let $\pi : C^d \to C_d$ be the quotient map from the ordinary d-fold product of C with itself to the symmetric product C_d, and let $\nu = \mu \circ \pi : C^d \to J(C)$ be the composition. Since

$$\pi_*[C^d] = d! \cdot [C_d],$$

(18.5) is equivalent to

$$(\nu^*(\alpha_I \cup \beta_J))[C^d] = \begin{cases} (-1)^{\binom{d}{2}} d!, & \text{if } I = J; \text{ and} \\ 0 & \text{otherwise;} \end{cases} \tag{18.6}$$

that is,

$$(\nu^*(\alpha_I \cup \beta_J))[C^d] = 0 \quad \text{if } I \neq J;$$

and

$$(\nu^* \eta_I)[C^d] = 1 \quad \text{for all } I.$$

In fact, this is straightforward, but we need (alas!) more notation, specifically for cohomology classes on the products C^d. □

18.2.2 Symmetric products as projective bundles

Consider now the Abel-Jacobi map

$$\mu = \mu_d : C_d \to J(C)$$

from the symmetric products of C to its Jacobian, for $d \geq 2g - 1$. By Abel's Theorem, the fibers of this map are just the complete linear systems of degree d on the curve C; by Riemann-Roch, they are all projective spaces of the same dimension $d - g$.

In fact, we can say more. Recall first that we have identifications

$$T^*_{\mu(D)} J = H^0(K_C)$$

of the cotangent space to the Jacobian at any point with the space of regular differentials on C, and

$$T^*_D C_d = \oplus T^*_{p_i} C$$

of the cotangent space to the symmetric product C_d at a point $D = \sum p_i$ with the p_i distinct with the direct sum of the cotangent spaces to C at the points p_i. Now, the Abel-Jacobi map is given by the abelian integrals

$$D = \sum p_i \mapsto \sum \int_{p_0}^{p_i};$$

differentiating with respect to the points p_i, we see that in terms of these identifications, the differential $d\mu_D$ of μ at a point $D \in C_d$ is given simply

as the transpose of the evaluation map

$$H^0(K_C) \to \oplus T^*_{p_i} C$$
$$\omega \mapsto \big(\omega(p_1), \ldots, \omega(p_d)\big);$$

in particular, *the cokernel of the differential $d\mu_D$ of μ at a point $D \in C_d$ is the annihilator of the subspace $H^0(K_C(-D)) \subset H^0(K_C)$ of differentials vanishing along D*. Since we are working in the range $d \geq 2g-1$, there are no such differentials; we conclude that the fibers of the Abel-Jacobi map are generically reduced, and hence (since they're Cohen-Macaulay) reduced. It follows that the map μ expresses the symmetric product C_d as a projective bundle over J in the analytic topology.

Two remarks here before moving on. First, the identification $T^*_D C_d = \oplus T^*_{p_i} C$ above can be extended over all of C_d (not just the open subset of reduced divisors) to an identification

$$T^*_D C_d = H^0(K_C / K_C(-D)),$$

and in these terms the differential of μ is again the transpose of the evaluation map. Thus we can see directly that μ is a submersion in case $d \geq 2g-1$; we don't need to invoke the Cohen-Macaulayness of the fibers of μ to deduce that these fibers are reduced. Secondly, we should point out that even in case $d \leq 2g-2$, Riemann-Roch tells us that the dimension of the kernel of the differential $d\mu$ at any point D—that is, the dimension of the cokernel of the evaluation map $H^0(K_C) \to H^0(K_C/K_C(-D))$—is exactly the dimension $r(D)$ of the fiber of μ through D; thus the fibers of μ are smooth in this case as well.

Now, recall our discussion in Chapter ** about projective bundles. We said there that if $f : X \to Y$ is any projective bundle in the analytic topology, then it's linearizable—that is, the projectivization of a vector bundle on Y, or equivalently a projective bundle in the Zariski topology—if and only if X has a line bundle L whose restriction to the fibers of f is $\mathcal{O}(1)$; and moreover the set of vector bundles E on Y with $X = \mathbb{P}E$ is naturally in one-to-one correspondence with the set of such line bundles on X. Invoking just the first statement, we see that $C_d \to J$ is indeed the projectivization of a vector bundle: for any point $p \in C$, the divisor

$$X_p = \{D : D - p \geq 0\} = C_{d-1} + p \subset C_d$$

in fact intersects the general fiber of μ in a hyperplane.

We can now the proceed in one of two ways to describe the relevant parts of the algebraic cohomology rings of C_d and J, and the maps between them induced by μ:

(a) The hard way. We can in fact construct a vector bundle E of rank $d-g+1$ on J whose fiber at each point $L \in J$ may be identified with the space $H^0(L)$ of sections of L, and whose projectivization will

correspondingly be the symmetric product C_d. We can realize E as the direct image of a line bundle $\mathcal{L}$ on $J \times C$, called the *Poincaré bundle*, whose restriction to the fiber $\{L\} \times C$ over each point $L \in J$ is the line bundle L. With a fair amount of effort, we can then use Grothendieck-Riemann-Roch to determine the Chern classes of E, which by ** gives us what we want.

(b) The easy way. We can avoid all this by getting cute: we *define* the bundle E to be the vector bundle whose projectivization is C_d, and such that the tautological bundle $\mathcal{O}_{\mathbb{P}E}(1)$ is the line bundle $\mathcal{O}_{C_d}(X_p)$ associated to the divisor X_p. We can then calculate the Segre classes of E as the pushforwards of powers of the class $[X_p] \in H^2_{\text{alg}}(C_d)$, and use these in turn to find the Chern class of E.

As you might expect, doing this the hard way yields a great deal more information, which can then be used to answer other questions about the geometry of linear series on curves. But this is all worked out in detail in Chapter 8 of [ACGH]; and, given that we are now 16 pages into this Chapter, doing things the easy way seems the better part of valor.

18.2.3 Chern classes of tautological bundles on Jacobians

So: as suggested above, we'll define the vector bundle E to be the vector bundle whose projectivization is C_d, and such that the tautological bundle $\mathcal{O}_{\mathbb{P}E}(1)$ is the line bundle $\mathcal{O}_{C_d}(X_p)$ associated to the divisor X_p. Concretely, this is the dual of the direct image $\mu_*(\mathcal{O}_{C_d}(X_p))$ of the line bundle on C_d associated to the divisor $X_p \subset C_d$, though we can't use that description to find its Chern classes without already knowing the relevant part of the cohomology ring of C_d. Instead, we'll calculate the Segre classes of E, which by ** are given as the pushforwards of powers of the class $\zeta = [X_p] \in H^2(C_d)$:

$$s_k(E) = \mu_*(\zeta^{k+d-g}).$$

Now, since we're working in $H^*(C_d)$ rather than $A^*(C_d)$, the class ζ is also the class of the divisor $X_q = C_{d-1} + q \subset C_d$ for any point $q \in C$. To represent the class ζ^{k+d-g}, then, we can just choose distinct points $p_1, \ldots, p_{k+d-g} \in C$ and consider the intersection

$$\begin{aligned} \bigcap_i X_{p_i} &= \{D \in C_d : D - p_i \geq 0 \ \forall i\} \\ &= C_{g-k} + E \ \subset \ C_d \end{aligned}$$

where $E = p_1 + \cdots + p_{k+d-g}$. This intersection is generically transverse—it is visibly transverse at a point $E + D'$ where D' consists of $g - k$ distinct points distinct from $p_1, \ldots, p_{k+d-g}$, and no component of the intersection

is contained in the complement of this locus—and so we have

$$\zeta^{k+d-g} = \big[C_{g-k} + E\big] \ \in \ H^{2k+2d-2g}(C_d).$$

We have then

$$\begin{aligned} s_k(E) &= \mu_* \big[C_{g-k} + E\big] \\ &= \big[W_{g-k}\big] \ \in \ H^{2k}(J). \end{aligned}$$

Applying Poincaré's formula, this yields

$$s_k(E) = \frac{\theta^k}{k!};$$

or, to express this compactly,

$$s(E) = e^{\theta}.$$

Apart from its compactness, this expression allows us to write down the Chern classes readily: it follows that

$$c(E) = \frac{1}{s(E)} = e^{-\theta}$$

or in other words,

$$c_k(E) = (-1)^k \frac{\theta^k}{k!}.$$

Finally, this allows us to give a particularly useful description of the cohomology ring of C_d: by **, we have

$$H^*(C_d) = H^*(J)[\zeta] / \big(\zeta^{d-g+1} - \theta\zeta^{d-g} + \cdots + \frac{\theta^2}{2}\zeta^{d-g-1}\cdots\big).$$

18.3 Application of Porteous; calculation of class of W_d^r

18.3.1 Construction of the Poincaré bundle and the bundle E

To set things up, we need to construct two bundles: one, for any d, a line bundle $\mathcal{L}$ on the product $J \times C$, called the *Poincaré bundle*, whose restriction to the fiber over $\{L\} \times C$ is isomorphic to L; and the other, for $d \geq 2g-1$, a vector bundle E of rank $d-g+1$ on J whose fiber over a point $\{L\}$ is naturally the space of sections $H^0(L)$.

It's important to recognize at the outset that neither of these conditions uniquely determines a bundle. For the first, if $\mathcal{L}$ is any bundle on $J \times C$ whose restriction to the fiber over $\{L\} \times C$ is isomorphic to L, then so is the tensor product of $\mathcal{L}$ with the pullback of any line bundle on J. For the

second, the condition that the fiber of E over a point $\{L\}$ is naturally the space of sections $H^0(L)$ is a priori meaningless; we can interpret it to mean in particular that the projectivization $\mathbb{P}E$ is isomorphic to the symmetric product C_d, but again this only determines E up to tensor product with a line bundle on J.

We can resolve both ambiguities with one normalization. To begin with, if we fix a base point $p_0 \in C$, we can normalize the Poincaré bundle $\mathcal{L}$ by requiring that the restriction of $\mathcal{L}$ to the cross-section $J \times \{p_0\}$ be trivial. **ref to theorem of the square, or just prove it here?** Then we can simply define the bundle E to be the direct image of $\mathcal{L}$ under the projection $J \times C \to J$.

To construct the Poincaré bundle CL, we first let $\mathcal{D} \subset C_d \times C$ be the tautological divisor of degree d: that is, the locus

$$\mathcal{D} = \{(D, p) : D - p \geq 0\}.$$

This is called the tautological divisor because its restriction to each fiber $\{D\} \times C$ of $C_d \times C$ over C_d is just the divisor D. Note also that its restriction to each fiber $C_d \times \{p\}$ of $C_d \times C$ over C is just the divisor $X_p = C_{d-1} + p \subset C_d$ introduced above.

Now, with $p_0 \in C$ a chosen base point, consider the divisor $\mathcal{D}_0 = X_{p_0} \times C \subset C_d \times C$

18.3.2 Calculation of the class of W_d^r

Now that we have the information we need about the algebraic cohomology of the Jacobian and the symmetric product, the bundles $\mathcal{L}$ and E and their Chern classes, we can proceed to the computation of the classes of the loci we're interested in.

To start with, we can use Porteous' formula to calculate the class of W_d^r as follows. To begin with, fix a divisor $D = p_1 + \cdots + p_m$ of degree $m \geq 2g - 1 - d$ on C (we'll choose the points p_i distinct for convenience, but this isn't strictly necessary); set $n = m + d$. Choosing D gives us an identification of Pic^d with Pic^n; what we'll now is actually describe the locus $W_d^r + D \subset \mathrm{Pic}^n$.

On the product $\mathrm{Pic}^n \times C$, consider the restriction map

$$\mathcal{L} \to \mathcal{L}|_\Gamma$$

where

$$\Gamma = \mathrm{Pic}^n \times D = \bigcup_{i=1}^{m} (\mathrm{Pic}^n \times \{p_i\})$$

is the union of the horizontal sections of $\mathrm{Pic}^n \times C$ over Pic^n corresponding to the points p_i. Taking the direct image of this map under the projection $\mathrm{Pic}^n \times C \to \mathrm{Pic}^n$, we have a map of vector bundles

$$\rho : E \to F = \bigoplus_{i=1}^{m} L_i,$$

where L_i is the restriction of $\mathcal{L}$ to the cross-section $\mathrm{Pic}^n \times \{p_i\}$. For each point $L \in \mathrm{Pic}^n$, this is the map

$$E_L = H^0(L) \to \bigoplus L_{p_i}$$

obtained by evaluation sections of L at the points p_i. In particular, the kernel of this map is the vector space $H^0(L(-D)) \subset H^0(L)$ of sections vanishing along D; so that *the locus $W^r_d + D \subset \mathrm{Pic}^n$ is just the locus where the map ρ has rank $n - g - r$ or less.*

That said, we can use Porteous' formula to calculate the class of this locus. We have already worked out the Chern classes of E. As for F, the line bundles L_{p_i} can all be deformed to the bundle $L_{p_0} = \mathcal{L}|_{\mathrm{Pic}^n \times \{p_0\}}$, which is trivial by our normalization of $\mathcal{L}$. The Chern classes $c_1(L_{p_i})$ are thus all algebraic equivalent to 0, hence 0 mod numerical equivalence, so that

$$c\,(F) = 0 \ \in \ A^*_{\mathrm{num}}(\mathrm{Pic}^n).$$

We have then

$$\frac{c(F)}{c(E)} = \frac{1}{e^{-\theta}} = e^{\theta}$$

and so Porteous' formula tells us that if W^r_d has pure dimension $\rho = g - (r+1)(g-d+r)$, its class is given as the determinant

$$\begin{aligned}[W^r_d] &= \begin{vmatrix} \frac{\theta^{g-d+r}}{(g-d+r)!} & \frac{\theta^{g-d+r+1}}{(g-d+r+1)!} & \cdots & \frac{\theta^{g-d+2r}}{(g-d+2r)!} \\ \frac{\theta^{g-d+r-1}}{(g-d+r-1)!} & \frac{\theta^{g-d+r}}{(g-d+r)!} & \cdots & \frac{\theta^{g-d+2r-1}}{(g-d+2r-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\theta^{g-d}}{(g-d)!} & \frac{\theta^{g-d+1}}{(g-d+1)!} & \cdots & \frac{\theta^{g-d+r}}{(g-d+r)!} \end{vmatrix} \\ &= D_{a,r} \cdot \theta^{(r+1)(g-d+r)} \end{aligned}$$

where $D_{a,r}$ is the $(r+1) \times (r+1)$ determinant

$$D_{a,r} = \begin{vmatrix} \frac{1}{a!} & \frac{1}{(a+1)!} & \cdots & \frac{1}{(a+r)!} \\ \frac{1}{(a-1)!} & \frac{1}{a!} & \cdots & \frac{1}{(a+r-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(a-r)!} & \frac{1}{(a-r+1)!} & \cdots & \frac{1}{a!} \end{vmatrix}$$

It remains to evaluate this last determinant. To do this, we clear denominators by multiplying the first column by $a!$, the second column by $(a+1)!$, and so on; we arrive at the expression

$$D_{a,r} = \prod_{i=0}^{r} \frac{1}{(a+i)!} \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ a & a+1 & \dots & a+r \\ a(a-1) & (a+1)a & \dots & (a+r)(a+r-1) \\ a(a-1)(a-2) & (a+1)a(a-1) & \dots & (a+r)(a+r-1)(a+r-2) \\ \vdots & \vdots & \ddots & \vdots \\ a\cdots(a-r+1) & (a+1)\cdots(a-r+2) & \dots & (a+r)\cdots a \end{vmatrix}.$$

We now make a series of row operations to this matrix: first, we add the second row to the third, so that the third row is

$$a^2, (a+1)^2, \dots, (a+r)^2.$$

Next, we add 3 times the third row to the fourth, and subtract 2 times the second from it, so that the fourth row now looks like

$$a^3, (a+1)^3, \dots, (a+r)^3.$$

Continuing in the way, we arrive at the van der Monde determinant

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a & a+1 & \dots & a+r \\ a^2 & (a+1)^2 & \dots & (a+r)^2 \\ \vdots & \vdots & \ddots & \vdots \\ a^r & (a+1)^r & \dots & (a+r)^r \end{vmatrix}$$

whose determinant is the product of the pairwise differences of the numbers $a, a+1, \dots, a+r$; that is,

$$\prod_{0 \le i < j \le r} (j-i) = \prod_{i=0}^{r} i!.$$

In sum, then, we have

$$D_{a,r} = \prod_{i=1}^{r} \frac{i!}{(a+i)!}$$

and correspondingly

$$[W_d^r] = \prod_{i=1}^{r} \frac{i!}{(g-d+r+i)!} \theta^{(r+1)(g-d+r)}$$

whenever W^r_d has pure dimension ρ. The key qualitative fact is simply that *this class is nonzero if* $\rho \geq 0$; thus W^r_d must be nonempty and we have established the existence half of Brill-Noether, that is,

Theorem 18.6. *If* $\rho = g-(r+1)(g-d+r) \geq 0$*, then every smooth curve of genus* g *has a linear system of degree* d *and dimension* r*.*

19

Families of Curves

Keynote Questions

(a) ????

For the most part, the varieties X whose intersection theory we've worked out in the book have been readily described: hypersurfaces and complete intersections, projective bundles, blow-ups and the like. In this final Chapter, we're going to encounter a new situation: we will be doing intersection theory on a variety whose very existence is nontrivial to establish, with the aim of deriving qualitative results about the geometry of that variety. We start, however, with a relatively straightforward application of the Grothendieck Riemann Roch formula to families of curves.

19.1 Invariants of families of curves

We've said (and illustrated) that much of the time that Grothendieck-Riemann-Roch is applicable, there are ad hoc methods for calculating the Chern classes of a pushforward that are much simpler and faster. But there still remains a fraction of such cases where no other method exists, and in these cases GRR is invaluable. We open this chapter with two examples of this type.

We are considering one-parameter families of curves of genus g; that is, a morphism $f : X \to B$ from a surface X to a curve B; for simplicity, we'll assume that X and B are smooth, and that the fibers X_b of f are irreducible with at worst nodes as singularities. The question we have is, *How can we measure how much the isomorphism class of X_b is varying with $b \in B$?*

There are, it seems to us, three natural ways of quantifying the nontriviality of the family:

(a) *The number of singular fibers.* A singular fiber is not isomorphic to a smooth fiber; therefore, the presence of singular fibers is a reflection of variation in the isomorphism class of X_b.[1] Thus, we can associate to a family $f : X \to B$ as above the number $\delta(f)$ of nodes appearing in the fibers of f.

(b) *The degree of the Hodge bundle.* To every smooth fiber X_b we can associate the g-dimensional vector space $H^0(K_{X_b})$, and over the open set $U \subset B$ where the fibers are smooth these vector spaces fit together into a vector bundle, called the *Hodge bundle* E of f. This bundle has a natural extension to all of B, described in the following paragraph. If the family were the trivial family $B \times C \to B$, this would be the trivial bundle; so we can think of the twisting of the Hodge bundle, as measured by its first Chern class, as a reflection of the nontriviality of the family. Accordingly we take our second invariant of the family to be

$$\lambda(f) = c_1(E).$$

To extend the Hodge bundle over all of B, we have to introduce what's called the *relative dualizing sheaf* $\omega_{X/B}$ of the family. This has many characterizations, which are discussed in more detail in **Harris-Morrison**; for our purposes, though, it will suffice to define $\omega_{X/B}$ to be the unique line bundle on X whose restriction to the smooth locus X° of f—that is, the complement of the finite set of nodes of fibers of f—is the vertical cotangent bundle. (Any line bundle on an open subset U of a smooth variety X can be extended to all of X; if the complement $X \setminus U$ has codimension 2 or more in X, this extension is unique.) More concretely, we have

$$\omega_{X/B} = K_X \otimes f^* K_B^{-1},$$

[1] This would not be true if we did not assume that the fibers were irreducible: you could take a trivial family $B \times C \to B$ and blow up a point in $B \times C$ to arrive at a family of curves of constant modulus with one singular fiber. The logic would be valid if we assumed all fibers to be *stable curves*; in the present context we avoid getting into a discussion of this notion by assuming all fibers irreducible.

since the sheaf on the right is locally free of rank 1, and agrees with the vertical cotangent bundle on X°. Having defined the relative dualizing sheaf of f, we can then define the Hodge bundle E to be its direct image

$$E = f_*(\omega_{X/B}).$$

(c) For our third invariant of f, we reason (if that's the word) as follows: again, if f were the trivial family $X = B \times C \to B$, the relative cotangent bundle $\omega_{X/B}$ would be simply the pullback of K_C via projection on the second factor; in particular, its self-intersection would be 0. We may thus think of this self-intersection as measuring the variation in the family, and accordingly we define our third invariant to be

$$\kappa(f) = \big(c_1(\omega_{X/B})\big)^2$$

We thus have three invariants of a family of curves as above, all of which may be thought of as measuring the variation in the family $\{X_b\}$ of curves. The question we ask now is, *what relations (if any) hold in general among these three?*

19.1.1 Examples

Before we launch into the answer, let's consider some examples in genus 3, to get an idea of what we might expect.

Pencils of quartics in the plane. Let's start with a family whose geometry we've already studied in earlier chapters: a general pencil $\{C_t = V(t_0F + t_1G) \subset \mathbb{P}^2\}$ of plane quartic curves. In this case, the total space of the family is the locus

$$X = \{(t, p) : p \in C_t\} \subset \mathbb{P}^1 \times \mathbb{P}^2.$$

We observe that X is smooth—the projection $\beta : X \to \mathbb{P}^2$ on the second factor expresses X as the blow-up of the plane at the 16 base points $F = G = 0$ of the pencil. Moreover, we've already seen that every fiber C_t of $\alpha : X \to \mathbb{P}^1$ is either smooth, or is irreducible with a single node; thus the family $\alpha : X \to \mathbb{P}^1$ satisfies the conditions of our discussion.

What are the invariants δ, λ and κ of f? Well, one we've already calculated in Chapter 7: the number of nodes in the fibers of f is

$$\delta(\alpha) = 27.$$

As for κ, this is a straightforward calculation in the Chow ring of X. To set it up, let $L \in A^1(X)$ be the pullback of the class of a line in $\mathbb{P}^2$, and let $E \in A^1(X)$ be the sum of the classes of the 16 exceptional divisors of β.

In these terms, the class of the fibers C_t of the projection α—that is, the pullback under α of the class η of a point in $\mathbb{P}^1$—is given by

$$[C_t] = \alpha^*(\eta) = 4L - E.$$

In particular, this implies that

$$\alpha^* K_{\mathbb{P}^1} = -2\alpha^*(\eta) = -8L + 2E.$$

On the other hand, from the blow-up map $\beta : X \to \mathbb{P}^2$ we see that

$$K_X = \beta^*(K_{\mathbb{P}^2}) + E = -3L + E.$$

Thus, the class of the relative dualizing sheaf $\omega_{X/\mathbb{P}^1}$ is

$$c_1(\omega_{X/\mathbb{P}^1}) = K_X - \alpha^* K_{\mathbb{P}^1} = 5L - E$$

and the invariant $\kappa(\alpha)$ is given by

$$\kappa(\alpha) = \big(c_1(\omega_{X/\mathbb{P}^1})\big)^2 = 25 - 16 = 9.$$

Next, the Hodge bundle can be described explicitly: since $5L - E = (4L - E) + L$, we can write the dualizing sheaf as

$$\begin{aligned} \omega_{X/\mathbb{P}^1} &= \mathcal{O}_X(5L - E) \\ &= \alpha^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \beta^*\mathcal{O}_{\mathbb{P}^2}(1). \end{aligned}$$

Now, the pushforward $\alpha_*(\beta^*\mathcal{O}_{\mathbb{P}^2}(1))$ is just the trivial bundle on $\mathbb{P}^1$ with fiber $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$; we have then

$$\begin{aligned} \alpha_*\big(\omega_{X/\mathbb{P}^1}\big) &= \alpha_*\big(\alpha^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \beta^*\mathcal{O}_{\mathbb{P}^2}(1)\big) \\ &= \mathcal{O}_{\mathbb{P}^1}(1) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \\ &\cong \mathcal{O}_{\mathbb{P}^1}(1)^3, \end{aligned}$$

and correspondingly

$$\lambda(\alpha) = c_1\big(\alpha_*(\omega_{X/\mathbb{P}^1})\big) = 3.$$

****David: if the above seems like trickery, here's a concrete version; feel free to delete either one****

To describe the Hodge bundle of our family, choose affine coordinates t on $\mathbb{A}^1 \subset \mathbb{P}^1$ and (x, y) on $\mathbb{A}^2 \subset \mathbb{P}^2$. We write the equation of C_t as $f_t(x, y)$, where f_t is a quartic polynomial in x and y whose coefficients are linear in t. In these terms, we can write down explicitly a basis for the space $H^0(K_{C_t})$ of regular differentials on C_t: we observe first that in the finite plane, the differential

$$\varphi_t = \frac{dx}{\frac{\partial}{\partial y} f_t(x, y)}$$

is regular and nowhere zero on the smooth locus of $C_t \cap \mathbb{A}^1$: since

$$df_t = \frac{\partial f_t}{\partial x}dx + \frac{\partial f_t}{\partial y}dy \equiv 0 \text{ on } C$$

and at a smooth point of C_t the partials $\frac{\partial f_t}{\partial x}$ and $\frac{\partial f_t}{\partial y}$ don't simultaneously vanish, the numerator vanishes to exactly the same order as the denominator. Moreover, changing coordinates on $\mathbb{P}^2$ we see that the differential φ will have zeroes at the points of intersection of C_t with the line $L_\infty = \mathbb{P}^2 \setminus \mathbb{A}^2$ at infinity; so that the differentials $x\varphi_t$ and $y\varphi_t$ are also regular on C_t. Finally, since φ_t, $x\varphi_t$ and $y\varphi_t$ are regular sections of the relative dualizing sheaf $\omega_{X/\mathbb{P}^1}$ on the smooth locus of α over $\mathbb{A}^1$, they extend to regular sections of $\omega_{X/\mathbb{P}^1}$ on all of $X_{\mathbb{A}^1}$. In sum, on $\mathbb{A}^1$, *the sections*

$$\varphi,\ x\varphi \ \textit{and}\ \ y\varphi \in \alpha_*(\omega_{X/\mathbb{P}^1})(\mathbb{A}^1)$$

give a trivialization of $\omega_{X/\mathbb{P}^1}$ *over* $\mathbb{A}^1$.

What happens over $t = \infty$? If we look again at the expression above for φ_t, we see that since the denominator $\frac{\partial f_t(x,y)}{\partial y}$ is linear in t, the section φ of $\alpha_*(\omega_{X/\mathbb{P}^1})$ extends to a regular section of $\alpha_*(\omega_{X/\mathbb{P}^1})$ over all of $\mathbb{P}^1$, having a simple zero at $t = \infty$, and the same is true of $x\varphi$ and $y\varphi$ as well. Moreover, making a change of coordinates on $\mathbb{P}^1$, we see that the multiples $t\varphi$, $tx\varphi$ and $ty\varphi$ form a frame for the bundle $\alpha_*(\omega_{X/\mathbb{P}^1})$ over $\mathbb{P}^1 \setminus \{0\}$; so that we have

$$\alpha_*(\omega_{X/\mathbb{P}^1}) \cong \mathcal{O}_{\mathbb{P}^1}(1)^3$$

and correspondingly

$$\lambda(\alpha) = c_1\big(\alpha_*(\omega_{X/\mathbb{P}^1})\big)n = 3.$$

To summarize, we have

$$\delta(\alpha) = 27, \quad \kappa(\alpha) = 9 \quad \text{and} \quad \lambda(\alpha) = 3.$$

Pencils of plane sections of a quartic surface. This is very similar to the preceding example; we'll just sketch the calculation. To set up, let $S \subset \mathbb{P}^3$ be a general quartic surface, and $\{H_t \subset \mathbb{P}^3\}_{t\in\mathbb{P}^1}$ a general pencil of planes; we let $C_t = S \cap H_t$. As we've seen **ref**, the curves C_t are all either smooth or irreducible with a single node; as we've also seen **ref**, the number of singular elements of the pencil $\{C_t\}$ is the degree of the dual surface $S^* \subset \mathbb{P}^{3*}$, which is

$$\delta = 36.$$

To calculate the other invariants, note first that the total space of our family is

$$X = \{(t, p) : p \in C_t\} \subset \mathbb{P}^1 \times S.$$

Via the projection $\beta : X \to S$, this is the blow-up of S at the four base points of the pencil. We'll let H and $E \in A^1(X)$ be the pullback of the hyperplane class on S and the sum of the classes of the exceptional divisors, respectively; we note that

$$H^2 = 4, \quad HE = 0 \quad \text{and} \quad E^2 = -4.$$

Since the canonical bundle K_S is trivial by adjunction, we have

$$K_X = E;$$

and since the fibers C_t of the projection map $\alpha : X \to \mathbb{P}^1$ on the first factor have class $H - E$, we have

$$c_1(\omega_{X/\mathbb{P}^1}) = K_X - f^* K_{\mathbb{P}^1} = E - (-2(H - E)) = 2H - E.$$

It follows that

$$\kappa(f) = (2H - E)^2 = 12.$$

To determine $\lambda(f)$, we use a variant of the method used above in the case of a pencil of plane quartics. Writing $2H - E = (H - E) + H$, we have

$$\begin{aligned} \omega_{X/\mathbb{P}^1} &= \mathcal{O}_X(2H - E) \\ &= \alpha^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \beta^* \mathcal{O}_S(1). \end{aligned}$$

Now, if $\mathbb{P}^3$ is the projectivization of a four-dimensional vector space V, and we think of the pencil of planes $\{H_t\}$ as a line $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{3*}$ in the dual projective space $\mathbb{P}^{3*} = \mathbb{P}(V^*)$, the pushforward $\alpha_*(\beta^* \mathcal{O}_S(1))$ is just the restriction to $\mathbb{P}^1$ of the universal quotient bundle Q. As we've seen many times, the first Chern class of the Q is the hyperplane class, whose restriction to a line $\mathbb{P}^1 \subset \mathbb{P}^{3*}$ has degree 1; it follows that

$$\begin{aligned} \alpha_*\big(\omega_{X/\mathbb{P}^1} &= \mathcal{O}_X(2H - E)\big) \\ &= \mathcal{O}_{\mathbb{P}^1}(1) \otimes \alpha_*\big(\beta^* \mathcal{O}_S(1)\big) \\ &= \mathcal{O}_{\mathbb{P}^1}(1) \otimes Q. \end{aligned}$$

Finally, applying our formula for the Chern class of a tensor product, we have

$$\lambda(\alpha) = c_1(\alpha_*\big(\omega_{X/\mathbb{P}^1}) = 4.$$

In sum, then, we have

$$\delta(\alpha) = 36, \quad \kappa(\alpha) = 12 \quad \text{and} \quad \lambda(\alpha) = 4.$$

Families of hyperelliptic curves. Finally, for a change let's consider a family of curves of genus 3 all of which are hyperelliptic. Such families are easy to write down: a hyperelliptic curve of genus 3 has equation

$$y^2 = f(x),$$

with f of degree 8; and we can vary the coefficients of f any way we like—for example, we can take them to be quadratic polynomials of a parameter $t \in \mathbb{P}^1$. In modern language, we are taking the double cover $X \to \mathbb{P}^1 \times \mathbb{P}^1$ branched over a general curve $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 8)$, and considering the composition

$$\eta : X \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$$

with the projection on the first factor. We note that X is smooth if B is; and if B has no more than simple tangency with any fiber of $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{P}^1$ the singularities of the fibers of η will be no worse than nodes; both conditions are satisfied for B general.

The numerical invariants of the family $\eta : X \to \mathbb{P}^1$ can be calculated readily. To begin with, the number of singular fibers is just the number of branch points of the projection $B \to \mathbb{P}^1$ on the first factor. By adjunction, the genus of B is

$$g(B) = (2-1)(8-1) = 7,$$

so by Riemann-Hurwitz the number of branch points of the 8-sheeted cover $B \to \mathbb{P}^1$ is

$$\delta = 2d + 2g - 2 = 28.$$

Alternatively, if we think of the defining equation $f(t, x)$ as an octic polynomial in x whose coefficients are quadratic polynomials in t, and recall that the discriminant of a polynomial of degree n is a polynomial of degree $2n - 2$ in its coefficients, we see that the discriminant of $f(t, x)$ is a polynomial of degree $2 \cdot 14 = 28$ in t.

To evaluate $\kappa(\eta)$, let E_0 and $F_0 \in A^1(\mathbb{P}^1 \times \mathbb{P}^1)$ be the classes of the fibers of the projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, and E and $F \in A^1(X)$ their pullbacks to X, so that

$$E^2 = F^2 = 0 \quad \text{and} \quad EF = 2.$$

Let $R \in A^1(X)$ be the class of the ramification divisor; note that since the preimage of the divisor $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ is $2R$, and $B \sim 2E_0 + 8F_0$, we have

$$2R = 2E + 8F,$$

so that

$$RE = 8, \quad RF = 2 \quad \text{and} \quad R^2 = 16.$$

Now, by Riemann-Hurwitz applied to the map $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$, we have

$$K_X = \pi^* K_{\mathbb{P}^1 \times \mathbb{P}^1} + R = -2E - 2F + R$$

and so

$$c_1(\omega_{X/\mathbb{P}^1}) = -2F + R;$$

it follows that

$$\kappa(\eta) = \big(c_1(\omega_{X/\mathbb{P}^1})\big)^2 = 8.$$

Finally, from the original equation $y^2 = f(t,x)$ of a fiber C_t, we see that a basis for the space of regular differentials is given by

$$\frac{dx}{y}, \quad x\frac{dx}{y} \quad \text{and} \quad x^2\frac{dx}{y};$$

these sections of the Hodge bundle $\eta_*(\omega_{X/\mathbb{P}^1})$ thus give a trivialization of this bundle over the affine open $t \neq \infty$ in $\mathbb{P}^1$. Moreover, since the coefficients of f are quadratic in t, and $y^2 = f$, these sections all vanish simply at the point $t = \infty$; and as before this gives an identification

$$\eta_*((\omega_{X/\mathbb{P}^1})) = \mathcal{O}_{\mathbb{P}^1}(1)^3.$$

In particular, we see that

$$\lambda(\eta) = c_1\big((\omega_{X/\mathbb{P}^1})\big) = 3.$$

In sum, then, the invariants of the family $\eta : X \to \mathbb{P}^1$ are

$$\delta(\eta) = 28, \quad \kappa(\eta) = 8 \quad \text{and} \quad \lambda(\eta) = 3.$$

We see from this in particular that the invariants δ, κ and λ are not in general fixed scalar multiples of each other.

19.1.2 The Mumford Relation

It's time now to see how the Grothendieck-Riemann-Roch theorem applies in this setting. Again, we assume that $f : X \to B$ is a morphism with X a smooth projective surface and B a smooth projective curve, with fibers irreducible curves of (arithmetic) genus g having at most nodes as singularities. Given that one of invariants we've introduced is the first Chern class of a pushforward, it's natural to see what GRR can tell us about it.

Before carrying out this calculation, we want to make two remarks. Recall that the statement of Kodaira-Serre duality for smooth curves says that for any invertible sheaf $\mathcal{F}$ on a smooth curve C, we have a natural identification

$$H^1(\mathcal{F}) = H^0(K_C \otimes \mathcal{F}^*).$$

Here the word "natural" means in particular that the identification applies simultaneously for all fibers in a family: that is, if $f : X \to B$ is a family of smooth, projective curves of genus g and $\mathcal{F}$ an invertible sheaf on X the ranks of whose cohomology groups on the fibers of f is constant, then we have an identification of sheaves

$$R^1 f_*(\mathcal{F}) = f_*(\omega_{X/B} \otimes \mathcal{F}^*).$$

(In fact, a version of this identification holds far more generally, but this is all we'll need for the following, and we want to keep this simple; for the

general statement, see **ref**.) In particular, if we apply this to the case of $\mathcal{F} = \omega_{X/B}$, we see that

$$R^1 f_*(\omega_{X/B}) = f_*(\mathcal{O}_X) = \mathcal{O}_B.$$

Now, the classical form of Kodaira-Serre duality applies just to smooth curves and families of smooth curves. But in fact we can extend it to our present context: if $f : X \to B$ is as above a family of curves with at most nodes as singularities, and $\omega_{X/B}$ is the relative dualizing sheaf as defined above, then it is still true that

$$R^1 f_*(\omega_{X/B}) = \mathcal{O}_B.$$

In the following calculation, we'll use this fact; for a proof, again see **ref**.

The second remark is simply a preliminary calculation. We recall first the *Hopf Index Theorem* invoked in Chapter 5: that the degree of the top Chern class $c_2(T_X)$ of the tangent bundle of the smooth projective surface X is the topological Euler characteristic of X. Moreover, the Generalized Riemann-Hurwitz formula (introduced in Chapter 7) says that the topological Euler characteristic of X is the product of the Euler characteristics $2-2g$ of the general fiber of f and the Euler characteristic of the base curve B, plus the total number δ of nodes in the fibers of f. Combining these, we see that

$$\begin{aligned} c_2(T_X) &= (2-2g)c_1(T_B) + \delta \\ &= -c_1(\omega_{X/B}) \cdot f^* c_1(T_B) + \delta. \end{aligned}$$

Abbreviating the expression $\omega_{X/B}$ for the relative dualizing sheaf to simply ω, we can thus express the ratio $c(T_X)/f^*c(T_B)$ as

$$\begin{aligned} \frac{c(T_X)}{f^*c(T_B)} &= \big(1 + c_1(T_X) + c_2(T_X)\big)\big(1 - f^*c_1(T_B)\big) \\ &= 1 - c_1(\omega) + \delta, \end{aligned}$$

and correspondingly

$$\frac{Td(T_X)}{f^*Td(T_B)} = 1 - \frac{c_1(\omega)}{2} + \frac{c_1(\omega)^2 + \delta}{12}.$$

We can now apply GRR to the pushforward of the sheaf $\omega_{X/B}$. Taking just the terms of degree 1 in $A^*(B)$ (and recalling that $c_1(R^1 f_* \omega_{X/B}) = 0$), we conclude that

$$\begin{aligned} \lambda(f) =& c_1(f_*\omega) \\ &= \left\{ f_*\big((1 + c_1(\omega) + \frac{c_1(\omega)^2}{2})(1 - \frac{c_1(\omega)}{2} + \frac{c_1(\omega)^2 + \delta}{12})\big) \right\}_{\dim B - 1} \\ &= f_* \left(\frac{c_1(\omega)^2}{2} - \frac{c_1(\omega)^2}{2} + \frac{c_1(\omega)^2 + \delta}{12} \right) \\ &= \frac{\kappa(f) + \delta(f)}{12}. \end{aligned}$$

This fundamental relation among the invariants λ, κ and δ is called the *Mumford relation*; as is suggested by the examples we calculated above in genus 3, it is the only linear relation satisfied by these three invariants in general.

19.1.3 A final word

At the outset of this discussion, we asked for ways of measuring the variation in a family $X \to B$ of curves, with X and B smooth and projective and the fibers irreducible with at worst nodes as singularities. While we can't prove this, we should at least indicate to what extent the invariants δ, κ and λ we introduced do this.

The answers are:

(a) The invariant δ does not necessarily detect variation in the fibers of $X \to B$: there exist families of curves with varying modulus, but with no singular fibers at all. By contrast,

(b) The invariants κ and λ do: if $f : X \to B$ is a family as above and either $\kappa(f) = 0$ or $\lambda(f) = 0$, then all the fibers are in fact isomorphic.

In general, it's not hard to see by constructing examples that the Mumford relation above is the only one satisfied universally by δ, κ and λ. The inequalities satisfied by these invariants, however, are more subtle. For a discussion of these questions and references, see **Harris-Morrison**.

19.2 The moduli space of curves

19.2.1 Moduli spaces in general; examples in genus 2

The notion of a parameter space has come up many times in this book. It's a natural idea: we want to give the set of geometric objects of a certain type the structure of a scheme in its own right, in a reasonably natural way. Moreover, when the objects in question are subschemes of a fixed scheme, this can always be done, by using the Hilbert scheme or otherwise.

We now want to consider a much more subtle notion: the idea of a space parametrizing *isomorphism classes* of geometric objects, for example abstract curves. Such spaces, called *moduli spaces*, are harder to get ahold of; they may not exist, or may only exist in a category strictly larger than that of schemes; and in any event they are hard to analyze directly in general.

Let's start with an example: smooth complete curves of genus 2. We know that any such curve can be expressed as a double cover of $\mathbb{P}^1$, branched at

six points, that is, as the curve associated to the polynomial

$$y^2 = \prod_{i=1}^{6}(x - a_i); \tag{19.1}$$

conversely, any six distinct points in $\mathbb{P}^1$ determine such a curve. Modulo the group PGL_2 of automorphisms of $\mathbb{P}^1$, we can take three of these points to be the points $0, 1$ and $\infty \in \mathbb{P}^1$; that is, associate to C the polynomial

$$y^2 = x(x-1)(x-b_1)(x-b_2)(x-b_3)$$

for some b_i distinct from each other and from $0, 1$ and ∞. Now, the group S_6 acts on the space of such triples—given any permutation σ we can send $a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}$ to $0, 1$ and ∞, and see where the remaining three wind up—and so we can naturally give the set of isomorphism classes of smooth curves of genus 2 the structure of the quotient

$$M_2 = \frac{\left(\mathbb{P}^1 \setminus \{0, 1, \infty\}\right)^3 \setminus \Delta}{S_6}.$$

Unfortunately, as the genus g gets larger, it becomes harder to write down a family of curves analogous to the family in 19.1 above—that is, including all curves of genus g, and given by a collection of polynomials whose coefficients vary freely—and eventually it becomes impossible, as we'll see later in the Chapter. That's not to say such spaces don't exist; just that it's harder to prove that they do, or to describe them explicitly.

19.2.2 Stacks

Before we go on, we should take a moment to talk about one of the vaguer phrases in the last section: the idea of giving the set of isomorphism classes of geometric objects (such as curves of genus g) the structure of a variety "naturally." What does that word mean in this context?

19.2.3 Stable curves, and the Deligne-Mumford compactification

As we've indicated, it's far from clear that there exists a scheme M_g parametrizing isomorphism classes of curves of genus g, even in the weaker sense of a coarse moduli space as described above. Moreover, there's another problem: however we do it, the resulting scheme M_g will not be proper. (There are families of smooth curves over a punctured disc Δ^* that cannot be extended to a family of smooth curves over the whole disc Δ; by the valuative criterion, this tells us that M_g cannot be proper.)

The best way to fix this would be to enlarge the set we're parametrizing—that is, throw in isomorphism classes of at least some singular curves. This, however, requires somewhat delicate balancing: if we include too few singular curves, and our space won't be proper; but if we include too many it won't even be separated.

Deligne and Mumford, in their landmark paper ****ref****, solve both problems in one stroke: they introduce a class of possibly singular curves, called *stable curves*, and prove that there exists a coarse moduli space for them that is proper (and in fact, it was proved later in ****ref****, projective).

19.2.4 Divisor classes on $\overline{M}_g$

There are a number of divisor classes on the

19.2.5 Recasting the Mumford relation

19.3 The Kodaira dimension of $\overline{M}_g$

19.3.1 Unirationality in low genus

We wrote down, in 19.1 above, a family of curves, parametrized by an open subset $U \subset \mathbb{A}^6$ of affine space $\mathbb{A}^6$, that included every smooth curve of genus 2. This gives us in particular a surjection

$$\varphi : U \to M_2,$$

and we may deduce in particular that M_2 *is unirational*.

We can similarly write down a dominant (if not surjective) map from an open subset of projective space to M_3: just consider the family of plane quartics given by

$$\sum_{i+j+k=4} a_{i,j,k} X^i Y^j Z^k,$$

where the homogeneous vector $\{a_{i,j,k}\} \in \mathbb{P}^{14}$ belongs to the open set $U \subset \mathbb{P}^{14}$ of equations of smooth quartics. (This is not surjective because it misses the hyperelliptic curves of genus 3.) Again, we conclude that M_3 is unirational.

We can write down similar families of curves of genus g—families given by polynomials whose coefficients vary freely, and including a general curve of genus g, so that we get a dominant rational map $\mathbb{P}^N \to M_g$ for some N—for other small values of g. For $g = 4, 5$ and 6 this is easy and explicit:

general canonical curves of genera 4 and 5 are complete intersections in $\mathbb{P}^3$ and $\mathbb{P}^4$; while in genus 6, a general canonical curve is a quadric section of the smooth del Pezzo surface $S \subset \mathbb{P}^5$. In genera 7-10, it's still easy though less explicit: we show that a general curve of those genera can be realized as a nodal plane curve whose singular locus is a general collection of points in $\mathbb{P}^2$ ****ref****. And in general 11, 12, 13 and 15, it's neither easy or explicit, but it can be done ****ref****.

Given this, it's natural to ask: is it true for all g that M_g is unirational? The answer, it turns out, is "no:" the current state of the art ****ref to Farkas**** is that for $g \geq 22$ the space M_g is actually of general type. In the remainder of this Chapter, we'll try to give an idea of how we can use enumerative techniques to prove at least some cases of this assertion.

****Do we need to discuss "general type?"****

19.3.2 The canonical class of $\overline{M}_g$

To say that a variety is of general type is to say that it has lots of pluricanonical forms; in particular, this means that multiples of the canonical divisor class are effective. If we're going to prove something like this, naturally, we need to start by determining the canonical divisor class. Happily, in the case of the moduli space $\overline{M}_g$, we can do this by combining two techniques we've already discussed: deformation theory, and the Grothendieck Riemann Roch formula.

The Reid-Tai criterion. In calculating the canonical divisor class of the moduli space $\overline{M}_g$, we could ignore the singularities of $\overline{M}_g$: since they occur in codimension 2, an equality of divisor classes on $\overline{M}_g$ that holds on the smooth locus must hold everywhere. Unfortunately, when we want to go from saying that the canonical divisor class of $\overline{M}_g$ is effective to saying that some desingularization of $\overline{M}_g$ possesses everywhere regular canonical forms, it's not so simple.

19.3.3 The Brill-Noether divisor

20

Appendix: Intersection Theory on Surfaces

20.1 Introduction

There were two major motivations for the development of intersection theory in the second half of the 19th and the beginning of the 20th centuries: enumerative geometry—the study of the parameter spaces of geometric objects, leading to the computation of the number of objects of specified types—and the theory of algebraic surfaces, where intersection theory leads to the qualitative classification of surfaces. As in the theory of curves, the central problems of the theory of surfaces can be formulated as problems of computing or estimating the dimension of the space of sections $\dim H^0(L)$ of a line bundle L. As a first approximation one computes the Euler characteristic $\chi(L) := \sum_j \dim H^j(L)$. This is done by the Riemann-Roch Theorem, whose inputs are the self-intersection of $c_1(L)$ and the intersection number of $c_1(L)$ with the canonical class:

Theorem 20.1. *Let S be a smooth surface with canonical divisor K_S. If L is a line bundle on S then*

$$\chi(L) = \frac{L^2 - LK_S}{2} + \chi(\mathcal{O}_S).$$

This result is just a sample of the ways in which intersection theory enters the theory of surfaces. We will give a proof in a special case at the end of this Appendix; The vast generalizations due to Hirzebruch and Grothendieck are discussed in Chapter 17.

****by the end of the section the notation is completely different; reconcile it. Also, in light of the treatment of line bundles above, maybe we could give a full proof.****

20.2 Riemann-Roch and Serre Duality for Curves

We begin by recalling the statement of the Riemann-Roch Theorem for a smooth projective curve (see for example Hartshorne [1977] Section 4.1):

Theorem 20.2. *If L is a line bundle on a smooth projective curve then*

$$\chi(L) = \deg L + \chi(\mathcal{O}_X).$$

An important complement is *Serre Duality*, which says that on a smooth variety X of dimension n there is a natural isomorphism, called the *residue* map $res : H^n\omega_X \cong K$, and that the multiplication maps

$$H^i(L) \otimes H^{n-i}(\omega_X \otimes L^{-1}) \to H^n(\omega_X) \cong K$$

are perfect pairings. In particular, $\chi(L) = (-1)^{\dim X}\chi(\omega_X \otimes L^{-1})$. This gives us the more classical statement:

Theorem 20.3. *If L is a line bundle on a smooth projective curve then*

$$h^0(L) - h^0(\omega_X \otimes L^{-1}) = \deg L + \chi(\mathcal{O}_X).$$

20.3 Curves and adjunction

Let X be an 1-dimensional projective scheme. We define the *arithmetic genus* of X to be

$$g(X) = 1 - \chi(\mathcal{O}_X).$$

When X is a smooth irreducible projective curve of genus g, we see by Serre duality that $1 - g = \chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = 1 - h^0(\omega_X)$, so $g = h^0(\omega_X)$, the number of independent differential forms on X, just as in Riemann's original definition. Also $\chi(\omega_X) = H^0(\omega_X) - H^1(\omega_X) = g - 1$, so from the Riemann-Roch formula we get $\deg \omega_X = 2g - 2$.

We often shorten the term "arithmetic genus" to "genus", but the reader should be aware of a different notion that is also useful: in case X is reduce of pure dimension 1, the *geometric genus* of X is defined to be the (arithmetic) genus of the normalization of X. The arithmetic genus of X is sometimes written $p_a(X)$, and the geometric genus $p_g(X)$.

To express the difference between the arithmetic and geometric genera of a (reduced, pure 1-dimensional) curve X, let $\nu : \tilde{X} \to X$ be its normalization, and consider the sequence

$$0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{\tilde{X}} \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is defined to be the quotient $\nu_* \mathcal{O}_{\tilde{X}} / \mathcal{O}_X$. The sheaf $\mathcal{F}$ is supported exactly at the singular points of X, and the length of the stalk $\mathcal{F}_p$ at a point $p \in X_{\text{sing}}$ is called the *δ-invariant* of the singularity $p \in X$. Comparing Euler characteristics in the sequence above (and noting that $\chi(\nu_* \mathcal{O}_{\tilde{X}}) = \chi(\mathcal{O}_{\tilde{X}})$, since the map ν is finite), we see that the difference between the arithmetic and geometric genera of X is just the sum of the δ-invariants of its singular points:

$$p_g(X) = p_a(\tilde{X}) = p_a(X) - \sum_{p \in X_{\text{sing}}} \delta_p.$$

We'll see in the next section a simple way of describing the δ-invariant of a curve singularity in terms of its geometric invariants.

If $L = \mathcal{O}_X(D)$ is the line bundle associated to an effective Cartier divisor $D \subset X$ of degree d, we see from the sequence

$$0 \to \mathcal{O}_X \to L = \mathcal{O}_X(D) \to L|_D \to 0$$

that

$$\chi(L) = \chi(L|_D) + \chi(\mathcal{O}_X) = d - g + 1,$$

the classical Riemann-Roch theorem for curves. We can extend this to line bundles of the form $L = \mathcal{O}_X(D - E)$ by considering the sequence

$$0 \to \mathcal{O}_X(D - E) \to \mathcal{O}_X(D) \to \mathcal{O}_X(D)|_E \to 0.$$

As a first sign that intersection theory might be useful in the theory of surfaces, we can compute the (arithmetic) genus of a curve on a surface via intersection theory:

Proposition 20.4. *If C is a curve on a smooth surface S with canonical class K_S, then*

$$g(C) = \frac{C^2 + K_S C}{2} + 1.$$

Proof. We have seen that $K_C = \big(K_S \otimes \mathcal{O}_S(C)\big)|_C$. Computing the degree on both sides, and equating the degree of the canonical bundle on C with $2g - 2$, we get the desired equation. □

20.3.1 Genus of a Curve in the Plane or on a Quadric Surface

Two cases where we often use adjunction are for curves in $\mathbb{P}^2$ and on a smooth quadric surface $Q \subset \mathbb{P}^3$; it's worth writing down the formulas in these cases.

We have seen that the canonical bundle $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$. If $C \subset \mathbb{P}^2$ is a smooth plane curve of degree d, then, the genus of C is given by

$$g = \frac{C^2 + K_{\mathbb{P}^2} C}{2} + 1 = \frac{d^2 - 3d}{2} + 1 = \binom{d-1}{2}.$$

Similarly, suppose $C \subset Q$ is a curve on a smooth quadric surface $Q \subset \mathbb{P}^3$. The surface Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (with the fibers of the projection maps $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ giving the two rulings by lines). Writing α and β for the classes of the fibers of the projection maps, we have $A(Q) = \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$. In particular, $\mathrm{Pic}(Q) = A^1(Q) = \mathbb{Z}\langle \alpha, \beta \rangle$. For example, the class of a hyperplane section of Q is $\alpha + \beta$. We say that a curve $C \subset Q$ is of type (a, b) if its divisor class is $a\alpha + b\beta$. Taking the intersections with the rulings we see that $a = \deg[C]\beta$ and $b = \deg[C]\alpha$; that is, a curve is of class (a, b) if it meets the general members of the two rulings in b and a points. To compute the degree of the curve as a curve in $\mathbb{P}^3$ we intersect with the hyperplane class, and get

$$\deg(C) = \deg([C](\alpha + \beta)) = a + b.$$

Since $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, its tangent bundle is the direct sum of the tangent bundles of the two factors pulled back to Q. Taking the dual to get Ω_Q, and then taking the exterior square, we get the tensor product of these two line bundles,

$$\omega_Q \cong \mathcal{O}_Q(-2).$$

(You can also check this by using the fact that Q is a divisor in $\mathbb{P}^3$.) In terms of divisors this means

$$K_Q = -2\alpha - 2\beta.$$

It follows that if $C \subset Q$ is a smooth curve of type (a, b), its genus is

$$g = \frac{C^2 + K_Q C}{2} + 1 = \frac{2ab - 2a - 2b}{2} + 1 = (a-1)(b-1).$$

Note that this is 0 when a or $b = 1$, as it should be; such a curve projects one-to-one onto one of the two $\mathbb{P}^1$ factors, and is thus rational.

20.3.2 Self-Intersection of a Curve on a Surface

The adjunction formula is an invaluable tool for computing intersections on surfaces:

Corollary 20.5. *If $C \subset S \subset \mathbb{P}^3$ is a curve of degree e and genus g on a smooth surface of degree d, then*

$$\deg[C]^2 = 2g - 2 - (d-4)e.$$

In particular, the degree of the self-intersection of a line on S is $2-d$.

The cases of a line in the plane (self-intersection 1), in a quadric surface (self-intersection 0) and in a cubic surface (self-intersection -1) may be familiar!

Proof. By adjunction we have

$$2g - 2 = \deg K_C = \deg[C]^2 + \deg K_S[C].$$

However, by adjunction for the surface S in $\mathbb{P}^3$ we have $K_S = S^2 + K_{\mathbb{P}^3}|_S = S^2 - 4H = (d-4)H$, where H is the hyperplane section of S. Since $\deg HC = e$, we are done. □

Exercise 20.6. Let $C \subset \mathbb{P}^3$ be a smooth curve of degree d and genus g; let S and $T \subset \mathbb{P}^3$ be smooth surfaces of degrees s and t, intersecting generically transversely along the union of C and another smooth curve D.

(a) Assuming C and D are nowhere tangent, in how many points do they meet?

(b) By Bezout, the degree of D is $st - d$. What is its genus?

(c) Check your answers in special cases: for example, take C a twisted cubic and S and T quadrics containing C; or C a plane curve and S a plane.

Finally, here's a simple example of *excess intersection*, that we can work out with just the results above:

Exercise 20.7. Let $L \subset \mathbb{P}^3$ be a line, and let S, T and $U \subset \mathbb{P}^3$ be surfaces of degrees s, t and u containing L, such that the scheme-theoretic intersection

$$S \cap T \cap U = L \cup \Gamma$$

with Γ reduced and zero-dimensional. What is the cardinality of Γ?

We'll discuss excess intersection in greater generality in Chapter 16.

20.4 Blow-ups of surfaces

In this section, which will take the form of a series of exercises, we'll describe the Chow ring of the blow-up of a surface S at a point; we'll use this to explore further the difference between the arithmetic and geometric genus of a singular curve in S. Throughout, S will be a smooth, projective surface, $p \in S$ a point and $C \subset S$ a reduced and irreducible curve. Let $\pi : \tilde{S} = Bl_p S \to S$ be the blow-up of S at p, and $E = \pi^{-1}(p) \subset \tilde{S}$ the exceptional divisor of the blow-up; let $e = [E] \in A^1(\tilde{S})$ be its class.

Exercise 20.8. (a) Show that the pushforward map gives an isomorphism

$$\pi_* : A^2(\tilde{S}) \xrightarrow{\cong} A^2(S).$$

(b) Let $\tilde{C} \subset \tilde{S}$ be a curve. Comparing $\tilde{C}$ with the cycle $\pi^*(\pi_* C)$, show that we have an orthogonal decomposition

$$A^1(\tilde{S}) \cong A^1(S) \oplus \mathbb{Z}\langle e \rangle.$$

with the inclusion $A^2(S) \hookrightarrow A^2(\tilde{S})$ given by the pullback map π^*, and the projection $A^2(\tilde{S}) \to A^2(S)$ given by the pushforward map π_*.

Exercise 20.9. Identifying E with the projectivization $\mathbb{P}T_p S \cong \mathbb{P}^1$, show that the differential $d\pi$ gives an isomorphism between the normal bundle $N_{E/\tilde{S}} = \mathcal{O}_{\tilde{S}}(E)|_E$ and the universal sub-line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, and deduce in turn that $e^2 = -1$.

Exercise 20.10. (a) Let η be a rational 2-form on S, regular and nonzero at p. Comparing the divisors of η and $\pi^*\eta$, show that the canonical class of $\tilde{S}$ is

$$K_{\tilde{S}} = \pi^* K_S + e.$$

(b) Applying the adjunction formula to $E \subset \tilde{S}$ and to curves in $\tilde{S}$ disjoint from E, prove the above formula for $K_{\tilde{S}}$ again.

Exercise 20.11. Suppose now that the curve $C \subset S$ has multiplicity m at p. If $\tilde{C}$ is the proper transform of C in $\tilde{S}$—that is, the closure of the preimage $\pi^{-1}(C \setminus \{p\})$—show that the class of $\tilde{C}$ is

$$[\tilde{C}] = \pi^*[C] - me.$$

Deduce that $\tilde{C}^2 = C^2 - m^2$; and applying adjunction and the formula above for $K_{\tilde{S}}$, deduce that

$$g(\tilde{C}) = g(C) - \binom{m}{2}.$$

As promised, this yields a geometric way of describing the δ-invariant of a singular point $p \in C$: if $p_1, \ldots, p_k \in \tilde{C}$ are the points of $\tilde{C}$ lying over p, we have

$$\delta_p = \binom{m}{2} + \sum_i \delta_{p_i}.$$

To calculate δ_p, then, all we have to do is to blow up p and keep blowing up until all the points lying over p are smooth; if the multiplicities of the points blown up are $m_1, m_2, \ldots$ then we have

$$\delta_p = \sum \binom{m_i}{2}.$$

Exercise 20.12. Using the process described above, find the δ-invariant of the following curve singularities:

(a) a triple point consisting of the union of three smooth branches, two of which are simply tangent;

(b) a *triple tacnode*: that is, a triple point consisting of the union of three smooth branches that are pairwise simply tangent;

(c) a fourfold point consisting of the union of four smooth branches, with the first two simply tangent and the last two simply tangent; and

(d) a fourfold point consisting of the union of two cusps with distinct tangent lines.

Exercise 20.13. Let $C \subset Q \subset \mathbb{P}^3$ be a smooth curve of type (a, b) on a smooth quadric surface $Q \subset \mathbb{P}^3$, and $p \in C$ a general point. Let $\pi_p : C \to \mathbb{P}^2$ be the projection from p, and $C_0 \subset \mathbb{P}^2$ its image.

(a) Describe the degree and the singularities of the curve C_0; and

(b) Using the formula above and the formula for the arithmetic genus of a plane curve, verify the formula for the genus of a curve on a quadric.

20.5 The Riemann-Roch Theorem for Surfaces

Proof of Theorem 20.1. We can combine the results of the preceding two sections to give a proof of a simple version of the Riemann-Roch theorem for surfaces.

The first step is to derive the formula for the line bundle $L = \mathcal{O}_S(C)$ associated to a smooth curve $C \subset S$ on a smooth projective surface S. From the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_S(C)|_C \to 0$$

we see that

$$\chi(\mathcal{O}_S(C)) = \chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(C)|_C).$$

As we saw in Section **??**, the degree of the line bundle $\mathcal{O}_S(C)|_C$ is the self-intersection C^2, and so by the Riemann-Roch formula on C,

$$\begin{aligned}\chi(\mathcal{O}_S(C)|_C) &= C^2 - \frac{C^2 + K_S C}{2} \\ &= \frac{C^2 - K_S C}{2};\end{aligned}$$

combining these, we arrive at the formula

$$\chi(\mathcal{O}_S(C)) = \frac{C^2 - K_S C}{2} + \chi(\mathcal{O}_S).$$

We claim now that *this formula holds for an arbitrary divisor C on the smooth surface S*. To see this, we again express the line bundle $L = \mathcal{O}_S(C)$ as a difference $L = M \otimes N^{-1}$ of two very ample line bundles M and N; we can say that M and N have regular sections whose zero divisors E and F are smooth curves on S. We then have an exact sequence

$$0 \to L = \mathcal{O}_S(E - F) \to M = \mathcal{O}_S(E) \to M|_F \to 0$$

and so the Euler characteristic $\chi(L)$ is the difference of the Euler characteristics of the other two terms. For the middle term $M = \mathcal{O}_S(E)$, we can apply the formula we just derived for line bundles associated to smooth curves: we have

$$\chi(M) = \frac{E^2 - K_S E}{2} + \chi(\mathcal{O}_S).$$

As for the right-hand term, $M|_F$ is a line bundle of degree EF on the curve F, which has genus

$$g(F) = \frac{F^2 + K_S F}{2} + 1;$$

by the Riemann-Roch formula for curves, then, we have

$$\chi(M|_F) = E \cdot F - \frac{F^2 + K_S F}{2}$$

and combining we have

$$\begin{aligned}\chi(L) &= \chi(M) - \chi(M|_F) \\ &= \frac{E^2 - K_S E}{2} + \chi(\mathcal{O}_S) - EF + \frac{F^2 + K_S F}{2} \\ &= \frac{E^2 - 2EF + F^2 - K_S E + K_S F}{2} + \chi(\mathcal{O}_S) \\ &= \frac{L^2 - K_S L}{2} + \chi(\mathcal{O}_S)\end{aligned}$$

□

21

Appendix: Other Cycle Theories

Over the complex numbers the cohomology ring of a smooth analytic manifold provides a very useful substitute for the Chow groups. Indeed, the desire to put intersection Theory on a firm footing was one of the great motivations for the development of cohomology theory in algebraic topology. There are other reasonable ways to define cycles and equivalence relations that for many purposes can also substitute for the theory of the Chow ring that we have used in this book. The various cohomology-like theories we will define—cycles mod algebraic equivalence, mod homological equivalence, or mod numerical equivalence, or simply topological cohomology—have different amounts of information to give, and are more or less calculable. In this Appendix we will explore a little of what is known about them and the comparisons between them.

21.1 Examples

It will be useful to have an example to follow through these comparisons, and for that quartic surfaces in $\mathbb{P}^3$—or more generally, K3 surfaces—will serve our purposes well. (For rational homogeneous varieties, like projective spaces and Grassmannians, all the cycle theories mentioned above in fact yield the same ring; for more general varieties, we don't know enough about them to make a useful comparison.) To begin with, let's say what we know about the Chow ring of a smooth quartic surface $S \subset \mathbb{P}^3$.

(a) $A^0(S)$ is the easiest, naturally; by part b of Proposition 1.6, this is freely generated by the fundamental class $[S]$ of S. In fact, this will be true of all the cycle theories we discuss; only the groups A^1 and A^2 will vary.

(b) $A^1(S)$ This is in some sense the most interesting Chow group of S. It is finitely generated and free, of rank between 1 and 20, and the lattice of this group (with the intersection product $(\alpha \cdot \beta) = \deg(\alpha\beta)$) plays a central role in the geometry of the surface. (See Barth et al. [2004] for an in-depth discussion of K3 surfaces and their associated lattices.)

The problem is that there is no algorithm for determining the group $A^1(S)$ for a given quartic surface S. What we do know is that for a very general quartic S the group $A^1(S)$ is freely generated by the class of a hyperplane section (from which it follows that every curve $C \subset S$ is a complete intersection of S with another surface T). But the locus, in the space U of smooth quartic surfaces, of those whose Chow group A^1 has rank strictly bigger than one is a countable dense union of subvarieties of U. In fact, the locus of S with $A^1(S)$ of rank $\geq \rho$ is dense for every $\rho \leq 20$. Thus we can't characterize the surfaces with Chow groups of rank at least ρ by a polynomial condition on the coefficients of their defining equations.

We should say that the fact that $A^1(S)$ is finitely generated is a reflection of the fact that a smooth surface in $\mathbb{P}^3$ is a *regular* surface, meaning its first Betti number is 0. If this were not the case for a given surface T, the group $A^1(T)$ would not be finitely generated. But it would still be describable as an extension of a finitely generated group by an abelian variety.

(c) $A^2(S)$, by contrast, is truly unknowable, at least within the framework of existing descriptions of groups. The basis for this assertion is a theorem of Mumford, who considers for any n the map $\mu_n : S_n \to A^2(S)$ from the nth symmetric power of S to $A^2(S)$, sending an n-tuple of points $p_1, \ldots, p_n \in S$ to the class $[p_1 + \cdots + p_n] \in A^2(S)$. Mumford's theorem asserts that the fiber of the map μ_n through a general point of S_n has dimension at most n; since the dimension of S_n is $2n$, this says not only that $A^2(X)$ cannot be finitely generated, but that it cannot be given the structure of a finite-dimensional algebraic group, or an extension of one by a finitely generated group.

We should say that Mumford's theorem applies to any surface S whose geometric genus $p_g(S)$ is positive; that is, that has a regular 2-form. There is also a converse to this which asserts that if $p_g(S) = 0$ then $A^2(S) \cong \mathbb{Z}$ (Bloch et al. [1976]), but this is itself somewhat mysterious. For example, while it follows that an Enriques surface S has $A^2(S) \cong \mathbb{Z}$, the proof is extremely indirect; no one has ever exhibited an explicit rational equivalence between two general points of S.

Does the unknowability of $A^2(S)$ mean that Chow theory is useless? Far from it: in almost all applications of intersection theory—and certainly in all applications to enumerative geometry—we are concerned just with the *degree* of a zero cycle, which is a much more manageable bit of information! The unmanageably infinite extra data provided by the Chow class of a zero-cycle doesn't interfere with the calculation of the manageably finite data of its degree.

21.2 Algebraic equivalence

Let's return to the discussion of possible cycle theories: the choice of what we define to be a cycle, and the choice of equivalence relation among these.

For the first, it seems logical enough, given that we're working in the category of varieties, to define cycles on a variety X to be (formal linear combinations of) subvarieties of X. We could have been more restrictive (consider only a special class of subvarieties) or more inclusive (for example, consider morphisms $f : A \to X$ to our variety, or, in a different direction, consider topological cycles). But defining $Z^*(X)$ to be linear combinations of subvarieties of X seems natural enough.

But the choice of equivalence relation is somewhat more problematic. Why, for example, do we define cycles to be equivalent only if there's a *rational* family of cycles interpolating between them—that is, a family parametrized by $\mathbb{P}^1$? What would happen if we instead allowed families parametrized by curves of any genus?

We can indeed define in this way a coarser equivalence relation, called *algebraic equivalence*, and corresponding groups $A^*_{alg}(X)$ with natural maps

$$A^*(X) \to A^*_{alg}(X).$$

Everything we've said above about Chow groups applies word-for-word to the groups of cycles mod algebraic equivalence; in particular, we have a ring structure and pushforward and pullback maps under exactly the same conditions.

Moreover, working mod algebraic equivalence immediately clears up the problem with the group of zero-cycles mod rational equivalence: for any connected, n-dimensional variety X we have $A^n_{alg}(X) \cong \mathbb{Z}$, with the isomorphism given by the degree map. The description of the group of cycles in codimension 1 also becomes somewhat simpler: if X is smooth and projective, we have an exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) = A^1(X) \to \mathrm{Pic}(X)/\mathrm{Pic}^0(X) \to 0,$$

where $\mathrm{Pic}^0(X)$ is an abelian variety, and $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is finitely generated. Working mod algebraic, rather than rational equivalence, kills the

connected component $\mathrm{Pic}^0(X)$ of the identity in the Chow group $A^1(X)$, so that

$$A^1_{alg}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$$

is finitely generated.

Unfortunately, that's where the good news ends. In codimensions between 1 and $\dim X$, the group of cycles mod algebraic equivalence remains in general as stubbornly unknowable as the Chow group. For example, for threefolds $X \subset \mathbb{P}^4$ of degree 5 or more, it's not known if the groups $A^2_{alg}(X)$ are finitely generated, or for that matter whether they differ from or coincide with the Chow groups $A^2(X)$ (which are themselves unknown).

The bottom line is that working mod algebraic equivalence is really not that much different from working mod rational equivalence; we usually do the latter simply because an equality of classes mod algebraic equivalence necessarily conveys less information than the same formula mod rational equivalence. But since the extra information is for the most part inaccessible, it really doesn't make much difference in practice.

21.3 Homological equivalence

For the duration of this subsection and the next, we'll take X to be a smooth projective variety of dimension n over the complex numbers $\mathbb{C}$; when we talk about the homology or cohomology groups of X (or of any subvariety of X) we'll mean the singular homology or cohomology with respect to the classical topology on X. Note that X is a compact complex manifold of dimension n, and hence a compact, oriented real manifold of dimension $2n$. In particular, X has a fundamental class, and Poincaré duality says that capping with the fundamental class provides an isomorphism $H_m(A, \mathbb{Z}) \cong H^{2n-m}(A, \mathbb{Z})$.

It is not obvious, however, that subvarieities of X are represented by classes; but in fact if A is any k-dimensional subvariety—smooth or singular—then A *has a fundamental class*; in other words,

$$H_{2k}(A, \mathbb{Z}) \cong \mathbb{Z}$$

with a distinguished generator $[A]$ corresponding to the natural orientation on the smooth locus of A.

This theorem is substantial. It certainly follows from the theorem (Hironaka [1975]; Lojasiewicz [1964]) that any projective variety over $\mathbb{C}$ admits a triangulation, in which the singular locus is a subcomplex—given this, since the simplices contained in the singular locus have real codimension at least 2, it follows that the sum of the simplices of top dimension is a

cycle—but that theorem is quite difficult. We can get a slightly weaker result—that A has a fundamental class in de Rham cohomology—by arguing that integration over A gives a well-defined linear functional on closed mod exact $2k$-forms on X; see **?**] for the argument.

Thus for any subvariety $A \subset X$ of codimension k we can define the *cohomology class* $\eta_A \in H^{2k}(X, \mathbb{Z})$ of A to be the Poincaré dual of the image $i_*[A] \in H_{2n-2k}(X, \mathbb{Z})$; and naturally we say that two subvarieties $A, B \subset X$ are *homologically equivalent* if their cohomology classes are equal. The group of cycles mod homological equivalence is denoted $H^*_{alg}(X)$. Since this is by definition a subgroup of the ordinary cohomology $H^*(X, \mathbb{Z})$—it's just the subgroup generated by the classes of algebraic subvarieties—and X is a compact manifold, we see that the group $H^*_{alg}(X)$ is always finitely generated.

Once more, the intersection of cycles (after moving into sufficiently transverse position) defines a ring structure on $H^*_{alg}(X)$, so that the natural quotient map $A^*(X) \to H^*_{alg}(X)$ is a ring homomorphism, and the inclusion $H^*_{alg}(X) \hookrightarrow H^*(X, \mathbb{Z})$ of $H^*_{alg}(X)$ into the ordinary cohomology ring of X is also a ring homomorphism. In fact, this is how cup products arose: as an attempt to rigorously define the intersection product for algebraic subvarieties—presented heuristically but very successfully by Schubert's calculus of conditions—on algebraic varieties.

Since homological equivalence coincides with algebraic equivalence in dimension 0 and in codimension 1, they agree for quartic surfaces $S \subset \mathbb{P}^3$. But for a general quintic threefold $X \subset \mathbb{P}^4$ they diverge. By the Lefschetz Hyperplane Theorem (see Appendix 22), we have

$$H^0(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z}) \cong H^4(X, \mathbb{Z}) \cong H^6(X, \mathbb{Z}) \cong \mathbb{Z},$$

and each of these groups is generated by the cohomology class of an algebraic subvariety (X itself, a hyperplane secrtion of X, a line on X and a point, respectively) so these are the algebraic cohomology groups $H^{2k}_{alg}(X)$ as well. In particular, two curves $C, C' \subset X$ will be homologically equivalent if and only if they have the same degree. On the other hand, as we'll see in Chapter 8, a general smooth quintic threefold X contains exactly 2875 lines; and, by a theorem of Ceresa and Collino (Ceresa and Collino [1983]), *these lines are linearly independent in* $A^2_{alg}(X)$.

21.4 Cohomology

Why not just go all the way and work with the topological cohomology groups $H^*(X, \mathbb{Z})$? In fact, this approach has a number of virtues: the cohomology of a smooth projective variety is finitely generated, behaves well

under continuous deformation and satisfies useful relations like the Kunneth formula. None of the other cycle theories we have discussed satisfies the second or third of these, and only the algebraic cohomology $H^*_{alg}(X)$ satisfies the first. These properties make $H^*(X,\mathbb{Z})$ relatively easy to compute. For example, the cohomology of all smooth quartics in $\mathbb{P}^3$ is the same, and not hard to calculate: as we'll see in Chapter 7,

$$\begin{aligned} &H^0(S,\mathbb{Z}) = H^4(S,\mathbb{Z}) = \mathbb{Z} \\ &H^1(S,\mathbb{Z}) = H^3(S,\mathbb{Z}) = 0; \quad \text{and} \\ &H^2(S,\mathbb{Z}) = \mathbb{Z}^{22}. \end{aligned}$$

One problem with using the cohomology ring is how to characterize the subring $H^*_{alg}(X) \hookrightarrow H^*(X,\mathbb{Z})$. Hodge theory provides a possible characterization up to torsion: the *Hodge Conjecture* asserts that the algebraic cohomology with rational coefficients $H^{2k}_{alg}(X,\mathbb{Q}) = H^{2k}_{alg}(X) \otimes \mathbb{Q}$ is the group of classes in $H^{2k}(X,\mathbb{Q})$ whose image in de Rham cohomology is representable by a differential form of type (k,k). The situation is not very satisfactory: the conjecture is only known for $k = 1$ and $n-1$, for abelian varieties and a few other classes, and the data necessary to apply it is difficult to compute in practice.

Still, there are situations where working with non-algebraic cohomology classes is useful in deriving relations among algebraic classes; a good example of this is the derivation of *Poincaré's formula* (Proposition 18.5 of Chapter 18). Working only with the topological cohomology groups $H^*(X,\mathbb{Z})$ of a variety is a viable option in characteristic zero, and some results can be extended to all characteristics using ètale cohomology, albeit with a lot more effort. For an intersection-theory based characteristic zero treatment of algebraic geometry that uses cohomology groups and doesn't even mention Chow groups see **?**].

21.5 Numerical equivalence

Another cycle theory that has some of the advantages of cohomology and is easily available in all characteristics is given by the the notion of *numerical equivalence*. We say that a class $\alpha \in A^k(X)$ is *numerically equivalent to 0* if the degree of the product $\alpha\beta$ is 0 for all classes $\beta \in A^{n-k}(X)$ of complementary dimension. The group of Chow classes mod numerical equivalence is denoted $N^*(X)$; again, intersection products give this group the structure of a ring, with the quotient map $A^*(X) \to N^*(X)$ a ring homomorphism.

In characteristic zero any class homologically equivalent to zero is evidently numerically equivalent to 0, so numerical equivalence represents a further coarsening of the notion of homological equivalence for varieties

over $\mathbb{C}$. Conjecturally it is not much coarser: if the Hodge conjecture is true, than by Poincaré duality and the Hard Lefschetz Theorem (see Appendix 22), the pairing

$$H^{2k}_{alg}(X,\mathbb{Q}) \times H^{2n-2k}_{alg}(X,\mathbb{Q}) \to \mathbb{Q}$$

given by intersection/cup product will be nondegenerate. Given this, it would follow that any class $\alpha \in H^*_{alg}(X)$ numerically equivalent to 0 is torsion in $H^*(X,\mathbb{Z})$, or in other words

$$N^k(X) = H^{2k}_{alg}(X)/\text{tors}.$$

Since the Hodge conjecture is known in codimension 1, this statement is known for $k = 1$ and $n-1$. This makes the notion of numerical equivalence particularly suitable in applications of intersection theory that deal largely with divisors and curves, such as the minimal model program; and most papers in that area work with numerical equivalence.

Note also that for our continuing examples—quartic surfaces in $\mathbb{P}^3$ and quintic threefolds in $\mathbb{P}^4$—this applies; and since there is no torsion in cohomology in either case, we have $N^*(X) = H^*_{alg}(X)$ in these cases.

21.6 Comparing the Theories

To summarize, we've defined a number of cycle theories, each of which is useful in certain situations. In a diagram, we have

$$\begin{array}{ccccccc} A^*(X) & \twoheadrightarrow & A^*_{alg}(X) & \twoheadrightarrow & H^*_{alg}(X) & \hookrightarrow & H^*(X,\mathbb{Z}) \\ & & & \searrow & \downarrow & & \\ & & & & N^*(X) & & \end{array}$$

with $N^*(X)$ conjecturally isomorphic to $H^*_{alg}(X)/\text{tors}$.

In fact, with small modifications to our arguments (and ignoring the necessary restriction to characteristic zero when using cohomology) we could use any of the five cycle theories we have described for almost every intersection we compute in this book. The class of smooth projective varieties for which the theories effectively coincide includes any variety with an affine stratification, and in particular all products of projective spaces and Grassmannians; it includes all homogeneous spaces for affine algebraic groups; it includes any projective bundle over a variety of this class, and all blow-ups of varieties in this class along subvarieties in this class. This class represents a tiny fraction of all varieties, but it's a large fraction of the set of varieties on which we can effectively carry out intersection theory, and it includes

all the varieties in whose Chow ring we calculate until Chapter 16 of this book.

****The following two sections were just moved from Ch 7. They need to be integrated.****

21.7 Chern classes in topology and differential geometry

Let M be a reasonably nice topological space, say a simplicial complex, and E a $\mathcal{C}^\infty$ a complex vector bundle on M. It is possible to associate, to any such pair (M, E), a class $c(E) \in H^*(M, \mathbb{Z})$, in a way that satisfies all the axioms of Chern classes in algebraic geometry. Moreover, this is compatible with the existing definition of Chern classes: if M is a smooth variety over $\mathbb{C}$, with the classical topology, and E an algebraic vector bundle on X, then the class associated to the underlying topological space and continuous vector bundle of M and E is just the image of $c(E) \in A^*(M)$ under the map $A^*(M) \to H^*(M, \mathbb{Z})$ described in Section 21. (This is not surprising, since the original topological definition appears to have been modeled on our definition in the algebraic setting, as we'll explain below.) Because of this compatibility, we will call this invariant of a continuous bundle its Chern class as well, and denote it $c(E)$.

21.7.1 Chern Classes and Obstructions

The Chern class was first defined in the context of topological spaces by Steifel and Whitney ****find reference in GH****, using *obstruction theory*. Very briefly, the motivation and construction went like this:

A complex vector bundle E of rank n is trivial if and only if it has n everywhere independent sections. That may be somewhat ambitious; let's adopt as a first measure of the nontriviality of the bundle the existence or nonexistence of a single everywhere nonzero section—that is, a section of the complement E^* of the zero-section in E. Obstruction theory provides a measure of the difficulty of finding such a section, called the *obstruction class*, as follows.

To begin with, we assume M given the structure of a simplicial complex. We start by choosing, at will, a section σ_0 of E^* over the 0-skeleton M_0 of M—that is, we select arbitrary nonzero vectors $\sigma_0(p) \in E_p^*$ in the fibers of E over the vertices $p \in M$ of our complex.

Next, we want to extend σ_0 to a section σ_1 of E^* defined over the 1-skeleton of M. We can always do this: since a closed 1-cell $\Delta_1 \subset M$ is con-

tractible, the restriction of E to Δ_1 will be trivial, so that finding a nonzero section of E over Δ_1 with the assigned values $\sigma_0|_{\partial\Delta_1}$ on the boundary $\partial\Delta_1$ amounts to finding an arc $\gamma : [0,1] \to \mathbb{C}^n \setminus \{0\}$ with given starting and ending points; since $\mathbb{C}^n \setminus \{0\}$ is connected, we can always do this.

****picture here****

We continue in this way, extending σ_1 to a section of E^* over successively larger skeleta of M: given a section σ_{k-1} of E^* over the $(k-1)$-skeleton M_{k-1}, for each k-simplex $\Delta_k \subset M$ we trivialize the bundle on Δ_k, and view the problem of extending σ_{k-1} over Δ_k as that of giving a homotopy of $\sigma_{k-1}|_{\partial\Delta_k}$ with the constant map; as long as $\pi_{k-1}(\mathbb{C}^n \setminus \{0\}) = 0$, we can do this.

We first encounter difficulty when $k = 2n$: since $\pi_{2n-1}(\mathbb{C}^n \setminus \{0\}) \cong \mathbb{Z}$, we may not be able to extend σ_{2n-1} over M_{2n}. As a measure of the obstruction, we define a simplicial $2n$-cochain $\alpha \in C^{2n}(M, \mathbb{Z})$, by

$$\alpha(\Delta_{2n}) = [\sigma_{2n-1}|_{\partial\Delta_k}] \in \pi_{2n-1}(\mathbb{C}^n \setminus \{0\}) \cong \mathbb{Z}.$$

One then proves two basic results:

(a) The cochain α is in fact a cocyle; and

(b) While the cocyle α depends on the choices made thus far in the process, its cohomology class does not.

We call the cohomology class $[\alpha] \in H^{2n}(M, \mathbb{Z})$ the *obstruction* to finding a nowhere zero section of E. (Note that the nonvanishing of $[\alpha]$ implies that we cannot have a nowhere zero section of E, but the converse is not true: the vanishing of $[\alpha]$ says only that we can find such a section over the $2n$-skeleton M_{2n}.) The topological Chern class $c_n(E)$ was originally defined ****reference!**** to be

$$c_n(E) = [\alpha] \in H^{2n}(M, \mathbb{Z}).$$

Now suppose that E is an algebraic bundle on a smooth variety M, and that E admits a regular section σ generically transverse to the zero section. In this case it is easy to prove that the topological Chern class $c_n(E)$ agrees with the algebraic Chern class we defined previously. To this end, we can choose our simplical complex on M to be transverse to the zero locus of σ, so that $V(\sigma)$ disjoint from the $(2n-1)$-skeleton of M, and intersects each $2n$-simplex of M transversely in a finite number of points. If we choose the section σ_{2n-1} of E^* over the $(2n-1)$-skeleton on M to be the restriction of σ to M_{2n-1}, then the obstruction cocyle α will associate to each $2n$-simplex Δ_{2n} the number of its points of intersection with $V(\sigma)$, counted with sign. The cohomology class of this cocyle is exactly the Poincaré dual of the fundamental class of $V(\sigma)$, completing the proof.

The other Chern classes were defined in the topological setting analogously. To measure the obstruction to finding k everywhere independent

sections of E, we introduce the bundle of k-frames in E: that is, the space

$$F_k(E) = \{(p, v_1, \ldots, v_k) : p \in M \text{ and } v_1, \ldots, v_k \in E_p \text{ are independent.}\}$$

The fibers of $F_k(E)$ over M are *frame manifolds*

$$F_k = \{v_1, \ldots, v_k \in \mathbb{C}^n : v_1 \wedge \cdots \wedge v_k \neq 0\}$$

and a simple calculation with the long exact sequence in homotopy shows that we have

$$\pi_i(F_k) = \begin{cases} 0, & \text{if } i < 2n - 2k + 1; \text{ and} \\ \mathbb{Z}, & \text{if } i = 2n - 2k + 1. \end{cases}$$

The obstruction to finding a section of $F_k(E)$ is thus a class in $H^{2n-2k+2}(M, \mathbb{Z})$, which is now called the Chern class $c_{n-k+1}(E)$.

21.7.2 Chern Classes and Curvature

In ****reference****, Chern found another way to characterize the classes that came to bear his name, at least in case the underlying space M was a differentiable manifold and the complex vector bundle E on M a differentiable bundle.

What Chern did was first to choose a Riemannian metric on the manifold M and a Hermitian metric on the vector bundle E—that is, a Hermitian inner product on each fiber of the bundle, varying differentiably with the point, so that in terms of a local trivialization of E, the entries of Hermitian matrix giving the inner product on E_p are $\mathcal{C}^\infty$ functions of p. One thing a metric on the bundle E does for us is to give us a notion of *parallel transport*: given an arc γ in M, there is a canonical way to identify the fibers of E over the points of the arc.

Now, suppose $p \in M$ is a point and $v, w \in T_pM$ a pair of tangent vectors at p. We can form a small geodesic square with a vertex at p, and sides at p geodesics of length ϵ in the directions v and w.

****picture here****

Parallel-transporting around the perimeter of this square yields an automorphism $\varphi(v, w, \epsilon)$ of the fiber E_p; letting $\epsilon \to 0$ we arrive at an endomorphism $A(v, w)$ of E_p,

$$A(v, w) = \lim_{\epsilon \to 0} \frac{\varphi(v, w, \epsilon) - I}{\epsilon},$$

which is the tangent vector at 0 to the arc $\{\varphi(v, w, \epsilon)\}$ in $Aut(E_p)$.

The endomorphism $A(v, w)$ is bilinear and skew-symmetric in v and w, so that we can think of A as an element of $Hom(\wedge^2 T_pM, End(E_p))$; as p

varies this gives us a global section of the bundle

$$\Omega^2_M \otimes End(E)—$$

that is, in terms of a local trivialization of E, and $n \times n$ matrix Θ of differentiable 2-forms on M, called the *curvature matrix* of the metric.

If we change our trivialization of E, the matrix Θ is replaced by its conjugate under the change of basis matrix. The coefficients of the characteristic polynomial of this matrix are thus well-defined global forms ω_{2k} on M for $k = 0, 1, \ldots, n$; what Chern showed was that

(a) the forms ω_{2k} are closed;

(b) the de Rham cohomology classes $[\omega_{2k}] \in H^{2k}_{dR}(M, \mathbb{C})$ are independent of the choice of metric; and

(c) these classes coincide with the images of the classes $c_k(E) \in H^{2k}(M, \mathbb{Z})$ defined by Steifel and Whitney, under the natural map $H^{2k}(M, \mathbb{Z}) \to H^{2k}(M, \mathbb{C}) = H^{2k}_{dR}(M, \mathbb{C})$.

In differential geometry texts, accordingly, the Chern classes are often defined to be the de Rham classes of the forms ω_k. It's much less clear than in the case of the obstruction-theoretic characterization of the Chern classes that these are in fact the same as the classes of degeneracy loci of generically transverse sections; a proof can be found in **.

22

Appendix: Lefschetz Hyperplane Theorem and Applications

22.1 Statement of the Lefschetz Hyperplane Theorem

The Lefschetz hyperplane theorem is a basic tool for studying the topology of projective varieties. It says that if $X \subset Y$ is a smooth, ample divisor, then in all but one dimension the cohomology of X is induced (in a sense we'll make precise below) from that of Y. This result offers the possibility of describing the cohomology of a variety Y by relating it to the cohomology of a hyperplane section, though in practice many of its applications go in the opposite direction, describing for example the cohomology of a hypersurface $X \subset \mathbb{P}^n$ via its relation to the known cohomology of $\mathbb{P}^n$.

Theorem 22.1 (Lefschetz Hyperplane Theorem)**.** *Let Y be a smooth projective variety, $X \subset Y$ a divisor. If X is smooth and ample, then the restriction maps*

$$H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

is an isomorphism for $k < \dim(X)$ and injective for $k = \dim(X)$. Likewise the maps

$$H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$$

is an isomorphism for $k < \dim(X)$ and surjective for $k = \dim(X)$.

By Poincaré duality, the groups $H^k(X, \mathbb{Z})$ for $k > \dim(X)$ are isomorphic to the groups $H_k(X, \mathbb{Z})$ with $k < \dim(X)$, so that this describes all the

cohomology groups of X in terms of those of Y except for one, the middle-dimensional cohomology of X.

We remark that neither of the two hypotheses on X can be weakened, at least as the statement stands. For example, if we replace Y by its blow up at a point away from X, we introduce an additional class in $H^2(Y, \mathbb{Z})$ that maps to zero under the restriction map to $H^2(X, \mathbb{Z})$. Since the divisor X on the blow-up is nearly ample—it's nef and big, if those words mean anything to you—this suggests that we can't replace the condition of ampleness on X with anything weaker.

Likewise, the condition that X be smooth is clearly necessary: we can take any projective variety $\tilde{X}$ of dimension $n \geq 2$ with nontrivial first homology $H_1(\tilde{X}, \mathbb{Z})$ take a generic projection $\tilde{X} \to \mathbb{P}^{n+1}$. The image $X \subset \mathbb{P}^{n+1}$ will still have nontrivial first homology, contradicting the conclusion of the Lefschetz theorem.

For a quick proof of the Theorem, see Andreotti and Frankel [1959]. Lefschetz' original proof ****What's the reference?**** applied only to very ample divisors $X \subset Y$, that is, divisors that could be realized as hyperplane sections of Y under some embedding $Y \subset \mathbb{P}^N$ in projective space (hence the name of the theorem). In this setting, Lefschetz took a general pencil of divisors $\{X_\lambda\}_{\lambda \in \mathbb{P}^1}$ including X, and so gave a map

$$\pi : \tilde{Y} = \mathrm{Bl}_\Gamma(Y) \to \mathbb{P}^1$$

from the blow-up of Y along the base locus $\Gamma = \cap X_\lambda$ of this pencil to $\mathbb{P}^1$. The map π is a fiber bundle except at the finite number of values $\lambda \in \mathbb{P}^1$ with X_λ singular; by studying the local geometry of the map around these points, Lefschetz was able to relate the cohomology of the variety $\tilde{Y}$, and hence that of Y, to that of X. Lefschetz' analysis has a great deal to tell us beyond the statement of the Lefschetz hyperplane theorem, and we will describe it in Section 22.4 below.

The simplest proof of the Lefschetz hyperplane theorem itself, however, is via a different approach altogether. The hypothesis that X is ample in Y says that the line bundle $L = \mathcal{O}_Y(X)$ has a Hermitian metric with positive curvature. If we let σ be the global section of L vanishing on X, then, the length/absolute value $\|\sigma\|$ of σ in terms of this metric is a Morse function on Y with minimum 0 along X; we may apply Morse theory and use the curvature statement to bound the index of the critical points of $\|\sigma\|$. We are led to the statement that Y *has the homotopy type of a space obtained by attaching cells of dimension* $\geq \dim(X)$ *to* X, from which the Lefschetz theorem follows. This argument is beautifully written up in [Milnor]. (** David: who did this first?)

22.2 Cohomology of hypersurfaces

As you might expect, the application of the Lefschetz hyperplane theorem to a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ (or more generally, any smooth complete intersection of dimension n) yields a great deal of information about the topology of X. It also yields quite a bit about the geometry of X, as we'll see.

The statement of the Lefschetz theorem is simple enough in this case: it says that for any $k < n$, the map

$$H_k(X, \mathbb{Z}) \longrightarrow H_k(\mathbb{P}^{n+1}, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even; and} \\ 0, & \text{if } k \text{ is odd} \end{cases}$$

induced by the inclusion $X \subset \mathbb{P}^{n+1}$ is an isomorphism; likewise, the restriction map

$$H^k(\mathbb{P}^{n+1}, \mathbb{Z}) \longrightarrow H^k(X, \mathbb{Z})$$

is an isomorphism. In case $k = n$, it says that the former map is surjective and the latter injective, which does not tell us so much: if n is odd, it says nothing; and if $n = 2m$ is even it simply says that the restriction to X of the m^{th} power ω^m of the hyperplane class is nonzero (equivalently, the cycle of intersection of X with a general $(n-m)$-plane in $\mathbb{P}^{n+1}$ is not homologous to zero) which we knew anyway: its square is the degree of $X \subset \mathbb{P}^{n+1}$, which is positive. We can draw one conclusion in general about the middle-dimensional cohomology of X, though: since we know from the above that $H^k(X, \mathbb{Z})$ and $H_k(X, \mathbb{Z})$ are torsion-free for all $k \neq n$, we may deduce tfrom the universal coefficient theorem that *the middle-dimensional cohomology $H^n(X, \mathbb{Z})$ is torsion-free as well.*

For $k < n$, the Lefschetz theorem tells us more. For example, it says that if if $Z \subset X$ is any subvariety of codimension $k < n/2$ in X, then the class $[Z] \in H^{2k}(X, \mathbb{Z})$ of Z is the restriction to X of an integral cohomology class on $\mathbb{P}^{n+1}$: in other words,

$$[Z] = (\alpha \cdot \omega^k)|_X$$

for some $\alpha \in \mathbb{Z}$. In particular, if X has degree d, this says that

$$\begin{aligned} \deg(Z) &= ([Z] \cdot \omega^{n-k})_X \\ &= (\alpha \cdot \omega^k \cdot \omega^{n-k} \cdot [X])_{\mathbb{P}^{n+1}} \\ &= (\alpha \cdot \omega^k \cdot \omega^{n-k} \cdot d\omega)_{\mathbb{P}^{n+1}} \\ &= \alpha \cdot d. \end{aligned}$$

so that we have as a first consequence the

Theorem 22.2. *Let X be a smooth complete intersection of dimension n and degree d in projective space. If $Z \subset X$ is any subvariety with* $\dim(Z) >$

$\dim(X)/2$, *then the degree of* Z *is divisible by* d, *and is properly divisible by* d *if it is not a plane section of* X.

A special case is the

Corollary 22.3. *A smooth complete intersection in projective space cannot contain a linear space of more than half its dimension.*

We have mentioned this before (** lines on hypersurfaces); it is also not hard to see by elementary considerations:

Exercise 22.4. Prove this corollary directly by looking at the partial derivatives of the defining equations of X along a linear subspace $\Lambda \subset X$. (This is easy for hypersurfaces X, slightly harder for general complete intersections.)

Putting the shoe on the other foot, we have another amusing consequence:

Corollary 22.5. *If* $Z \subset \mathbb{P}^{n+1}$ *is any nondegenerate subvariety of dimension* $> n/2$ *and prime degree, then* Z *is contained in no smooth hypersurface.*

Thus, for example, a cubic scroll $S \subset \mathbb{P}^4$ lies on no smooth threefold in $\mathbb{P}^4$.

Exercise 22.6. Explain why this does not contradict the Bertini theorem. More generally, prove a sort of converse: that if $Z \subset \mathbb{P}^{n+1}$ is any smooth subvariety of dimension $\leq n/2$, then Z is contained in a smooth hypersurface.

We can strengthen this substantially in the case of subvarieties $Z \subset X$ of codimension 1, where we have the theory of divisors and line bundles to help us. For the following, then, let X be a smooth complete intersection of dimension $n \geq 3$ in projective space, and let $Z \subset X$ be any subvariety of dimension $n-1$. Since $n > 1$, to begin with, we have

$$H^1(X, \mathbb{Z}) = 0,$$

so that in particular every line bundle on X is determined by its topological Chern class. Since $n > 2$, moreover, every cohomology class of degree 2 on X is the restriction of a class on projective space. It follows that every line bundle on X is the restriction to X of a line bundle on projective space, that is, *every line bundle* L *on* X *is* $\mathcal{O}_X(m)$ *for some* m.

**David: this is awkward. How much do we assume the reader knows about the Picard group? Should we invoke the exponential sheaf sequence here? Do we need a separate argument that the maps $H^k(\mathcal{O}_{\mathbb{P}}) \to H^k(\mathcal{O}_X)$ are isomorphisms for $k < n$?

Now suppose for the moment that X is a hypersurface of degree d in $\mathbb{P}^{n+1}$. From the sheaf sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n}+1}(m-d) \to \mathcal{O}_{\mathbb{P}^{n}+1}(m) \to \mathcal{O}_X(m) \to 0$$

and the fact that $H^1(\mathcal{O}_{\mathbb{P}^{n}+1}(m-d)) = 0$, we deduce that every global section of $\mathcal{O}_X(m)$ is the restriction to X of a global section of $\mathcal{O}_{\mathbb{P}^{n}+1}(m)$—that is, a homogeneous polynomial of degree m. The same argument, applied inductively, yields the same statement for complete intersections.

Putting this together, we see that if $Z \subset X$ is a divisor, the line bundle $\mathcal{O}_X(Z) = \mathcal{O}_X(m)$ for some m, and moroever the global section of $\mathcal{O}_X(Z)$ vanishing on Z is the restriction to X of a homogeneous polynomial of degree m. In other words, we have proved the

Theorem 22.7. *If X is a smooth complete intersection of dimension $n \geq 3$ in projective space, any subvariety $Z \subset X$ of codimension 1 in X is the intersection of X with a hypersurface.*

We can thus strengthen Corollary ** above substantially in some cases: for example, we have the

Corollary 22.8. *Let $S \subset \mathbb{P}^4$ be a smooth surface. If S is not a complete intersection, it lies on no smooth threefold in $\mathbb{P}^4$; if $S = X \cap Y$ is a complete intersection of threefolds X and Y of degrees e and f, it lies on on no smooth threefold of degree $d \neq e, f$ in $\mathbb{P}^4$.*

At the other end of the spectrum, we note another corollary of the Lefschetz theorem:

Corollary 22.9. *Let X be a smooth complete intersection of dimension n in projective space. If $Y, Z \subset X$ are subvarieties of the same dimension $k \neq n/2$ and degree d, then Y and Z are homologous in X.*

Thus, for example, the $2,875$ lines we have found on a smooth quintic threefold are all homologous to one another. This provides the simplest example of a discrepancy between the notions of homological and algebraic equivalence: while Lefschetz assures us that these lines are cobordant to each other in the category of $\mathcal{C}^\infty$ submanifolds of X, they can't be deformed into each other as algebraic subvarieties. In fact, though it's not readily apparent, they are linearly independent in the group of cycles mod algebraic equivalence on X (** ref).

As a final application of the Lefschetz theorem for hypersurfaces in projective space, we will use it to determine the Betti numbers of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d. This is straightforward: Lefschetz tells us all the Betti numbers of X except the middle one $b_n(X)$, so that knowing the Euler characteristic of X will determine b_n. The Euler characteristic in

turn is given to us as the top chern class of the tangent bundle of X, which we can calculate from the standard normal bundle sequence

$$0 \to T_X \to (T_{\mathbb{P}^{n+1}})|X \to N_{X/\mathbb{P}^{n+1}} \to 0.$$

As we have seen, the normal bundle $N_{X/\mathbb{P}^{n+1}}$ of X in $\mathbb{P}^{n+1}$ is simply $\mathcal{O}_X(d)$; so, letting ω as usual denote both the hyerplane class in $\mathbb{P}^{n+1}$ and its restriction to X, we have

$$\begin{aligned} c(T_X) &= \frac{c(T_{\mathbb{P}^{n+1}})|X}{N_{X/\mathbb{P}^{n+1}}} \\ &= \frac{(1+\omega)^{n+2}}{1+d\omega} \\ &= \left(1+(n+2)\omega+\binom{n+2}{2}\omega^2+\dots\right)\left(1-d\omega+d^2\omega^2+\dots\right). \end{aligned}$$

Taking the top chern class, then, and remembering that $(\omega^n)_X = d$ we have

$$\chi(X) = c_n(T_X) = \sum_{k=0}^{n}(-1)^k\binom{n+2}{k+2}d^{k+1}.$$

The middle Betti number b_n of X is then either this number minus n, if n is even, or $n-1$ minus this number if n is odd.

In table 22.1 we give the results of this calculation in a few of the cases where it is actually used.

X	$\chi(X)$	$b_n(X)$
quadric surface	4	2
cubic surface	9	7
quartic surface	24	22
quintic surface	55	53
quadric threefold	4	0
cubic threefold	−6	10
quartic threefold	−56	60
quintic threefold	−200	204
quadric fourfold	6	2
cubic fourfold	27	23

TABLE 22.1. Euler Characteristics of Favorite Hypersurfaces

Note that some of these we already knew: the quadric surface is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$, from which we can see directly both the Euler characteristic and the second Betti number; the quadric fourfold may also be viewed as the Plücker embedding of the Grassmannian $\mathbb{G}(1,3)$, whose middle cohomology has basis given by the two Schubert cycles $\sigma_{1,1}$ and σ_2.

Exercise 22.10. Find the Betti numbers of the smooth intersection of a quadric and a cubic hypersurface in $\mathbb{P}^4$, and of the intersection of three quadrics in $\mathbb{P}^5$. You should recognize the answer(s): both of these are examples of *K3 surfaces*, which are diffeomorphic to a smooth quartic surface in $\mathbb{P}^3$.

Exercise 22.11. Find the Betti numbers of the smooth intersection of a two quadrics in $\mathbb{P}^5$. This is the famous *quadric line complex*, about which you can read more than you want to know in [GH], Chapter 6.

Exercise 22.12. Find the Betti numbers of a smooth hypersurface of bidegree (a, b) in $\mathbb{P}^m \times \mathbb{P}^n$.

Exercise 22.13. Show that the cohomology groups of a smooth quadric threefold $Q \subset \mathbb{P}^4$ are isomorphic to those of $\mathbb{P}^3$ ($\mathbb{Z}$ h in even dimensions, 0 in odd), but its cohomology ring is different (the square of the generator of $H^2(Q, \mathbb{Z})$ is twice the generator of $H^4(Q, \mathbb{Z})$). (This is a useful example if you're ever teaching a course in algebraic topology.)

22.3 Application: points on the universal hypersurface

There are many application of the Lefschetz hyperplane theorem that may not be obvious unless you're in the habit of looking for them. We give a representative one here.

The problem in question is a completely elementary algebraic one: we ask simply whether the generic polynomial

$$\sum a_{i_1,i_2,\dots,i_n} z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}$$

of degree d in n variables can be solved in rational functions of the coefficients. In other words, the question is whether exist rational functions $z_\alpha = z_\alpha(a_I)$ such that

$$\sum a_{i_1,i_2,\dots,i_n} z_1(a_I)^{i_1} z_2(a_I)^{i_2} \dots z_n(a_I)^{i_n}$$

vanishes identically as a rational function of the a_I. There are many solutions when $d = 1$, of course; and we can also answer the question when $d = 2$ and $n = 1$: the solutions of the quadric polynomial

$$ax^2 + bx + c$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and so are not rational functions of the coefficients a, b and c. But even in the next simplest case $d = n = 2$, where we are asking whether there exist rational functions x, y of six variables a, b, c, d, e, f such that

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

the answer is not immediately clear.

In fact, the answer is that no such solution exists whenever $d > 1$, and we will prove it now. This may seem like a highly unlikely application of a theorem on the topology of algebraic varieties;[1] but it doesn't seem to be easy to prove algebraically (for one way, see (** Zinovy R.).

To set this up, we projectivize and express the question in geometric language. First, let $\mathbb{P}^N$ be the projective space parametrizing polynomials of degree at most d in n variables, or equivalently homogeneous polynomials of degree d in $n+1$ variables, mod scalars. Next, in the product $\mathbb{P}^N \times \mathbb{P}^n$ we introduce the incidence correspondence, called the *universal hypersurface of degree d in $\mathbb{P}^n$*, given by

$$\Phi = \{(X, p) : p \in X\} \subset \mathbb{P}^N \times \mathbb{P}^n.$$

The fibers of Φ over a point X in $\mathbb{P}^N$ is a copy of the hypersurface $X \subset \mathbb{P}^n$. What we are asking in our present problem is whether there exists a rational section of Φ over $\mathbb{P}^N$: that is, a rational map

$$\sigma : \mathbb{P}^N \to \Phi$$

such that the composition $\pi \circ \sigma$ of σ with the projection $\pi : \Phi \to \mathbb{P}^N$ is the identity as a rational map. To put it another way, we are asking whether it is possible, at least over an open subset $U \subset \mathbb{P}^N$, to pick out one point on each hypersurface $X \in U$.

Suppose that there were such a map. Its image (the image of a rational map is defined in general to be the closure of the image of the map where defined) would then be a subvariety $Z \subset \Phi$ that would map generically 1-1 onto $\mathbb{P}^N$—in other words, a subvariety of Φ of codimension $n-1$ that would have intersection number 1 with the fiber of Φ over $\mathbb{P}^N$.

To see whether this is possible, we look at the cohomology of Φ. We observe first that Φ is smooth: its fibers over $\mathbb{P}^N$ may be singular, but it is a projective bundle over $\mathbb{P}^n$). Next, we observe that it is an ample divisor in $\mathbb{P}^N \times \mathbb{P}^n$ (any divisor in a product of two projective spaces other than a pullback from one factor is in fact very ample). Thus the Lefschetz theorem may be applied; and since as long as $d > 1$ we have

$$n - 1 < (N + n - 1)/2$$

[1] though it could be pointed out that the fundamental theorem of algebra is proved by topology

it says that the restriction map

$$H^{n-1}(\mathbb{P}^N \times \mathbb{P}^n, \mathbb{Z}) \to H^{n-1}(\Phi, \mathbb{Z})$$

is an isomorphism. Now, the cohomology of $\mathbb{P}^N \times \mathbb{P}^n$ is generated by the pullbacks α and β to $\mathbb{P}^N \times \mathbb{P}^n$ of the hyperplane classes on $\mathbb{P}^N$ and $\mathbb{P}^n$. Thus, if $Z \subset \Phi$ is any subvariety of codimension $n-1$, its class may be written as

$$[Z] = \left(c_0\alpha^{n-1} + c_1\alpha^{n-2}\beta + \cdots + c_{n-2}\alpha\beta^{n-2} + c_{n-1}\beta^{n-1}\right)|_\Phi$$

with $c_i \in \mathbb{Z}$. The intersection number of Z with a fiber of Φ over $\mathbb{P}^N$ may then be calculated by

$$\begin{aligned}([Z]\cdot\alpha^N)_X &= \left((c_0\alpha^{n-1} + \cdots + c_{n-1}\beta^{n-1})\cdot\alpha^N\cdot(\alpha+d\beta)\right)_{\mathbb{P}^N\times\mathbb{P}^n} \\ &= d\cdot c_{n-1}.\end{aligned}$$

In other words, *the degree over $\mathbb{P}^N$ of any codimension $n-1$ subvariety of Φ is divisible by d*, and hence no section exists. □

22.4 Extensions and generalizations

There are numerous ways of extending the basic Lefschetz theorem: we may try to weaken some of the hypotheses (such as the smoothness of the divisor $X \subset Y$, or its codimension; we can try to prove analogous statements for quasi-projective varieties, and we can ask whether similar statements hold for homotopy as well as homology groups. All of these are the subject of theorems, which we will now describe.

To begin with, perhaps the simplest extension is from looking at the zero locus X of a section of an ample line bundle on Y to looking at the zero locus of a section of an ample vector bundle. The statement here is a direct generalization of Lefschetz:

Theorem 22.14. *Let Y be a projective variety, E an ample vector bundle of rank r on Y, and $X \subset Y$ the zero locus of a section of E. If X is smooth and of codimension r in Y, then the restriction map*

$$H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

is an isomorphism for $k < \dim(X)$ and injective for $k = \dim(X)$.

A harder question is whether we can extend the Lefschetz theorem to arbitrary smooth subvarieties $X \subset Y$ of low codimension. The problem here is that we need some sort of ampleness condition on X, and it's not clear how to express this. One case where this is automatic, however, is

when the ambient space Y is projective space; there every subvariety is ample in every sense. In this case we have the Barth Theorem:

Theorem 22.15. *If $X \subset \mathbb{P}^{n+r}$ is any smooth subvariety of codimension r in projective space, then the restriction map*

$$H^k(\mathbb{P}^{n+r}, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

is an isomorphism for $k < \dim(X) - r$ and injective for $k = \dim(X) - r$.

22.5 Open problems

****what are they?****

23

Solutions to Selected Exercises

23.1 Chapter 1

Exercise 23.1. 1.44 Let $\nu = \nu_{2,2} : \mathbb{P}^2 \to \mathbb{P}^5$ be the quadratic Veronese map. If $C \subset \mathbb{P}^2$ is a plane curve of degree d, show that the image $\nu(C)$ has degree $2d$. (In particular, this means that the Veronese surface $S \subset \mathbb{P}^5$ contains only curves of even degree!)

Solution to Exercise 1.44. □

Exercise 23.2. 1.45 More generally, let $\nu = \nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$ be the degree d Veronese map. If $X \subset \mathbb{P}^n$ is a variety of dimension k and degree e, show that the image $\nu(X)$ has degree $d^k e$.

Solution to Exercise 1.45. □

Exercise 23.3. 1.46 Let $\sigma = \sigma_{r,s} : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{(r+1)(s+1)-1}$ be the Segre map, and let $X \subset \mathbb{P}^r \times \mathbb{P}^s$ be a subvariety of codimension k. If the class $[X] \in A^k(\mathbb{P}^r \times \mathbb{P}^s)$ is given by

$$[X] = c_0\alpha^k + c_1\alpha^{k-1}\beta + \cdots + c_k\beta^k$$

(where α and $\beta \in A^1(\mathbb{P}^r \times \mathbb{P}^s)$ are the pullbacks of the hyperplane classes, and we take $c_i = 0$ if $i > s$ or $k - i > r$),

(a) Show that all $c_i \geq 0$.

(b) Calculate the degree of the image $\sigma(X) \subset \mathbb{P}^{(r+1)(s+1)-1}$; and, using this and the first part,

(c) Show that any linear space $\Lambda \subset \Sigma_{r,s} \subset \mathbb{P}^{(r+1)(s+1)-1}$ contained in the Segre variety lies in a fiber of a projection map.

Solution to Exercise 1.46. □

Exercise 23.4. 1.47 Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ be the rational map given by

$$\varphi : (x_0, x_1, x_2) \mapsto \left(\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}\right),$$

or, equivalently,

$$\varphi : (x_0, x_1, x_2) \mapsto (x_1x_2, x_0x_2, x_0x_1)$$

and let $\Gamma_\varphi \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the graph of φ. Find the class

$$[\Gamma_\varphi] \in A^2(\mathbb{P}^2 \times \mathbb{P}^2).$$

Solution to Exercise 1.47. □

Exercise 23.5. 1.48 Let $X_1, \ldots, X_n \subset \mathbb{P}^n$ be hypersurfaces of degrees $d_1, \ldots, d_n$. Let $p \in \mathbb{P}^n$ be a point, and suppose that the hypersurface X_i has multiplicity m_i at p; suppose moreover that the intersection of the projective tangent cones $\mathbb{P}TC_pX_i$ to X_i at p is empty. Use the description of the Chow ring of the blow-up of $\mathbb{P}^n$ at p to show that the number of points of intersection of the X_i away from p is

$$\#\left(\bigcap (X_i \setminus \{p\})\right) = \prod d_i - \prod m_i.$$

Solution to Exercise 1.48. □

Exercise 23.6. 1.49 Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. Suppose that X has an ordinary double point (that is, a point $p \in X$ such that the projective tangent cone $\mathbb{P}TC_pX$ is a smooth quadric) and is otherwise smooth. What is the degree of the dual hypersurface $X^* \subset \mathbb{P}^{n*}$?

Solution to Exercise 1.49. □

Exercise 23.7. 1.50 Let $X \subset \mathbb{P}^n$ be a variety of degree d and dimension k; suppose that $p \in X$ is a point of multiplicity m. Let $B = Bl_p(\mathbb{P}^n)$ be the blow-up of $\mathbb{P}^n$ at the point p, and $\tilde{X} \subset B$ the proper transform of X in B. Find the class $[\tilde{X}] \in A^{n-k}(B)$.

Solution to Exercise 1.50. □

Exercise 23.8. 1.51 Let $p \in X \subset \mathbb{P}^n$ be as in the preceding Exercise, and suppose that the projection map $\pi_p : X \to \mathbb{P}^{n-1}$ is birational onto its image. What is the degree of $\pi_p(X)$?

Solution to Exercise 1.51. □

Exercise 23.9. 1.52Show that if X is an irreducible plane cubic with a node, then $c_1 : Pic(X) \to A_1(X)$ is not a monomorphism as follows. Show that there is no birational map from X to $\mathbb{P}^1$. Use this to show that if $p \neq q \in X$ are smooth points then the line bundles $\mathcal{O}_X(p)$ and $\mathcal{O}_X(q)$ are non-isomorphic. Show, however, that the zero loci of their unique sections, the points p and q, are rationally equivalent.

Solution to Exercise 1.52. □

Exercise 23.10. 1.53 Let X be a quadric cone in $\mathbb{P}^3$ and let B be the blowup of X at the origin. Show that B is a smooth surface. Show that if $X \subset \mathbb{P}^3$ a quadric cone with vertex p then $A_1(X) = \mathbb{Z}$, generated by the class of a line, and the image of $c_1 : Pic(X) \to A_1(X)$ is $2\mathbb{Z}$ by showing that the image consists of the subgroup of classes of curves lying on X that have even degree as curves in $\mathbb{P}^3$. In particular, the class of a line on X is not in the image. Do this by showing that any curve C passing through the vertex of X no curve $C \subset X$ of odd degree can be a Cartier divisor on X: if such a curve meets the general line of the ruling of X at δ points away from p and has multiplicity m at p, then intersecting C with a general plane through p we see that $\deg(C) = 2\delta + m$; it follows that m is odd, and hence that C cannot be Cartier at p. Thus, the class $[M]$ of a line of the ruling cannot be $c_1(L)$ for any line bundle L.

Solution to Exercise 1.53. □

Exercise 23.11. 1.54 Show that the Chow ring of a product of projective spaces $\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k}$ is

$$\begin{aligned} A(\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k}) &= \bigotimes A(\mathbb{P}^{r_i}) \\ &= \mathbb{Z}[\alpha_1, \ldots, \alpha_k]/(\alpha_1^{r_1+1}, \ldots, \alpha_k^{r_k+1}), \end{aligned}$$

where $\alpha_1, \ldots, \alpha_k$ are the pullbacks of the hyperplane classes from the factors. Use this to calculate the degree of the image of the Segre embedding

$$\sigma : \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k} \hookrightarrow \mathbb{P}^{(r_1+1)\cdots(r_k+1)-1}$$

corresponding to the multilinear map $V_1 \times \cdots \times V_k \to V_1 \otimes \cdots \otimes V_k$.

Solution to Exercise 1.54. The description of the Chow ring of $\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_t}$ follows by exactly the same logic as in the two-factor case. Similarly, if $\zeta \in A^1(\mathbb{P}^{(r_1+1)\cdots(r_k+1)-1})$ is the hyperplane class in the target projective

space, we see that $\sigma^*\zeta = \alpha_1 + \cdots + \alpha_k$, and the degree of the image is correspondingly

$$\deg\,(\alpha_1 + \cdots + \alpha_k)^{r_1+\cdots+r_k} = \binom{r_1 + \cdots + r_k}{r_1, \ldots, r_k} = \frac{(r_1 + \cdots + r_k)!}{r_1! \cdots r_k!}.$$

□

Exercise 23.12. 1.55 For $t \neq 0$, let $A_t : \mathbb{P}^r \to \mathbb{P}^r$ be the automorphism

$$[X_0, X_1, X_2, \ldots, X_r] \mapsto [X_0, tX_1, t^2X_2, \ldots, t^rX_r].$$

Describe the limit, as $t \to 0$, of the graph of A_t in $\mathbb{P}^r \times \mathbb{P}^r$: that is, let $\Phi \subset \mathbb{A}^1 \times \mathbb{P}^r \times \mathbb{P}^r$ be the closure of the locus

$$\tilde{\Phi} = \{(t, p, q) : t \neq 0 \text{ and } q = A_t(p)\}.$$

Describe the fiber Φ_0 of Φ over the point $t = 0$, and deduce once again the formula of Section 1.2.3 for the class of the diagonal in $\mathbb{P}^r \times \mathbb{P}^r$.

Solution to Exercise 1.55. In terms of coordinates $([X_0, \ldots, X_r], [Y_0, \ldots, Y_r])$ on $\mathbb{P}^r \times \mathbb{P}^r$, the limit is the locus

$$\Gamma = \bigcup_{i=0}^{r} V(X_0, \ldots, X_{i-1}) \times V(Y_{i+1}, \ldots, Y_r) \cong \bigcup_{i=0}^{r} \mathbb{P}^{r-i} \times \mathbb{P}^i,$$

whose class is visibly $\sum \alpha^i \beta^{r-i}$. To see the inclusion $\Phi_0 \subset \Gamma$, observe that the ideal of Φ includes the polynomials $X_iY_j - t^{j-i}X_jY_i$ for all $0 \leq i < j \leq r$; so the ideal of Φ_0 includes X_iY_j for all $i < j$. To see the inclusion in the opposite direction, show that

$$\big(0, [0, \ldots, 0, 1, X_{i+1}, \ldots, X_r], [Y_0, \ldots, Y_{i-1}, 1, 0, \ldots, 0]\big) = \lim_{t\to 0}\big(t, p, A_t(p)\big)$$

where

$$p = \big[t^iY_0, t^{i-1}Y_1, \ldots, tY_{i-1}, 1, X_{i+1}, \ldots, X_r\big].$$

□

Exercise 23.13. 1.56 Let

$$\Psi = \{(p, q, r) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n \mid p, q \text{ and } r \text{ are collinear in } \mathbb{P}^n\}.$$

(Note that this includes all diagonals.) Show that this is a closed subvariety of codimension $n - 1$ in $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$.

Solution to Exercise 1.56. If the points p, q and r are given by the homogeneous vectors $(X_0, \ldots, X_n)$, $(Y_0, \ldots, Y_n)$ and $(Z_0, \ldots, Z_n)$, then Ψ is the zero locus of the 3×3 minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & \ldots & X_n \\ Y_0 & Y_1 & \ldots & Y_n \\ Z_0 & Z_1 & \ldots & Z_n \end{pmatrix}.$$

These are homogeneous trilinear forms on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, from which we see that Ψ is indeed a closed subset of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$. Moreover, from the form of the equations—the 3×3 minors of a $3\times(n+1)$ matrix—we see that every component of Ψ has codimension at most $n-1$ (see for example Exercise 10.9 of Eisenbud [1995]). Since the projection $\Psi \to \mathbb{P}^n \times \mathbb{P}^n$ onto any two factors is surjective, with fibers $\mathbb{P}^1$ away from the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ and fibers $\mathbb{P}^n$ over Δ we can see that it's irreducible, and hence a subvariety of codimension $n-1$. □

Exercise 23.14. 1.57 With $\Psi \subset \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ as in the preceding exercise, use the method of undetermined coefficients to find the class

$$\psi = [\Psi] \in A^{n-1}\left(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n\right).$$

Assume characteristic 0, and use Kleiman's theorem as necessary.

Solution to Exercise 1.57. As suggested, we write

$$\psi = \sum_{i+j+k=n-1} c_{i,j,k}\alpha_1^i\alpha_2^j\alpha_3^k$$

and determine the coefficients by intersecting with cycles in complementary dimension: specifically, we see that

$$c_{i,j,k} = \deg(\psi \cdot \alpha_1^{n-i}\alpha_2^{n-j}\alpha_3^{n-k}).$$

To calculate this, we show that by Kleiman's theorem, for general subspaces $\mathbb{P}^i, \mathbb{P}^j$ and $\mathbb{P}^k \subset \mathbb{P}^n$ the product $\mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^k$ will intersect Ψ generically transversely, so that

$$\deg(\psi \cdot \alpha_1^{n-i}\alpha_2^{n-j}\alpha_3^{n-k}) = \#\big(\Psi \cap (\mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^k)\big) = 1$$

We conclude that

$$\psi = [\Psi] = \sum_{i+j+k=n-1} \alpha_1^i\alpha_2^j\alpha_3^k$$

□

Exercise 23.15. 1.58 To extend the result of the preceding exercise to characteristic p, show in arbitrary characteristic that for general subspaces $\mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^k \subset \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, the intersection $\Psi \cap \mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^k$ is transverse.

Solution to Exercise 1.58. There are many ways to do this, including simply writing everything out in suitably chosen coordinates. We'll use a trick here: if $\mathbb{P}^n$ is the projectivization of a vector space V, we let $U \subset V \times V \times V$ be the complement of the union $(\{0\}\times V\times V)\cup(V\times\{0\}\times V)\cup(V\times V\times\{0\})$, so that U maps to $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ with two-dimensional fibers. Now let

$$W = \{(v_1, v_2, v_3) : v_1 + v_2 + v_3 = 0\} \subset V \times V \times V$$

and let $P = A \times B \times C$ be the product of general linear subspaces of dimensions $i+1$, $j+1$ and $k+1$ in V; thus $W \cap U$ and $P \cap U$ map down to Ψ and $\mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^k$ with one- and two-dimensional fibers respectively.

The point is, W and P are simply linear subspaces of $V \times V \times V$, whose intersection has dimension $(i+1)+(j+1)+(k+1)-(n+1) = 1$ (and whose span is all of $V \times V \times V$). It follows that their images Ψ and $\mathbb{P}^i \times \mathbb{P}^j \times \mathbb{P}^k$ intersect transversely at one point. □

Exercise 23.16. 1.59 Suppose that $(F_0, \dots, F_r)$ and $(G_0, \dots, G_r)$ are general $(r+1)$-tuples of homogeneous polynomials in $r+1$ variables, of degrees d and e respectively, so that in particular the maps $f : \mathbb{P}^r \to \mathbb{P}^r$ and $g : \mathbb{P}^r \to \mathbb{P}^r$ sending x to $(F_0(x), \dots, F_r(x))$ and $(G_0(x), \dots, G_r(x))$ are regular. For how many points $x = (x_0, \dots, x_r) \in \mathbb{P}^r$ do we have $f(x) = g(x)$?

Solution to Exercise 1.59. Let Γ_f and $\Gamma_g \subset \mathbb{P}^r \times \mathbb{P}^r$ be the graphs of f and g, and γ_f and $\gamma_g \in A^r(\mathbb{P}^r \times \mathbb{P}^r$ the classes of their graphs. By Proposition 1.29, we have

$$\gamma_f = \sum_{i=0}^{r} d^i \alpha^i \beta^{r-i} \quad \text{and} \quad \gamma_f = \sum_{i=0}^{r} e^i \alpha^i \beta^{r-i}$$

So, after verifying transversality either by Kleiman in characteristic 0 or by calculation of tangent spaces in characteristic p, we have the answer

$$\#(\Gamma_f \cap \Gamma_g) = \deg(\gamma_f \cdot \gamma_g) = \sum_{i=0}^{r} d^i e^{r-i}.$$

□

Exercise 23.17. 1.61 Consider the locus $\Phi \subset (\mathbb{P}^2)^4$ of fourtuples of collinear points. Find the class $\varphi = [\Phi] \in A^2((\mathbb{P}^2)^4)$ of Φ by the method of undetermined coefficients; that is, by intersecting with cycles of complementary dimension.

Solution to Exercise 1.61. As suggested, we write

$$\varphi = [\Phi] = \sum_{i=1}^{4} c_i \alpha_i^2 + \sum_{1 \le i < j \le 4} d_{i,j} \alpha_i \alpha_j.$$

We then have

$$c_1 = \deg(\varphi \cdot \alpha_2^2 \alpha_3^2 \alpha_4^2) = \#\big(\Phi \cap (\mathbb{P}^2 \times \{p_0\} \times \{q_0\} \times \{r_0\})\big) = 0$$

for general (and in particular non-colinear) p_0, q_0 and $r_0 \in \mathbb{P}^2$; likewise, $c_i = 0$ for all i. After a transversality check, we have similarly

$$d_{1,2} = \deg(\varphi \cdot \alpha_1 \alpha_2 \alpha_3^2 \alpha_4^2) = \#\big(\Phi \cap (L \times M \times \{q_0\} \times \{r_0\})\big) = 1$$

for general lines $L, M \subset \mathbb{P}^2$ and points $q_0, r_0 \in \mathbb{P}^2$; likewise, $d_{i,j} = 1$ for all $i < j$. In sum, $\varphi = \sum_{i<j} \alpha_i \alpha_j$. □

Exercise 23.18. 1.62 With $\Phi \subset (\mathbb{P}^2)^4$ as in the preceding problem, calculate the class $\varphi = [\Phi]$ by using the result of Exercise 1.57 on the locus $\Psi \subset (\mathbb{P}^2)^3$ of triples of collinear points, and considering the intersection of the loci $\Psi_{1,2,3}$ and $\Psi_{1,2,4}$ of fourtuples (p_1, p_2, p_3, p_4) with p_1, p_2, p_3 colinear and with p_1, p_2, p_4 colinear.

Solution to Exercise 1.62. Let $\Psi_{i,j,k}$ be the locus where p_i, p_j and p_k are colinear, and $\psi_{i,j,k}$ its class. By Exercise 1.57 (or the calculation in Section 1.2.5),

$$\psi_{1,2,3} = \alpha_1 + \alpha_2 + \alpha_3 \quad \text{and} \quad \psi_{1,2,4} = \alpha_1 + \alpha_2 + \alpha_4$$

We observe that the loci $\Psi_{1,2,3}$ and $\Psi_{1,2,4}$ have intersection

$$\Psi_{1,2,3} \cap \Psi_{1,2,4} = \Phi \cup \Delta_{1,2}$$

where $\Delta_{1,2} = \{p_1 = p_2\}$ is the preimage of the diagonal under projection on the first two factors. Checking that this intersection is generically transverse (Kleiman won't help us here; we have to calculate tangent spaces), we have

$$\varphi + [\Delta_{1,2}] = (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_4) = \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2 + \sum i < j \alpha_i \alpha_j$$

and the result follows. □

Exercise 23.19. 1.63 Let A, B and $C : \mathbb{P}^2 \to \mathbb{P}^2$ be three general automorphisms. For how many points $p \in \mathbb{P}^2$ are the points $p, A(p), B(p)$ and $C(p)$ collinear?

Solution to Exercise 1.63. In $(\mathbb{P}^2)^4$, let Γ_A, Γ_B and Γ_C be the pullbacks, via the projections $\pi_{1,2}, \pi_{1,3}$ and $\pi_{1,4}$, of the graphs of A, B and C; let γ_A, γ_B and γ_C be their classes. Kleiman's theorem implies that the intersection $\Phi \cap \Gamma_A \cap \Gamma_B \cap \Gamma_C$ is transverse, so that the answer to the exercise is given by

$$\begin{aligned}
\#(\Phi \cap \Gamma_A \cap \Gamma_B \cap \Gamma_C) &= \deg(\varphi\gamma_A\gamma_B\gamma_C) \\
&= \deg\left(\sum_{i<j} \alpha_i\alpha_j \prod_{k=2}^{4} (\alpha_1^2 + \alpha_1\alpha_k + \alpha_k^2)\right) \\
&= \deg\left(\alpha_2\alpha_3\alpha_4 \sum_{i<j} \alpha_i\alpha_j \prod_{k=2}^{4} (\alpha_1 + \alpha_k)\right) \\
&= 6
\end{aligned}$$

□

Exercise 23.20. 1.65 Let B be the blowup of $\mathbb{P}^n$ at a point p as in Section 1.2.4 above, with classes λ_k, γ_k and e_k as described. Use the relation $e_{n-1} = \lambda_{n-1} - \gamma_{n-1}$ to describe the classes γ_k in terms of λ_k and e_k and vice versa.

Solution to Exercise 1.65. A k-plane contained in the exceptional divisor E is the transverse intersection of E with the proper transform of a $(k+1)$-plane in $\mathbb{P}^n$ containing p; that is,

$$e_k = e\gamma_{k+1}.$$

Now write $e = \lambda - \gamma$ and use the relations derived in Section 1.2.4 to express this as

$$e_k = (\lambda - \gamma)\gamma_{k+1} = \lambda_k - \gamma_k.$$

Put another way, the difference between the class of the preimage of a k-plane not containing p and the class of the proper transform of a k-plane containing p is the class of a k-plane contained in E. Note that except in case $k = n-1$ this is not directly visible! □

The next few exercises deal with the blow-up of $\mathbb{P}^3$ along a line. To fix notation, let $\pi : X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along a line $L \subset \mathbb{P}^3$; that is, the graph $X \subset \mathbb{P}^3 \times \mathbb{P}^1$ of the rational map $\pi_L : \mathbb{P}^3 \to \mathbb{P}^1$ given by projection from L; let $\alpha : X \to \mathbb{P}^1$ be projection on the second factor.

Exercise 23.21. 1.66 Let $H \subset \mathbb{P}^3$ be a plane containing L, and $\tilde{H} \subset X$ its proper transform. Let $J \subset \mathbb{P}^3$ be a plane transverse to L, $\tilde{J} \subset X$ its proper transform (which equals its preimage in X), and let $M \subset J$ be a line not meeting L. Show that the subvarieties

$$X, \quad \tilde{H}, \quad \tilde{J}, \quad \tilde{J} \cap \tilde{H}, \quad M, \quad M \cap \tilde{H}$$

are the closed strata of an affine stratification of X, with open strata isomorphic to affine spaces. In particular, since only one $(M \cap \tilde{H})$ is a point, deduce that $A^3(X) \cong \mathbb{Z}$.

Exercise 23.22. 1.67 Let $h = [\tilde{H}]$, $j = [\tilde{J}] \in A^1(X)$ and $m = [M] \in A^2(X)$ be the classes of the corresponding strata. Show that

$$h^2 = 0, \quad j^2 = m, \quad \text{and} \quad \deg(jm) = \deg(hm) = 1.$$

Conclude that

$$A(X) = \mathbb{Z}[h, j]/(h^2, j^3 - hj^2).$$

Solution to Exercise 1.67. As in the calculations in Section 1.2.4, there are lots of effective cycles representing these classes, so it's not hard to evaluate the intersection products in question. For example, h is the class of the proper transform $\tilde{H}$ of a plane $H \subset \mathbb{P}^3$ containing L; if $H' \subset \mathbb{P}^3$ is another such plane, with proper transform $\tilde{H}'$, then $\tilde{H}$ and $\tilde{H}'$ will be disjoint, whence $h^2 = 0$. Similarly, two general planes not containing L will intersect transversely in a line disjoint from L, so $j^2 = m$, and so on. □

Exercise 23.23. 1.68 Now let $E \subset X$ be the exceptional divisor of the blow-up, and $e = [E] \in A^1(X)$ its class. What is the class e^2?

Solution to Exercise 1.68. In contrast to the last problem, the class e has E as its only effective representative cycle. To calculate e^2, we observe that the preimage in X of a plane $H \subset \mathbb{P}^3$ containing L is the sum of its proper transform and the exceptional divisor, so that $j = h + e$ or equivalently $e = j - h$. It follows then that

$$e^2 = (j-h)^2 = m - 2hj.$$

□

Exercise 23.24. 1.70 Let $\mathbb{P}^5$ be the space of conic curves in $\mathbb{P}^2$.

(a) Find the dimension and degree of the locus of double lines (in characteristic $\neq 2$).

(b) Find the dimension and degree of the locus Δ of singular conics (that is, line pairs and double lines).

Solution to Exercise 1.70. For the first part, observe that the locus of double lines is the image of the map $\mathbb{P}^2 \to \mathbb{P}^5$ sending a linear form to its square; in characteristic $\neq 2$, this is the Veronese embedding.

The second part can be done in many ways. We can do it in the manner of Section 1.2.5: we realize the locus of singular conics as the image of $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$ sending a pair of linear forms to their product. This is the Segre map $\sigma_{2,2} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ followed by a linear projection. Since the projection has degree 2 onto its image, the degree of the image is $\frac{1}{2}\binom{4}{2} = 3$. Alternatively, if $L \subset \mathbb{P}^5$ is a general line—that is, a general pencil of conics—then L consists of all conics containing the base locus of the pencil, which consists of four points $p_1, \ldots, p_4$, no three collinear. The points of $L \cap \Delta$ correspond to line pairs containing $\{p_1, \ldots, p_4\}$, of which there are visibly 3. Finally, if we realize $\mathbb{P}^5$ as the space of 3×3 symmetric matrices, Δ is the zero locus of the determinant, which is a homogeneous cubic polynomial in the entries. □

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Exercise 23.25. 1.71 Let $\mathbb{P}^9$ be the space of plane cubics, and $\Gamma \subset \mathbb{P}^9$ the locus of reducible cubics. Let L and $C \subset \mathbb{P}^2$ be a line and a smooth conic intersecting transversely at two points $p, q \in \mathbb{P}^2$; let $L + C$ be the corresponding point of Γ. Show that Γ is smooth at $L + C$, with tangent space

$$\mathbb{T}_{L+C}\Gamma = \mathbb{P}\{\text{homogeneous cubic polynomials } F : F(p) = F(q) = 0\}.$$

Solution to Exercise 1.71. Let $\tau : \mathbb{P}^2 \times \mathbb{P}^5 \to \Gamma \subset \mathbb{P}^9$ be as in Section 1.2.5. Since (L, C) is the unique point in $\tau^{-1}(L + C)$, we need to show that the differential $d\tau$ is injective at (L, C), with image contained in the subspace

specified. Letting G and H be the homogeneous linear and quadratic polynomials defining L and C, we can represent a tangent vector to $\mathbb{P}^2$ at L as $G + \epsilon G'$, where G' is a linear form taken modulo G, and likewise we can represent a tangent vector to $\mathbb{P}^5$ at C as $H + \epsilon H'$, where H' is a quadratic form taken modulo H. Multiplying, we have

$$d\tau_{(L,C)}(G + \epsilon G', H + \epsilon H') = GH + \epsilon(G'H + GH').$$

Thus the image of $d\tau_{(L,C)}$ contains all cubics of the form $G'H + GH'$, which is to say all cubics vanishing at $L \cap C$, and the result follows. □

Exercise 23.26. 1.72 Using the preceding exercise, show that if $p_1, \ldots, p_7 \in \mathbb{P}^2$ are general points, and $H_i \subset \mathbb{P}^9$ is the hyperplane of cubics containing p_i, then the hyperplanes $H_1, \ldots, H_7$ intersect Γ transversely—that is, the degree of Γ is the number of reducible cubics through $p_1, \ldots, p_7$.

Solution to Exercise 1.72. Since the points p_i are general, no three are colinear and no six lie on a conic; thus if $L + C$ is any reducible cubic containing all 7, the line L must contain 2 and the conic C five. Say C contains $p_1, \ldots, p_5$; since p_6 and p_7 are general with respect to $p_1, \ldots, p_5$, L must be transverse to C and the points of $L \cap C$ disjoint from $p_1, \ldots, p_7$.

Now let p_8 and p_9 be the points of intersection $L \cap C$. By the preceding exercise, we need to show that $L + C$ *is the unique cubic containing* $\{p_1, \ldots, p_9\}$. But any such cubic contains the seven points $p_1, \ldots, p_5, p_8, p_9$ of C and the four points $9_6, \ldots, p_9$ of L, and so must contain both. □

Exercise 23.27. 1.73 Calculate the number of reducible plane cubics passing through 7 general points $p_1, \ldots, p_7 \in \mathbb{P}^2$, and hence, by the preceding exercise, the degree of Γ.

Solution to Exercise 1.73. By the preceding exercises, this is $\binom{7}{2} = 21$. □

Exercise 23.28. 1.74 We can also calculate the degree of the locus $\Sigma \subset \mathbb{P}^9$ of triangles (that is, totally reducible cubics) directly, as in the series of exercises starting with (1.71). To start, show that if $C = L_1 + L_2 + L_3$ is a triangle with the three vertices—that is, points $p_{i,j} = L_i \cap L_j$ of pairwise intersection—distinct, then Σ is smooth at C with tangent space

$$\mathbb{T}_{L+C}\Sigma = \mathbb{P}\{\text{homogeneous cubic polynomials } F : F(p_{i,j}) = 0 \ \forall\ i, j\}.$$

Solution to Exercise 1.74. We can represent the locus of triangles as the image of the map $\sigma : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$ sending a triple of linear forms to their product; we claim that at a point $(L_1, L_2, L_3) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ as in

the exercise, the differential $d\sigma$ is injective, with image as specified. As in Exercise 1.71, if $L_i = V(H_i)$ we write

$$d\sigma_{(L_1,L_2,L_3)}(H_1 + \epsilon H_1', H_2 + \epsilon H_2', H_3 + \epsilon H_3') \\ = H_1H_2H_3 + \epsilon(H_1'H_2H_3 + H_1H_2'H_3 + H_1H_2H_3');$$

the key is to see that the pairwise products H_iH_j generate the ideal of all polynomials vanishing on the points $p_{i,j}$. □

Exercise 23.29. 1.75 Using the preceding exercise,

(a) Show that if $p_1, \ldots, p_6 \in \mathbb{P}^2$ are general points, then the degree of Σ is the number of triangles containing $p_1, \ldots, p_6$; and

(b) Calculate this number directly.

Solution to Exercise 1.75. The fact that for $p_1, \ldots, p_6$ general the hyperplanes H_{p_i} intersect Σ transversely now follows as in the solution of Exercise 1.72: any triangle containing $\{p_1, \ldots, p_6\}$ must (after possibly reordering the points) consist of the union C of the lines $\overline{p_1, p_2}$, $\overline{p_3, p_4}$ and $\overline{p_5, p_6}$; we check that the points of pairwise intersection of these lines are distinct from each other and from $p_1, \ldots, p_6$, and finally deduce that the only cubic containing these three points in addition to $p_1, \ldots, p_6$ is C.

It follows that the degree of Σ is the number of triangles containing six general points, which is simply the number of ways of breaking up the points into three pairs; that is, $\frac{1}{6}\binom{6}{2,2,2} = 15$. □

Exercise 23.30. 1.76 Consider a general asterisk—that is, the sum $C = L_1 + L_2 + L_3$ of three distinct lines all passing through a point p. Show that the variety $\Sigma \subset \mathbb{P}^9$ of triangles is smooth at C, with tangent space the space of cubics double at p. Deduce that the space $A \subset \mathbb{P}^9$ of asterisks is also smooth at C.

Solution to Exercise 1.76. For the first part, this is the same calculation as in Exercise 1.74, the difference being that here the quadrics H_iH_j generate the ideal of polynomials double at p. For the second part, we observe that A is the image, under the restriction of the map $\sigma : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$, of the locus $\Phi \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ of triples of concurrent lines, and argue (by homogeneity, for example) that Φ is smooth at (L_1, L_2, L_3). □

Exercise 23.31. 1.77 Let $p_1, \ldots, p_5 \in \mathbb{P}^2$ be general points. Show that any asterisk containing $\{p_1, \ldots, p_5\}$ consists, after possibly relabelling the points, of the sum of the line $L_1 = \overline{p_1, p_2}$, the line $L_2 = \overline{p_3, p_4}$ and the line $L_3 = \overline{p_5, (L_1 \cap L_2)}$.

Solution to Exercise 1.77. Since the points p_i are general, no line contains three of them; hence two must contain two and one one. □

Exercise 23.32. 1.78 Using the preceding two exercises, show that for $p_1, \ldots, p_5 \in \mathbb{P}^2$ general points, the hyperplanes H_{p_i} intersect the locus $A \subset \mathbb{P}^9$ of asterisks transversely, and calculate the degree of A accordingly.

Solution to Exercise 1.78. This is slightly trickier than the preceding examples involving reducible cubics and triangles, since we don't have an explicit description of the tangent space $\mathbb{T}_C A \subset \mathbb{P}^9$. To set up, let $C = L_1 + L_2 + L_3$ be as in Exercise 1.77, and suppose that $C' \in \mathbb{T}_C A \cap H_{p_1} \cap \cdots \cap H_{p_5}$. Since $\mathbb{T}_C A$ is contained in the locus of cubics double at p, the cubic C' must contain L_1 and L_2, so any tangent vector to A contained in the hyperplanes H_{p_i} must be of the form

$$d\sigma_{(L_1,L_2,L_3)}(H_1, H_2, H_3 + \epsilon H_3') = H_1 H_2 H_3 + \epsilon(H_1 H_2 H_3').$$

Since the tangent vector $(H_1, H_2, H_3 + \epsilon H_3')$ lies in $T_{(L_1,L_2,L_3)}\Phi$, the linear form H_3' must vanish at p and so must be a multiple of H_3; thus $C' = C$.

Thus the degree of A is simply the number of asterisks containing $p_1, \ldots, p_5$; by Exercise 1.77 this is $\frac{1}{2}\binom{5}{2,2,1} = 15$. □

Exercise 23.33. 1.79 Show that (in characteristic $\neq 3$) the locus $Z \subset \mathbb{P}^9$ of triple lines is a cubic Veronese surface, and deduce that its degree is 9.

Solution to Exercise **??**. □

Exercise 23.34. 1.80 Let $X \subset \mathbb{P}^9$ be the locus of cubics of the form $2L + M$ for L and M lines in $\mathbb{P}^2$.

(a) Show that X is the image of $\mathbb{P}^2 \times \mathbb{P}^2$ under a regular map such that the pullback of a general hyperplane in $\mathbb{P}^9$ is a hypersurface of bidegree $(2, 1)$.

(b) Use this to find the degree of X.

Solution to Exercise 1.80. X is the image of $\mathbb{P}^2 \times \mathbb{P}^2$ under the map τ sending a pair of linear forms (H, J) to the cubic form $H^2 J$, which is quadratic in the coefficients of H and linear in the coefficients of J; hence the first part.

Now, if $\alpha, \beta \in A^1(\mathbb{P}^2 \times \mathbb{P}^2)$ are the pullbacks of the hyperplane classes on $\mathbb{P}^2$ via the two projections, and $\zeta \in A^1(\mathbb{P}^9)$ the hyperplane class there, by the first part we have

$$\tau^*(\zeta) = 2\alpha + \beta.$$

Since the map τ is birational onto its image, it follows that the degree of $X \subset \mathbb{P}^9$ is given by

$$\deg X = \deg\left((2\alpha + \beta)^4\right) = 2^2 \binom{4}{2} = 24.$$

□

Exercise 23.35. 1.81 If you try to find the degree of the locus X of the preceding problem by intersecting X with hyperplanes $H_{p_1}, \ldots, H_{p_4}$, where

$$H_p = \{C \in \mathbb{P}^9 : p \in C\},$$

you get the wrong answer (according to the preceding problem). Why? Can you account for the discrepancy?

Solution to Exercise 1.81. First, the cardinality of the intersection $X \cap H_{p_1} \cap \cdots \cap H_{p_4}$ is easy to find: any nonreduced cubic containing all four points must be (after reordering) the cubic $2 \cdot \overline{p_1, p_2} + \overline{p_3 p_4}$. The number of such cubics is $\binom{4}{2} = 6$.

What's wrong? The problem is that, unlike the other examples of this technique, in this case *the hyperplanes H_{p_i} do not intersect the locus X transversely*. By the calculation analogous to those above, the tangent space to X at the image of (H, J) is the space of cubics in the ideal (HJ, H^2), which is contained in the hyperplanes H_{p_1} and H_{p_2}. □

Exercise 23.36. 1.82 Let $\mathbb{P}^2$ denote the space of lines in the plane, and $\mathbb{P}^5$ the space of plane conics. Let $\Phi \subset \mathbb{P}^2 \times \mathbb{P}^5$ be the closure of the locus of pairs

$$\{(L, C) : \ C \text{ is smooth, and } L \text{ is tangent to } C\}.$$

Show that Φ is a hypersurface; and, assuming characteristic 0, find its class $[\Phi] \in A^1(\mathbb{P}^2 \times \mathbb{P}^5)$.

Solution to Exercise 1.82. To see that Φ is a hypersurface, let $F = \{(p, l) : p \in l\} \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$, and observe that Φ is the image of the incidence correspondence

$$\Omega = \{(C, L, p, l) : C \text{ and } L \text{ are tangent to } l \text{ at } p\} \subset \mathbb{P}^5 \times \mathbb{P}^{2*} \times F;$$

the dimension and irreducibility of Ω are readily seen via projection on the third factor.

As for the calculation of $[\Phi]$, this is probably best done by the method of undetermined coefficients: if α and $\beta \in A^1(\mathbb{P}^2 \times \mathbb{P}^5)$ are the pullbacks of hyperplane classes, then we can write

$$[\Phi] = c\alpha + d\beta$$

for some $c, d \in \mathbb{Z}$. If $\mathcal{C} \subset \mathbb{P}^2$ and $\mathcal{D} \subset \mathbb{P}^5$ are general pencils of lines and conics respectively, the integers c and d are then given as the cardinality of the intersection of Φ with the curves $\mathcal{C} \times \{C_0\}$ and $\{L_0\} \times \mathcal{D} \subset \mathbb{P}^2 \times \mathbb{P}^5$ respectively—that is, the number of lines in a general pencil that are tangent to a given conic, and the number of conics in a general pencil that

a tangent to a given line. The answer in both cases is the number of branch points of a general degree 2 map from $\mathbb{P}^1$ to $\mathbb{P}^1$; that is, 2. Thus

$$[\Phi] = 2\alpha + 2\beta.$$

□

Exercise 23.37. 1.83 Now let $\mathbb{P}^9$ be the space of plane cubic curves as before, and let $Y \subset \mathbb{P}^9$ be the closure of the locus of reducible cubics consisting of a smooth conic and a tangent line. Use the result of the first part to determine the degree of B.

Solution to Exercise 1.83. Y is the image, under the map $\tau : \mathbb{P}^2 \times \mathbb{P}^5 \to \mathbb{P}^9$ of Section 1.2.5, of the locus $\Phi \subset \mathbb{P}^2 \times \mathbb{P}^5$. Since this map is birational onto its image, and the pullback of the hyperplane class $\zeta \in A^1(\mathbb{P}^9)$ is given by $\tau^*(\zeta) = \alpha + \beta$, we have

$$\begin{aligned} \deg(Y) &= \deg\left(\tau^*(\zeta)^5[\Phi]\right) \\ &= \deg\left((\alpha+\beta)^5(2\alpha+2\beta)\right) \\ &= 42 \end{aligned}$$

□

Exercise 23.38. 1.84 Let $\mathbb{P}^{14}$ be the space of quartic curves in $\mathbb{P}^2$, and let $\Sigma \subset \mathbb{P}^{14}$ be the closure of the space of reducible quartics. What are the irreducible components of Σ, and what are their dimensions and degrees?

Solution to Exercise 1.84. First, Σ has two irreducible components, one whose general point corresponds to a union of a line and a cubic, and one whose general point corresponds to a union of two conics. These are images of $\mathbb{P}^2 \times \mathbb{P}^9$ and $\mathbb{P}^5 \times \mathbb{P}^5$ respectively, under maps that are given by bihomogeneous forms of bidegree $(1,1)$. The first map is birational onto its image, and so the first component has dimension 11 and degree $\binom{11}{3} = 1650$; the second map is finite of degree 2 onto its image, which thus has dimension 10 and degree $\frac{1}{2}\binom{10}{5} = 126$. □

Exercise 23.39. 1.85 Find the dimension and degree of the locus $\Omega \subset \mathbb{P}^{14}$ of totally reducible quartics (that is, quartic polynomials that factor as a product of four linear forms).

Solution to Exercise 1.85. Here Ω is the image of the map $\mu : (\mathbb{P}^2)^4 \to \mathbb{P}^{14}$ sending a fourtuple of linear forms to their product. Again, if $\alpha_i \in A^1\left((\mathbb{P}^2)^4\right)$ are the pullbacks of the hyperplane class from $\mathbb{P}^2$ via the four

projections, and $\zeta \in A^1(\mathbb{P}^{14})$ is the hyperplane class there, then since the map μ is finite of degree $4! = 24$ we have

$$\begin{aligned}\deg(\Omega) &= \frac{1}{24}\deg\left(\tau^*(\zeta)^8\right)\\ &= \frac{1}{24}\deg\left((\alpha_1+\alpha_2+\alpha_3+\alpha_4)^8\right)\\ &= \frac{1}{24}\binom{8}{2,2,2,2}\\ &= 105\end{aligned}$$

□

Exercise 23.40. 1.86 Again let $\mathbb{P}^{14}$ be the space of plane quartic curves, and let $\Theta \subset \mathbb{P}^{14}$ be the locus of sums of four concurrent lines. Using the result of Exercise 1.61, find the degree of Θ.

Solution to Exercise 1.86. Here Θ is the image, under the map μ of the preceding solution, of the locus $\Phi \subset (\mathbb{P}^2)^4$ described in Exercise 1.61. Having calculated the class of Φ to be $\sum_{i<j}\alpha_i\alpha_j$ in that exercise, we have

$$\begin{aligned}\deg(\Theta) &= \frac{1}{24}\deg\left(\tau^*(\zeta)^6[\Phi]\right)\\ &= \frac{1}{24}\deg\left((\alpha_1+\alpha_2+\alpha_3+\alpha_4)^6\sum_{i<j}\alpha_i\alpha_j\right)\\ &= \frac{1}{24}\cdot 6\cdot\binom{6}{2,2,1,1}\\ &= 45\end{aligned}$$

□

Exercise 23.41. 1.87 Find the degree of the locus $A \subset \mathbb{P}^{14}$ of the preceding problem, this time by calculating the number of sums of four concurrent lines containing six general points $p_1, \ldots, p_6 \in \mathbb{P}^2$, assuming transversality.

Solution to Exercise 1.87. Given that no three of the points p_i are colinear, if a sum of four lines contains them all then either

(a) two of the lines will each contain two of the points, and the remaining two lines one each; or

(b) three of the lines will each contain two of the points.

If the four lines are concurrent (and the points general), the latter case can't occur; thus the sums of four collinear lines containing $p_1, \ldots, p_6$ correspond to the decompositions of the set $\{p_1, \ldots, p_6\}$ into two sets of two and two

sets of 1 (take two of the lines to be the spans of the two pairs; take the remaining lines to be the lines spanned by the point of intersection of the first two and each of the two remaining points). The number is thus

$$\frac{1}{2^2}\binom{6}{2,2,1,1} = 45.$$

□

A natural generalization of the locus of asterisks, or of sums of four concurrent lines, would be the locus, in the space $\mathbb{P}^N$ of hypersurfaces of degree d in $\mathbb{P}^n$, of *cones*. We will indeed be able to calculate the degree of this locus in general, but it will require more advanced techniques than we have at our disposal here. ******** Joe will write this into Section 12.3********.

Exercise 23.42. 1.88 More generally, let $\mathbb{P}^N$ be the space of hypersurfaces of degree d in $\mathbb{P}^n$.

(a) Find the dimension and degree of the locus $\Gamma \subset \mathbb{P}^N$ of reducible and/or nonreduced hypersurfaces. (Note that this locus is in general reducible; by its degree we mean the degree of the union of its components of maximal dimension.)

(b) Find the dimension and degree of the locus $\Sigma \subset \mathbb{P}^N$ of totally reducible hypersurfaces.

Solution to Exercise 1.88. For the first, the largest-dimensional component of the locus of reducible hypersurfaces of degree d is the component Φ whose general point corresponds to the union of a hyperplane and a hypersurface of degree $d-1$; this has dimension $\binom{n+d-1}{n} - 1 + n$ and degree

$$\deg(\Phi) = \binom{\binom{n+d-1}{n} - 1 + n}{n},$$

except when $d = 2$, in which case the degree is half that. □

Exercise 23.43. 1.89 Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d and $L \subset S$ a line. Calculate the degree of the self-intersection of the class $\lambda = [L] \in A^1(S)$ by considering the intersection of S with a general plane $H \subset \mathbb{P}^3$ containing L.

Solution to Exercise 1.89. A general plane H containing L will intersect S in the union of L and a curve $C \subset H$ of degree $d-1$; applying Bertini, we can see that C intersects L transversely. If $\zeta \in A^1(S)$ is the restriction to S of the hyperplane class in $\mathbb{P}^3$, then we have

$$\lambda = \zeta - \gamma$$

where γ is the class of C; and

$$\deg(\lambda^2) = \deg\big(\lambda(\zeta - \gamma)\big) = 1 - (d-1) = 2 - d.$$

□

Exercise 23.44. 1.90 Let S be a smooth surface. Show that if $C \subset S$ is any curve, and the corresponding point in the Hilbert scheme $\mathcal{H}$ of curves on S lies on a positive-dimension irreducible component of $\mathcal{H}$, then the degree $\deg(\gamma^2)$ of the self-intersection of the class $\gamma = [C] \in A^1(S)$ is nonnegative. Using this and the preceding Exercise, prove the statement made in Section 1.4.3 that *a smooth surface $S \subset \mathbb{P}^3$ of degree 3 or more can contain only finitely many lines.*

Solution to Exercise 1.90. Let $B \subset \mathcal{H}$ be a curve, with the point $b \in B$ corresponding to C. If B is rational, we are essentially done: the universal family of curves over $B \subset \mathcal{H}$ gives a rational equivalence between C and the curve C' corresponding to any point $b' \in B$, and since C and C' have no common components their intersection number is nonnegative.

In general, whatever the genus of B, by Riemann-Roch a high multiple of the the point $b \in B$ will be rationally equivalent to a linear combination $\sum m_i b_i$ of other points of B; if $C_i \subset S$ is the curve corresponding to b_i we see similarly that $mC \sim \sum m_i C_i$; by the same token $\deg((m\gamma)^2)$, and hence $\deg(\gamma^2)$, will be nonnegative. (If B is singular we may work with its normalization.) □

Exercise 23.45. 1.91 Let $C \subset \mathbb{P}^3$ be a smooth quintic curve. Show that

(a) if C has genus 2, it must lie on a quadric surface;

(b) if C has genus 1, it cannot lie on a smooth quadric surface (in fact, it can't lie on any quadric); and

(c) if C has genus 0, it may or may not lie on a quadric surface (that is, some rational quintic curves do lie on quadrics and some don't).

Solution to Exercise 1.91. For the first part, look at the restriction maps

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2)).$$

$h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$, and by Riemann-Roch, $h^0(\mathcal{O}_C(2)) = 10 - g + 1$. So if $g = 2$, the restriction map must have a nonzero kernel; that is, C lies on a quadric.

The fact that C can't lie on a smooth quadric follows from the genus formula for $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Finally, there do exist smooth rational quintics lying on a quadric: these are the curves of type $(1, 4)$. But a dimension count shows that the space of all rational quintic curves has dimension 20 (4-tuples of quintic polynomials

form a 24-dimensional vector space; mod scalars and PGL_2, the space of image curves in $\mathbb{P}^3$ is 24-4=20-dimensional); while the dimension of the space of curves of type $(1,4)$ on a quadric is $9 + 2 \cdot 5 - 1 = 18$ (for the quadric, which is uniquely determined by the curve, and the space of curves of type $(1,4)$ on Q). □

Exercise 23.46. 1.94 Let $C \subset \mathbb{P}^3$ be a smooth quintic curve. Show that C lies on a quadric surface Q and a cubic surface S with intersection $Q \cap S$ consisting of the union of C and a line.

Solution to Exercise 1.94. As in the last exercise, we start by looking at the restriction maps

$$r_2 : H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2)) \quad \text{and} \quad r_3 : H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_C(3)).$$

As we found in Exercise 1.94, the map r_2 must have a kernel, corresponding to a quadric Q containing C (unique, by Bezout, and necessarily irreducible). Similarly, the kernel of r_3 must have dimension at least 5; since the space of products LQ has vector space dimension 4, there must be a cubic surface S containing C and not containing Q. Q being irreducible, the intersection $Q \cap S$ will have dimension 1; and by Bezout it must consist of the union of C and a line. □

Exercise 23.47. 1.93 Use the result of Exercise 1.94—showing that a smooth quintic curve of genus 2 is linked to a line in the complete intersection of a quadric and a cubic—to find the dimension of the subset of the Hilbert scheme corresponding to smooth curves of degree 5 and genus 2.

Solution to Exercise 1.93. Here we want to look at the incidence correspondence Φ of four-tuples (Q, S, C, L) consisting of a quadric surface Q, a cubic surface S, a smooth quintic curve of genus 2 and a line L such that

$$Q \cap S = C \cup L.$$

We can calculate the dimension of Φ via its projection to the last factor, and from that find that the space of smooth quintics of genus 2 has dimension 20. □

Exercise 23.48. 1.95 Use the Poincaré-Hopf Theorem to compute the topological Euler characteristic of a smooth variety $Y = X_1 \cap X_2 \subset \mathbb{P}^n$ where X_i is a hypersurface of degree d_i.

23.2 Chapter 2

Exercise 23.49. 4.23 Let X be a variety, and let $f : Y \to X$ be a finite flat map, so that both f^* and f_* are well defined. Let $d := (\kappa(Y) : \kappa(X))$

be the degree of f. Show that the composition $f_* \circ f^* : A_*(X) \to A_*(X)$ is multiplication by d.

Solution to Exercise 4.23. □

Exercise 23.50. 4.24 Show that rational equivalence is the finest equivalence relation that is defined by graded subgroups $R_*(Y) \subset Z_*(Y)$ with the following properties:

(a) $R_*(Y) \subset Z_*(Y)$ is defined for all schemes algebraic over a field k.

(b) $R_*(Y)$ is preserved by proper push-forward and flat pull-back.

(c) $R_*(\mathbb{P}^1)$ contains $\langle 0 \rangle - \langle \infty \rangle$.

Solution to Exercise 4.24. □

Exercise 23.51. 4.25 The following example shows that the assertion of Theorem 4.14 can fail if f is an open immersion: Let C be a conic in $\mathbb{P}^2$, and let U be the open set $\mathbb{P}^2 \setminus C$ in $\mathbb{P}^2$. Show that U is an affine variety, with Chow groups $A_0(U) = 0,\ A_1(U) = \mathbb{Z}/(2),\ A_2(U) = \mathbb{Z}$. There is a map on cycles $\alpha : Z_*(U) \to Z_*(\mathbb{P}^2)$ taking the class of a curve in U to the class of its closure in $\mathbb{P}^2$. If this induced a map $\alpha : A_1(U) \to A_1(\mathbb{P}^2)$, it would split the surjection $A_*(\mathbb{P}^2) \to A_*(U)$ defined in Proposition 4.11(c). Give an example of two rationally equivalent cycles Z_1, Z_2 on U such that $\alpha(Z_1)$ is not rationally equivalent to $\alpha(Z_2)$.

Solution to Exercise 4.25. □

Exercise 23.52. 4.26 Show by example that Theorem 4.14 is false without the hypothesis "proper" even when f is a birational map between smooth curves.

Solution to Exercise 4.26. □

Exercise 23.53. 4.27 Show that if L is a line bundle on a quasi-projective scheme X, then L may be embedded as a subsheaf of the total quotient sheaf $\kappa(X)$ of X as a subsheaf locally generated by one non-zerodivisor. (The result is true much more generally, but there are subtleties in the most general setting—see Kleiman [1979].)

Solution to Exercise 4.27. □

Exercise 23.54. 4.28 Suppose that X is quasi-projective and reduced. Show that if L and M are two line bundles on X embedded in $\kappa(X)$ as above, and $\varphi : L \to M$ is an isomorphism, then φ extends to an isomorphism $\kappa(X) \to \kappa(X)$, and that such an isomorphism is given by multiplication with a rational function. Conclude that $Pic(X)$ is the same as

the group of Cartier divisors modulo "principal" Cartier divisors—those defined by invertible rational functions on X.

Solution to Exercise 4.28. □

Exercise 23.55. 4.29 Suppose that X is quasi-projective and reduced. Krull's Principal Ideal Theorem implies that in a Noetherian ring every prime ideal minimal over the ideal generated by a non-zerodivisor has codimension 1. Extend this idea to define a map from the multiplicative group of Cartier divisors on X to the group of codimension 1 cycles $Z^1(X)$, to make a commutative diagram

$$\begin{array}{ccccccccc} \mathcal{O}_X^* & \longrightarrow & \kappa(X)^* & \longrightarrow & \{\text{Cartier Divisors on } X\} & \longrightarrow & Pic(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & \kappa(X)^* & \longrightarrow & Z_{n-1}(X) & \longrightarrow & A_{n-1}(X) & \longrightarrow & 0 \end{array}$$

with exact rows. Conclude that there is a natural map $Pic(X) \to A_{n-1}(X)$.

Solution to Exercise 4.29. □

Exercise 23.56. 4.30 Suppose that X is quasi-projective and normal. Using the result from commutative algebra that a normal domain is the intersection of discrete valuation rings (see for example Eisenbud [1995] §11.2), show that the map from the group of Cartier divisors to the additive group of one-cycles on X is a monomorphism. Conclude that the map $Pic(X) \to A_{n-1}(X)$ is a monomorphism in this case.

Solution to Exercise 4.30. □

Exercise 23.57. 4.31 Suppose that X is quasi-projective and regular—that is, that every local ring of X is a regular local ring (it would be enough that every local ring of X is factorial.) Show that the map $Pic(X) \to A_{n-1}(X)$ is an isomorphism.

Solution to Exercise 4.31. □

********The following exercise is actually false: $Pic(X)$ is positive-dimensional. Do you want to make X the union of the three coordinate axes in $\mathbb{P}^3$?********

Exercise 23.58. 4.32 Let X consist of three concurrent lines in $\mathbb{P}^2$. Show that $A_1(X) = \mathbb{Z}$ but $Pic(X) = \mathbb{Z}^3$. (Hint: Use the isomorphism $PicX = H^1(\mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the sheaf of invertible functions on X.)

Solution to Exercise 4.32. □

Exercise 23.59. 4.33 Let X be the cone over a smooth plane conic C. Prove that the natural comparison map $Pic(X) = \mathbb{Z} \to \mathbb{Z} = A_1(X)$ is multiplication by 2.

Solution to Exercise 4.33. □

Exercise 23.60. 4.34 (For those who know what a Gorenstein ring is). Extend Exercises 4.28 and 4.29 to the case where X is quasiprojective and generically Gorenstein. Let X be the triple line in $\mathbb{P}^2_k$, where k is a field. Show that $A_1(X) = \mathbb{Z}$, but that $Pic(X) = \mathbb{Z} \oplus k^+$, the additive group of k, and that the map $Pic(X) \to A_*(X)$ is the projection on $\mathbb{Z}$ followed by multiplication by 3. (Hint: Use $Pic(X) = H^1(\mathcal{O}_X^*)$.)

****Are we assuming $k = \overline{k}$ here? Do we need to?****

Solution to Exercise 4.34. □

The following exercises deal with some examples of the construction of dynamic projection. In each, we suppose $X \subset \mathbb{P}^n$ is a variety; we let A and $R \subset \mathbb{P}^n$ be complementary linear subspaces general with respect to X, and Ψ the action on the multiplicative group G_m on $\mathbb{P}^n$ described in Section 4.1.1, with attractor A and repellor R. In each case we let X_0 be the limit, as $t \to 0$, of the images $\psi_t(X)$.

Exercise 23.61. 4.35 Let $X \subset \mathbb{P}^3$ be a twisted cubic curve, and consider the dynamic projection of X with attractor a point $p \in \mathbb{P}^3$ and repellor a plane. Show that X_0 is the reduced union of three noncoplanar lines through p.

Solution to Exercise 4.35. □

Exercise 23.62. 4.36 Again let $X \subset \mathbb{P}^3$ be a twisted cubic curve, and now consider the dynamic projection of X with attractor a point $L \subset \mathbb{P}^3$ and repellor another line. Show that $X_0 = V(I_L^2)$ is the scheme defined by the square of the ideal of L.

Solution to Exercise 4.36. □

Exercise 23.63. 4.37 Now let $X \subset \mathbb{P}^3$ be a nondegenerate quartic curve, and consider the dynamic projection of X with attractor a point $p \in \mathbb{P}^3$ and repellor a plane.

(a) If X has genus 1, show that X_0 is the reduced union of four noncoplanar lines through p.

(b) If X has genus 0, show that X_0 is the union of four noncoplanar lines through p, with an embedded point of multiplicity 1 at p.

Solution to Exercise 4.37. □

Exercise 23.64. 4.38 There are two isomorphism classes of subschemes $X \subset \mathbb{P}^3$ with Hilbert polynomial $p_X(m) = 3m + 1$ consisting of a planar triple line and an embedded point p: one where the embedded point is spatial (that is, $\dim T_p X = 3$), such as

$$X = V(x_0^2, x_0x_1, x_0x_2, x_1^3);$$

and one where the embedded point is planar (that is, $\dim T_p X = 2$), such as

$$X = V(x_0, x_1^3, x_1^4).$$

Show that the former is a flat limit of twisted cubics, and that the latter is not.

Solution to Exercise 4.38. □

Exercise 23.65. 4.39 Consider the G_m action on $\mathbb{P}^n$ given by

$$\varphi_t : (x_0, \ldots, x_n) \mapsto (t^{w_1}x_1, \ldots, t^{w_n}x_n)$$

for $t \in G_m$, with weights $w_0 > w_1 > \cdots > w_n \in \mathbb{Z}$. Let $X \subset \mathbb{P}^n$ be a subvariety. Prove analogs for each part of Proposition 4.5 for each pair of spaces $R = \{x_0 = \cdots = x_i = 0\}$ and $A = \{x_{i+1} = \cdots = x_n = 0\}$. If X has dimension m, and the coordinate system is chosen general with respect to X, then show that the limit of tX as t approaches 0 is supported on the m-plane $x_{m+1} = \cdots x_n = 0$.

Hint for Exercise 4.39: Show as in Proposition 4.5 that for any point not in R or A has G_m orbit closure meeting both of these.

Solution to Exercise 4.39. □

23.3 Chapter 3

Exercise 23.66. 5.28 Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d, and $L \subset S$ a line. Find the degree of the self-intersection $[L]^2 \in A^2(S)$ by linkage, considering the intersection of S with a general plane containing L.

Solution to Exercise 5.28. Let $H \subset \mathbb{P}^3$ be a general plane containing L, and write $H \cap S = L \cup C$; let h, l and $c \in A^1(S)$ be the classes of $H \cap S$, L and C, so that in particular $h = l + c$. We have $\deg(h^2) = d$, $\deg(hl) = 1$

and $\deg(hc) = d-1$. (Verify the last equality by showing directly that C intersects L transversely in $d-1$ points.) In any case, we have

$$1 = \deg(hl) = \deg(cl) + \deg(l^2) = d - 1 + \deg(l^2)$$

so $\deg(l^2) = 2-d$. (Check this in case $d = 1, 2$ or 3! Also, calculate similarly that $\deg(c^2) = 0$, and verify this by observing that if H' is another general plane containing L, with $H' \cap S = L \cup C'$, then C and C' are disjoint.) □

Exercise 23.67. 5.29 Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface, and C a twisted cubic curve lying on Q; let $c \in A^1(Q)$ be its class. Find the self-intersection c^2 in the Chow ring of Q.

Solution to Exercise 5.29. We can do this by using the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the calculation of the Chow ring of products of projective spaces in Theorem 1.27: if α and $\beta \in A^1(\mathbb{P}^1 \times \mathbb{P}^1$ are the pullbacks of the point classes from the two factors, we can calculate $[C] = 2\alpha + \beta$ or $\alpha + 2\beta$; thus

$$\deg c^2 = \deg(2\alpha + \beta)^2 = 4.$$

Alternatively (and more in keeping with the spirit of this chapter), let $Q' \subset \mathbb{P}^3$ be another quadric containing C, and write $Q \cap Q' = C \cup L$; by Bezout, L is a line in Q. Let h and $l \in A^1(Q)$ be the classes of a plane section of Q and of L; we have then $2h = c + l$, so that

$$\deg(cl) = \deg(2hl) - \deg(l^2) = 2$$

(the self-intersection of a line on a quadric is 0, either by the preceding exercise or by direct examination); and hence

$$\deg(c^2) = \deg(2hc) - \deg(lc) = 2 \cdot 3 - 2 = 4.$$

□

Exercise 23.68. 5.30 Now let $S \subset \mathbb{P}^3$ be a smooth cubic surface, and C a twisted cubic curve lying on S; again, let $c \in A^1(S)$ be its class. Find the self-intersection c^2 in the Chow ring of S.

Solution to Exercise 5.30. Here we have no choice but to follow the second approach of the preceding problem (unless we know a lot more about the Chow ring of S than we're letting on). Let $Q \subset \mathbb{P}^3$ be a general quadric containing C, and write $S \cap Q = C \cup D$; let $h \in A^1(S)$ be the class of a plane section, and $d = [D]$. We have then ****this solution needs to be completed****

$$\deg(cd) =$$

□

Exercise 23.69. 5.31 Show that if $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ is the d-th veronese mapping and $L \subset \mathbb{P}^N$ is a linear subspace of dimension $e < d$, then the length of the scheme $L \cap X$ is at most $e + 1$.

Solution to Exercise 5.31. □

Exercise 23.70. ?? Show that if $0 \leq l \leq k$ and $l \leq p \leq q$, then the condition of meeting a given k-plane in dimension at least l is a condition of codimension equal to $\max(0,\ (l+1)(q-p+l-k))$ on the elements of $\mathbb{G}(p,q)$. (Hint: Show first that if $q - p + l - k \leq 0$ then every p-plane in $\mathbb{P}^q$ meets the given k-plane in dimension $\geq l$. Next do the case $k = l$ by showing that the set of p-planes satisfying the condition is isomorphic (by projection from the k-plane) to the Grassmannian of $p - k - 1$-planes in $q - k - 1$-space. Finally, if $q - p + l - k \geq 0$, consider the incidence correspondence

$$I := \{(\Gamma, \Lambda) \in \mathbb{G}(p,q) \times \mathbb{G}(l,k) \mid \Lambda \subset \Gamma\}.$$

Show that the projection of I to the first factor is generically one-to-one onto the set of planes meeting our condition; and compute the dimension of I by projecting to the second factor.

Solution to Exercise **??**. □

Exercise 23.71. 5.32 In Example 5.27, calculate the length of the Tor modules in Serre's formula, and verify that the correct multiplicity $m_p(X_2, V)$ is 2. (Hint: this can be calculated directly, using the fact that all the ideals involved are monomial; alternatively, we can take the Koszul resolution of X_2 and tensor with $\mathcal{O}_{V,p}$.)

Solution to Exercise 5.32. □

Exercise 23.72. 5.33 Let $f : Y \to X$ be a generically finite surjective morphism of smooth projective varieties. Show that the composition

$$f_* f^* : A_k(X) \to A_k(X)$$

is multiplication by the degree of f; deduce in particular that the map $f^* : \mathbb{Q} \otimes A_k(X) \to \mathbb{Q} \otimes A_k(Y)$ is injective. Show by example that $f^* f_* \alpha$ may not be a multiple of α, so no analogous statement holds.

Solution to Exercise 5.33. □

23.4 Chapter 4

Exercise 23.73. 2.10 Let $e_1, \dots, e_4$ be a basis for a vector space V, and let

$$\eta = \sum_{1 \le a < b \le 4} p_{a,b} e_a \wedge e_b \in \wedge^2 V.$$

Show that the polynomial in the $p_{a,b}$ corresponding to the equation

$$\eta \wedge \eta = 0 \in \wedge^4 V$$

is $2g$, where g is as in (2.1). Thus, in characteristic not 2, the equation $\eta^2 = 0$ characterizes decomposable vectors in the exterior square $\wedge^2 V$, and more generally defines the Grassmannian $\mathbb{G}(1, \mathbb{P}V) \subset \mathbb{P}(\wedge^2 V)$ for a vector space V of any dimension.

In fact, even over the integers, there is an element $\eta^{(2)} \in \wedge^4 V$ called the *divided square* of η, such that $\eta \wedge \eta = 2\eta^{(2)}$, and in every characteristic η is decomposable if and only if $\eta^{(2)} = 0$. The coefficients of $\eta^{(2)}$, when written out in terms of the basis of $\wedge^4 V$, are called the 4×4 Pfaffians of η; they are the polynomials $g_{a,b,c,d}$ of Exercise **??**. The square of the ideal of 4×4 Pfaffians of η is the ideal of 4×4 minors of the generic alternating matrix $(p_{i,j})$. See Eisenbud [1995] Appendix A2.4.

Solution to Exercise 2.10. □

Exercise 23.74. 2.112.11 Let Λ and $\Gamma \in G$ be two points in the Grassmannian $G = G(k, V)$. Show that the line $\overline{\Lambda, \Gamma} \subset \mathbb{P}(\wedge^k V)$ is contained in G if and only if the intersection $\Lambda \cap \Gamma \subset V$ of the corresponding subspaces of V has dimension $k - 1$.

Solution to Exercise 2.11. □

Exercise 23.75. 2.12 Suppose that $L_1, \dots, L_m \subset V$ are k-planes contained in a vector space, and that for every $i \neq j$ we have $\dim L_i \cap L_j = k - 1$ (equivalently, $\dim L_i + L_j = k + 1$.) Show that either all the L_i contain a subspace M of dimension $k - 1$ or all L_i are contained in a subspace N of dimension $k + 1$.

Solution to Exercise 2.12. □

Exercise 23.76. 2.13 Using Exercises 2.11 and 2.12, show that the linear subspaces of $\mathbb{P}(\wedge^k V)$ that are maximal among linear spaces contained in $G(k, V)$ are precisely those defined in one of two ways:

(a) $\Gamma_M = \{L \mid M \subset L\}$ for some subspace $M \subset V$ of dimension $k - 1$, so that $\Gamma_M \cong \mathbb{P}(V/M)$; or

(b) $\Sigma_N = \{L \mid L \subset N\}$ for some subspace $N \subset V$ of dimension $k+1$, so that $\Sigma_N \cong \mathbb{P}(N)$.

Solution to Exercise 2.13. □

Notice that unless $\dim V = 2k$, these two kinds of maximal linear subspaces have different dimensions. Indeed, Exercise (2.13) is key in establishing a basic fact: that for $\dim V \neq 2k$, the automorphism group of the Grassmannian $G = G(k, V)$ is the same as the automorphism group $PGL(V)$ of $\mathbb{P}V$: assuming $2k < \dim V$, we argue that an automorphism of $G(k, V)$ must be projective, that is, induced by an automorphism of the vector space $\wedge^k V$; hence must preserve linear spaces of $\mathbb{P}(\wedge^k V)$ contained in $G(k, V)$; and so induces an automorphism of the Grassmannian $G(k-1, V)$. Continuing in this way, we arrive at an automorphism of $\mathbb{P}V$ that induces the original automorphism of $G(k, V)$. The situation when $\dim V = 2k$ is similar, except that there do exist automorphisms of $G(k, V)$ exchanging the two families of maximal linear subspaces (for example, we can choose a nondegenerate bilinear form on V and send each k-plane Λ to $\Lambda^\perp$); we arrive at an exact sequence

$$1 \to PGL(V) \to Aut(G) \to \mathbb{Z}/2 \to 1.$$

Exercise 23.77. 2.14 Let $\mathbb{G}(1,3) \subset \mathbb{P}^5$ be the Grassmannian of lines in $\mathbb{P}^3$, embedded in $\mathbb{P}^5$ by the Plücker embedding.

(a) Write down the equations in Plücker coordinates of the maximal linear spaces on $\mathbb{G}(1,3)$, that is, the locus of lines containing a point and the locus lines contained in a 2-plane.

(b) Write down the equations in Plücker coordinates of the subvariety of $\mathbb{G}(1,3)$ consisting of lines meeting a given line.

These spaces are special cases of what we will call *Schubert cycles*.

Solution to Exercise 2.14. □

Exercise 23.78. 2.15 Using the fact that the Grassmannian

$$G = G(k, V) \subset \mathbb{P}(\wedge^k V)$$

is cut out by quadratic equations, show that if $[\Lambda] \in G$ is the point corresponding to a k-plane Λ then the tangent plane $\mathbb{T}_{[\Lambda]}G \subset \mathbb{P}(\wedge^k V)$ intersects G in the locus

$$G \cap \mathbb{T}_{[\Lambda]}G = \{[\Gamma] : \dim(\Gamma \cap \Lambda) \geq k-1\};$$

that is, the locus of k-planes meeting Λ in codimension 1.

Solution to Exercise 2.15. □

Exercise 23.79. 2.16 Consider the universal k-plane over $G = \mathbb{G}(k, \mathbb{P}V)$:

$$\Phi = \{(\Lambda, p) : p \in \Lambda\} \subset G \times \mathbb{P}V.$$

whose fiber over a point $[\Lambda] \in G$ is the k-plane $\Lambda \subset \mathbb{P}V$. Show that this is a closed subvariety of $G \times \mathbb{P}V$ of dimension $k + (k+1)(n-k)$, and that it's cut out by bilinear forms on $\mathbb{P}(\wedge^k V) \times \mathbb{P}V$.

Solution to Exercise 2.16. □

Exercise 23.80. 2.17 Use the preceding exercise to show that if $X \subset \mathbb{P}^n$ is any subvariety of dimension $l < n - k$, then the locus

$$\Gamma_X = \{\Lambda : X \cap \Lambda \neq \varnothing\} \subset \mathbb{G}(k, n)$$

of k-planes meeting X is a closed subvariety of $\mathbb{G}(k, n)$ of codimension $n - k - l$.

Solution to Exercise 2.17. □

Exercise 23.81. 2.18 Let $l < k < n$, and consider the locus of nested pairs of linear subspaces of $\mathbb{P}^n$ of dimensions l and k:

$$\mathbb{F}(l, k; n) = \{(\Gamma, \Lambda) : \Gamma \subset \Lambda\} \subset \mathbb{G}(l, n) \times \mathbb{G}(k, n).$$

Show that this is a closed subvariety of $\mathbb{G}(l, n) \times \mathbb{G}(k, n)$, and calculate its dimension. (These are examples of a further generalization of Grassmannians called *flag manifolds*.)

Solution to Exercise 2.18. □

Exercise 23.82. 2.19 Again let $l < k < n$, and for any $m \leq l$ consider the locus of pairs of linear subspaces of $\mathbb{P}^n$ of dimensions l and k intersecting in dimension at least m:

$$\mathbb{F}(l, k; n) = \{(\Gamma, \Lambda) : \dim(\Gamma \cap \Lambda) \geq m\} \subset \mathbb{G}(l, n) \times \mathbb{G}(k, n).$$

Show that this is a closed subvariety of $\mathbb{G}(l, n) \times \mathbb{G}(k, n)$, and calculate its dimension.

Solution to Exercise 2.19. □

Exercise 23.83. 2.20 Different hyperplane sections of $G(k, n)$ can have different properties; not all are given by the vanishing of a Plücker coordinate.

(a) Use (for example) Exercise 2.15 to show that a hyperplane section of $G(2, 4) = \mathbb{G}(1, 3)$ defined by a Plücker coordinate is the cone over a nonsingular quadric in $\mathbb{P}^3$. What other varieties appear as a hyperplane section of $\mathbb{G}(1, 3)$?

(b) In general, how many isomorphism classes of hyperplane sections of $\mathbb{G}(k,n)$ are there?

Solution to Exercise 5.33. □

Exercise 23.84. 2.21 We gave a description of the Plücker relations in the special case $\mathbb{G}(1,n)$ following Example 2.1, and in Exercise 2.10 above. Show that the polynomials above, obtained by expanding the determinants of submatrices of A, are the same as the polynomials described there.

Solution to Exercise 2.21. □

Exercise 23.85. 2.22 By Theorem 2.4, a tangent vector to the Grassmannian $G(k,V)$ at a point $[\Lambda]$ corresponds naturally to a homomorphism $\varphi : \Lambda \to V/\Lambda$, and so we can associate to it its rank. Show that an automorphism of $G(k,n)$ carries tangent vectors to tangent vectors of the same rank, and hence that in case $1 < k < n$ the group of automorphisms of $G(k,n)$ cannot act transitively on nonzero tangent vectors (so that in particular Kleiman's Theorem does not apply in characteristic p). Show, on the other hand, that the group of automorphisms of $G(k,n)$ *does* act transitively on tangent vectors of a given rank.

Solution to Exercise 2.22. □

Exercise 23.86. 2.23 In Example 2.6, we showed that the open Schubert cell $\tilde{\Sigma}_1 = \Sigma_1 \setminus (\Sigma_2 \cap \Sigma_{1,1})$ is isomorphic to affine space $\mathbb{A}^3$. For each of the remaining Schubert indices a, b, show that the Schubert cell $\tilde{\Sigma}_{a,b} \subset \mathbb{G}(1,3)$ is isomorphic to affine space of dimension $4 - a - b$.

Solution to Exercise 2.23. □

Exercise 23.87. 2.24 Consider the Schubert cycle

$$\Sigma_1 = \{\Lambda : \Lambda \cap L \neq \varnothing\} \subset G.$$

Suppose $\Lambda \in \Sigma_1$ and that $\Lambda \neq L$, so that $\Lambda \cap L$ is a point q and the span $\overline{\Lambda, L}$ a plane K. Show that Λ is a smooth point of Σ_1, and that its tangent space is

$$T_\Lambda(\Sigma_1) = \{\varphi \in \mathrm{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda}) : \varphi(\tilde{q}) \subset \tilde{K}/\tilde{\Lambda}\}.$$

Solution to Exercise 2.24. □

Exercise 23.88. 2.25 Consider the Schubert cycle

$$\Sigma_{2,1} = \Sigma_{2,1}(p,H) = \{\Lambda : p \in \Lambda \subset H\} \subset G.$$

Show that $\Sigma_{2,1}$ is smooth, and that its tangent space at a point Λ is

$$T_\Lambda(\Sigma_{2.1}) = \{\varphi \in \mathrm{Hom}(\tilde{\Lambda}, V/\tilde{\Lambda}) : \varphi(\tilde{p}) = 0 \text{ and } \mathrm{Im}(\varphi) \subset \tilde{H}/\tilde{\Lambda}\}.$$

Solution to Exercise 2.25. □

Exercise 23.89. 2.26 Use the preceding two exercises to show in arbitrary characteristic that general Schubert cycles Σ_1 and $\Sigma_{2,1} \subset \mathbb{G}(1,3)$ intersect transversely, and deduce the equality $\deg(\sigma_1 \cdot \sigma_{2,1}) = 1$.

Solution to Exercise 2.26. □

Exercise 23.90. 2.27 Let L_1, L_2 and $L_3 \subset \mathbb{P}^3$ be pairwise disjoint lines (Figure 2.5).

(a) Show that the union $L_1 \cup L_2 \cup L_3$ lies on a unique quadric surface Q, and that Q is smooth;

(b) Observe that any line in $\mathbb{P}^3$ meeting all three lines L_1, L_2 and L_3 must lie in Q; conclude that Q is unique, and consists of the union of the lines meeting L_1, L_2 and L_3;

(c) Show that through any point of Q there is a unique line meeting L_1, L_2 and L_3; and finally

(d) If $L_4 \subset \mathbb{P}^3$ is a general line, deduce that there are exactly two lines meeting L_1, L_2, L_3 and L_4.

Solution to Exercise 2.27. □

Exercise 23.91. 2.28 Let $L_1, \ldots, L_4 \subset \mathbb{P}^3$ be four pairwise skew lines, and $\Lambda \subset \mathbb{P}^3$ a line meeting all four; set

$$p_i = \Lambda \cap L_i \quad \text{and} \quad H_i = \overline{\Lambda, L_i}.$$

Use the result of Exercise 2.24 to show that $[\Lambda] \in G$ fails to be a transverse point of intersection of the Schubert cycles $\Sigma_1(L_i)$ exactly when the cross-ratio of the four points $p_1, \ldots, p_4 \in \Lambda$ equals the cross-ratio of the four planes $H_1, \ldots, H_4$ in the pencil of planes containing Λ.

Solution to Exercise 2.28. □

The following series of Exercises deals with a question raised in Section 2.3.2: if $C_1, \ldots, C_4 \subset \mathbb{P}^3$ are general translates of four curves in $\mathbb{P}^3$, do the corresponding cycles $\Gamma_{C_i} \subset \mathbb{G}(1,3)$ of lines meeting the C_i intersect transversely?

To start with, we have to identify the smooth locus of the cycle $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting a given curve C, and its tangent spaces at these points; this is the content of the first exercise, which is a direct generalization of Exercise 2.24 above.

Exercise 23.92. 2.29 Let $C \subset \mathbb{P}^3$ be any curve, and $L \subset \mathbb{P}^3$ a line meeting C at one smooth point p of C and not tangent to C. Show that the cycle $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting C is smooth at the point $[L]$, and that its tangent space at $[L]$ is the space of linear maps $\tilde{L} \to \mathbb{C}^4/\tilde{L}$ carrying the one-dimensional subspace $\tilde{p} \subset \tilde{L}$ to the one-dimensional subspace $(\tilde{\mathbb{T}}_p C + \tilde{L})/\tilde{L}$ of $\mathbb{C}^4/\tilde{L}$, where we have written $\mathbb{T}_p C$ for the projective line in $\mathbb{P}^3$ that is tangent to C at p.

Solution to Exercise 2.29. □

Next, we have to verify that, for general translates C_i of any four curves, the corresponding cycles Γ_{C_i} are smooth at each of the points of their intersection. A key fact will be the irreducibility of the relevant incidence correspondence:

Exercise 23.93. 2.30 Let $B_1, \ldots, B_4 \subset \mathbb{P}^3$ be four irreducible curves, and let $\varphi_1, \ldots, \varphi_4 \in PGL_4$ be four general automorphisms of $\mathbb{P}^3$; let $C_i = \varphi_i(B_i)$. Show that the incidence correspondence

$$\Phi = \left\{(\varphi_1, \ldots, \varphi_4, L) \,:\, L \cap \varphi_i(B_i) \neq \varnothing \ \forall i\right\} \subset (PGL_4)^4 \times \mathbb{G}(1,3)$$

is irreducible.

Solution to Exercise 2.30. □

Using this, we can prove the following Exercise—asserting that for general translates C_i of four given curves and any line L meeting all four, the cycles Γ_{C_i} are smooth at $[L]$—simply by exhibiting a single collection $(\varphi_1, \ldots, \varphi_4, L)$ satisfying the conditions in question:

Exercise 23.94. 2.31 Let $B_1, \ldots, B_4 \subset \mathbb{P}^3$ be four curves, and $\varphi_1, \ldots, \varphi_4 \subset PGL_4$ four general automorphisms of $\mathbb{P}^3$; let $C_i = \varphi_i(B_i)$. Show that the set of lines $L \subset \mathbb{P}^3$ meeting C_1, C_2, C_3 and C_4 is finite; and that for any such L

(a) L meets each C_i at only one point p_i;

(b) p_i is a smooth point of C_i; and

(c) L is not tangent to C_i for any i

Solution to Exercise 2.31. □

Exercise 23.95. 2.32 Let $C_1, \ldots, C_4 \subset \mathbb{P}^3$ be any four curves, and $L \subset \mathbb{P}^3$ a line meeting all four and satisfying the conclusions of Exercise **??**. Use the result of Exercise **??** to give a necessary and sufficient condition that the four cycles $\Gamma_{C_i} \subset \mathbb{G}(1,3)$ intersect transversely at $[L]$, and show directly that this condition is satisfied when the C_i are general translates of given curves.

Solution to Exercise 2.32. □

The following three exercises deal with the approach, described in Section 2.3.3, to calculating the class of the variety $\Sigma_C \subset \mathbb{G}(1,3)$ of lines incident to a space curve $C \subset \mathbb{P}^3$ by specialization. Recall from that section that we choose a general plane $H \subset \mathbb{P}^3$ meeting C at d points p_i and a general point $q \in \mathbb{P}^3$, and let $\{A_t\}$ be the one-parameter subgroup of PGL_4 with attractor q and repellor H; we let $C_t = A_t(C)$ and take $\Psi \subset \mathbb{A}^1 \times \mathbb{G}(1,3)$ to be the closure of the locus

$$\tilde{\Psi} = \{(t, \Lambda) \mid t \neq 0 \text{ and } \Lambda \cap C_t \neq \varnothing\}.$$

Exercise 23.96. 2.33 Show that the support of the fiber Ψ_0 is exactly the union of the Schubert cycles $\Sigma_1(\overline{p_i, q})$

Solution to Exercise 2.33. □

Exercise 23.97. 2.34 Show that Ψ_0 has multiplicity 1 at a general point of each Schubert cycle $\Sigma_1(\overline{p_i, q})$

Solution to Exercise 2.34. □

Exercise 23.98. 2.35 Suppose now that $C \subset \mathbb{P}^3$ is a general rational quartic curve. Describe the flat limit of the family of cycles $\Gamma_{C_t} \subset \mathbb{G}(1,3)$, and in particular its embedded components.

Solution to Exercise 2.35. □

Exercise 23.99. 2.36 Let $C \subset \mathbb{P}^r$ be a smooth curve. Show that the rational map $\varphi : C^{(2)} \to \mathbb{G}(1,r)$ sending a pair of distinct points $p, q \in C$ to the line $\overline{pq}$ actually extends to a regular map on all of $C^{(2)}$ by sending the pair $2p$ to the projective tangent line $\mathbb{T}_p C$. Use this to show that the image of φ coincides with the locus of lines $L \subset \mathbb{P}^r$ such that the scheme-theoretic intersection $L \cap C$ has degree at least 2.

Solution to Exercise 2.36. □

Exercise 23.100. 2.37 Show by example that the conclusion of the preceding exercise is false in general if we do not assume $C \subset \mathbb{P}^r$ smooth. Is it still true if we allow C to have mild singularities, such as nodes?

Solution to Exercise 2.37. □

Exercise 23.101. 2.38 Similarly, show by example that the conclusion of Exercise 2.36 is false if we consider higher-dimensional secant planes: for example, the image of the map

$$\begin{aligned} \varphi : C^{(3)} &\to \mathbb{G}(2, r) \\ p + q + r &\mapsto \overline{p, q, r} \end{aligned}$$

need not coincide with the locus of 2-planes $\Lambda \subset \mathbb{P}^r$ whose scheme-theoretic intersection with C has degree at least 3.

Solution to Exercise 2.38. □

Exercise 23.102. 2.39 Show that the smooth locus of $S = S_1(C)$ contains the locus of lines $L \subset \mathbb{P}^3$ such that the scheme-theoretic intersection $L \cap C$ consists of two reduced points, and for such a line L identify the tangent plane $T_L S$ as a subspace of $T_L \mathbb{G}$. (When is a tangent line to C a smooth point of $S_1(C)$?)

Solution to Exercise 2.39. □

Exercise 23.103. 2.40 Use the result of the preceding Exercise to show that if C and $C' \subset \mathbb{P}^3$ are two general twisted cubic curves, then the varieties $S_1(C)$ and $S_1(C') \subset \mathbb{G}(1, 3)$ of chords to C and C' intersect transversely.

Solution to Exercise 2.40. □

Exercise 23.104. 2.41 Let $C \subset \mathbb{P}^3$ be a smooth, nondegenerate curve, and let L and $M \subset \mathbb{P}^3$ be general lines.

(a) Find the number of chords to C meeting both L and M by applying the result above; and

(b) Verify this count by considering the product morphism

$$\pi_L \times \pi_M : C \to \mathbb{P}^1 \times \mathbb{P}^1$$

(where $\pi_L, \pi_M : C \to \mathbb{P}^1$ are the projections from L and M) and comparing the arithmetic and geometric genera of the image curve.

Solution to Exercise 2.41. □

Exercise 23.105. 2.42 Let $C \subset \mathbb{P}^3$ be a twisted cubic curve. Show that the locus $S_1(C) \subset \mathbb{G}(1, 3) \subset \mathbb{P}^5$ of chords to C is the Veronese surface.

Solution to Exercise 2.42. □

Exercise 23.106. 2.43 Show that the twisted cubic and the elliptic quartic curves are the only smooth, irreducible, nondegenerate curves $C \subset \mathbb{P}^3$ such that $S_1(C)$ is smooth. (Hint: if C has no trisecant lines, the projection $\pi_p : C \to \mathbb{P}^2$ of C to a plane from a point $p \in C$ gives an isomorphism of C with a smooth plane curve of degree $d-1$; now use adjunction in the plane to argue that if this is the case for all $p \in C$ then either C is rational or $d = 4$.)

Solution to Exercise 2.43. □

Exercise 23.107. 2.44 Let $C \subset \mathbb{P}^3$ be a smooth, irreducible nondegenerate curve of degree d. In Exercise **??** above, we constructed a family of curves $C_t \subset \mathbb{P}^3$ with $C_1 = C$ and C_0 the union of d lines through a point. If we use the corresponding family of cycles $S_1(C_t) \subset \mathbb{G}(1,3)$ to describe a rational equivalence between $S_1(C)$ and a union of Schubert cycles, one of the two coefficients in the formula above for the class of $S_1(C)$ emerges, but the other doesn't. What's going on?

Solution to Exercise 2.44. □

Exercise 23.108. 2.45 Let $C \subset \mathbb{P}^3$ be a smooth nondegenerate curve of degree d and genus g, and let $T(C) \subset \mathbb{G}(1,3)$ be the locus of its tangent lines. Find the class $[T(C)] \in A^3(\mathbb{G}(1,3))$ of $T(C)$ in the Grassmannian $G(1,3)$.

Solution to Exercise 2.45. □

Exercise 23.109. 2.46 Let $C \subset \mathbb{P}^3$ be a smooth nondegenerate curve of degree d and genus g, and let $S \subset \mathbb{P}^3$ be a general surface of degree e. How many tangent lines to C are tangent to S?

Solution to Exercise 2.46. □

23.5 Chapter 5

Exercise 23.110. 3.11 Suppose that $\varphi : \Lambda \to V$ is an inclusion of a k-dimensional vector space Λ into an n-dimensional vector space V and let $\psi : V \to \Lambda' = V/\varphi(\Lambda)$ be the quotient map, so that

$$0 \longrightarrow \Lambda \xrightarrow{\varphi} V \xrightarrow{\psi} \Lambda' \longrightarrow 0$$

is exact. Choose bases, so that φ and ψ are represented by matrices. Show that there is a nonzero scalar λ such that the $k \times k$ minor of φ involving rows with indices $i_1 < \cdots < i_k$ is equal to λ times the $(n-k)\times(n-k)$ minor

of ψ involving the complementary set of column indices $j_1 < \cdots < j_{n-k}$. Conclude that the set-theoretic isomorphism $G(k,V) \cong G(n-k,V^*)$ taking a subspace of V to its annihilator in V^* is an algebraic isomorphism as well.

Solution to Exercise 3.11. Multiply φ by an automorphism of Λ and reorder the basis of V to make the "first" $k \times k$ submatrix of φ equal to 1. Show that the complementary "last" minor of ψ must be nonzero, and multiply ψ by an automorphism of Λ' to make it the identity matrix as well. Show that it suffices to prove the assertion for these new φ and ψ. Writing

$$\varphi = \begin{pmatrix} I_k \\ \varphi_1 \end{pmatrix}$$

show that

$$\psi = \begin{pmatrix} -{}^t\varphi_1 & I_{n-k} \end{pmatrix}.$$

Show directly that each $k \times k$ minor of φ is equal to the complementary $(n-k) \times (n-k)$ minor of ψ. □

Exercise 23.111. 3.12 Complete the proof of Corollary 3.2 without invoking duality—that is, prove directly that σ^k_{n-k} is the class of a point in $G(k,n)$.

Solution to Exercise 3.12. If $W \subset V$ is a 1-dimensional subspace, then $\Sigma_{n-k}(W)$ is the set of subspaces containing W. Let $W_1, \ldots, W_k$ be k independent 1-dimensional subspaces. The only k-dimensional subspace containing all of them is their span, $\Lambda = \overline{W_1, \ldots, W_k}$, so

$$\bigcap \Sigma_{n-k}(W_i) = \{[\Lambda]\}.$$

We must prove that the subvarieties $\Sigma_{n-k}(W_i)$ intersect transversely at $[\Lambda]$. By Theorem 3.1 the tangent space to $\Sigma_{n-k}(W_i)$ is the set of homomorphisms $\varphi : \Lambda \to V/\Lambda$ sending W_i to zero. Since the W_i span Λ, the intersection of these tangent spaces is zero; that is, the codimension of their intersections is the sum of their codimensions, as required. □

Exercise 23.112. 3.13 Use the description of the Chow ring $A(\mathbb{G}(1,4))$ given in Section 3.3 to answer the question: given 6 general 2-planes in $\mathbb{P}^4$, how many lines meet all 6? Note that this number is the same as the degree of the Grassmannian $\mathbb{G}(1,4) = G(2,5)$ in the Plücker embedding.

Solution to Exercise 3.13. □

Exercise 23.113. 3.14 Let $S \subset \mathbb{P}^4$ be a surface of degree d, and $\Gamma_S \subset \mathbb{G}(1,4)$ the variety of lines meeting S.

(a) Find the class $\gamma_S = [\Gamma_S] \in A^1(\mathbb{G}(1,4))$.

(b) Use this to answer the question: if $S_1, \ldots, S_6 \subset \mathbb{P}^4$ are surfaces of degrees $d_1, \ldots, d_6$, how many lines in $\mathbb{P}^4$ will meet all six?

Solution to Exercise 3.14. □

Exercise 23.114. 3.15 Let $C \subset \mathbb{P}^4$ be a curve of degree d, and $\Gamma_C \subset \mathbb{G}(1,4)$ the variety of lines meeting C.

(a) Find the class $\gamma_C = [\Gamma_C] \in A^2(\mathbb{G}(1,4))$.

(b) Use this to answer the question: if C_1, C_2 and $C_3 \subset \mathbb{P}^4$ are curves of degrees d_1, d_2 and d_3, how many lines in $\mathbb{P}^4$ will meet all three?

Solution to Exercise 3.15. □

Exercises 3.16-3.19 deal with the geometry of the surface described in Keynote Question (b), whose degree we worked out in Section 3.2.1: the surface $X \subset \mathbb{P}^3$ swept out by the lines corresponding to a general twisted cubic $C \subset \mathbb{G}(1,3)$.

Exercise 23.115. 3.16 To start, use the fact that the dual of $\mathbb{G}(1,3) \subset \mathbb{P}^5$ has degree 2 to show that a general twisted cubic $C \subset \mathbb{G}(1,3)$ lies on two Schubert cycles $\Sigma_1(L)$ and $\Sigma_1(M)$ for some pair of skew lines $L, M \subset \mathbb{P}^3$

Solution to Exercise 3.16. □

Exercise 23.116. 3.17 Show that for skew lines L and $M \subset \mathbb{P}^3$, the intersection $\Sigma_1(L) \cap \Sigma_1(M)$ is isomorphic to $L \times M$ via the map sending a point $[\Lambda] \in \Sigma_1(L) \cap \Sigma_1(M)$ to the pair $(\Lambda \cap L, \Lambda \cap M) \in L \times M$, and that it is the intersection of $\mathbb{G}(1,3)$ with the intersection of the hyperplanes spanned by $\Sigma_1(L)$ and $\Sigma_1(M)$

Solution to Exercise 3.17. □

Exercise 23.117. 3.18 Finally, suppose that $C \subset \Sigma_1(L) \cap \Sigma_1(M)$ is a twisted cubic curve. Using the fact that its bidegree in $\Sigma_1(L) \cap \Sigma_1(M) \cong L \times M \cong \mathbb{P}^1 \times \mathbb{P}^1$ (possibly after switching factors) is $(2,1)$, show that for some degree 2 map $\varphi : L \to M$, the family of lines corresponding to C may be realized as the locus

$$C = \{\overline{p, \varphi(p)} : p \in L\}.$$

Show correspondingly that the surface

$$S = \bigcup_{[L] \in C} L \subset \mathbb{P}^3$$

swept out by the lines of C is a cubic surface double along a line, and that it's the projection of a rational normal surface scroll $X_{1,2} \subset \mathbb{P}^4$.

Solution to Exercise 3.18. □

Exercise 23.118. 3.19 For what special twisted cubic curves $C \subset \mathbb{G}(1,3)$ will the conclusion of the preceding exercise be false?

Solution to Exercise 3.19. □

Exercise 23.119. 3.20 Now let $G = \mathbb{G}(1,3) \subset \mathbb{P}^5$ be as in the preceding Exercise, and let $C \subset G$ be a general rational normal quartic curve. Can you describe the surface S swept out by the lines of C? In particular, what is the singular locus of S?

Solution to Exercise 3.20. □

In Section 3.3 we calculated the number of lines meeting four general n-planes in $\mathbb{P}^{2n+1}$. In the following two Exercises, we'll see another way to do this (analogous to the alternative count of lines meeting four lines in $\mathbb{P}^3$ given in Exercise 2.27), and a nice geometric sidelight.

Exercise 23.120. 3.21 Let $\Lambda_1, \ldots, \Lambda_4 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ be four general n-planes. Calculate the number of lines meeting all four by showing that the union of the lines meeting Λ_1, Λ_2 and Λ_3 is a Segre variety $S_{1,n} = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ and using the calculation above **ref Chapter 1** for the degree of $S_{1,n}$.

Solution to Exercise 3.21. □

Exercise 23.121. 3.22 By the preceding exercise, we can associate to a general configuration $\Lambda_1, \ldots, \Lambda_4$ of k-planes in $\mathbb{P}^{2k+1}$ an unordered set of $k+1$ cross-ratios. Show that two such configurations $\{\Lambda_i\}$ and $\{\Lambda_i'\}$ are projectively equivalent if and only if the corresponding sets of cross-ratios coincide.

Solution to Exercise 3.22. □

The next two Exercises deal with the example of dynamic specialization given in Section 3.3.1, and specifically with the family Φ of cycles described there.

Exercise 23.122. 3.23 Show that the support of Φ_0 is all of $\Sigma_{2,2}(P) \cup \Sigma_{3,1}(p_0, H_0)$.

Solution to Exercise 3.23. □

Exercise 23.123. 3.24 Verify the last assertion made in the calculation of σ_2^2; that is, show that Φ_0 has multiplicity 1 along each component. [Hint: argue that by applying a family of automorphisms of $\mathbb{P}^4$ we can assume that the plane H_t is constant and use the calculation of the preceding chapter.]

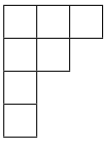
Solution to Exercise 3.24. □

Exercise 23.124. 3.25 A further wrinkle in the technique of dynamic specialization is that to carry out the calculation of an intersection of Schubert cycles we may have to specialize in stages. To see an example of this, use dynamic specialization to calculate the intersection σ_2^2 in the Grassmannian $\mathbb{G}(1,5)$. [Hint: you have to let the two 2-planes specialize first the a pair intersecting in a point, then to a pair intersecting in a line.]

Solution to Exercise 3.25. □

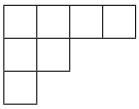
Exercise 23.125. 3.26 Suppose that the Schubert class $\sigma_a = \sigma_{a_1,\ldots,a_k} \in A^*(G(k,n))$ corresponds to the Young diagram Y in a $k \times (n-k)$ box B. Show that under the duality $G(k,n) \cong G(n-k,n)$, the class σ_a is taken to the Schubert class $\sigma_b = \sigma_{b_1,\ldots,\beta_{n-k}}$ corresponding to the Young diagram Z that is the *transpose* of Y, that is, the diagram obtained by flipping Y around a 45° line running northwest-southeast. For example if

$$\sigma_{3,2,1,1} \in A^*(G(4,7)) \quad \longleftrightarrow$$



then the corresponding Schubert cycle in $G(3,7)$ is

$$\sigma_{4,2,1} \in A^*(G(3,7)) \quad \longleftrightarrow$$



Solution to Exercise 3.26. □

Exercise 23.126. 3.27 Show that the map $i^* : A^d(G,(k,n+1)) \to A^d(G(k,n))$ is a monomorphism if and only if $n-k \geq d$, and that $i^* : A^d(G,(k+1,n+1)) \to A^d(G(k,n))$ is a monomorphism if and only if $k \geq d$. (Thus, for example, the formula

$$\sigma_1^2 = \sigma_2 + \sigma_{11},$$

which we established in $A^*(\mathbb{G}(1,3))$, *holds true in every Grassmannian*).

Solution to Exercise 3.27. □

Exercise 23.127. 3.28 Let $C \subset \mathbb{P}^r$ be a smooth, irreducible, nondegenerate curve of degree d and genus g, and let $S_1(C) \subset \mathbb{G}(1,r)$ be the variety of chords to C, as defined in Section 2.3.4 above. Find the class $[S_1(C)] \in A_2(\mathbb{G}(1,r))$.

Solution to Exercise 3.28. □

Exercise 23.128. 3.29 Prove that the Schubert cycle

$$\Sigma := \Sigma_{a_1,\ldots,a_k}(\mathcal{V}) \subset G(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$$

is defined scheme-theoretically by the polynomials vanishing on $G(k,n)$ set of Plücker coordinates $p_{i_1,\ldots,i_k}$ with $i_j > n-k-a_j$ for some j. (These equations even generate the homogenous ideal of the Schubert cycle; see for example **?**].) To show that these equations vanish on Σ you can follow the ideas of Section 2.2.2. To show that they generate the ideal of the Grassmannian, pass to an open affine set on which the Plücker coordinates look like determinants of various sizes of matrices in the variables.)

Solution to Exercise 3.29. □

Exercise 23.129. 3.30 Let Q, Q' and Q'' be three general quadrics in $\mathbb{P}^8$. How many 2-planes lie on all three? [Warning: this involves a substantially harder calculation of intersections of Schubert classes. The reader is encouraged to try it now, but should be aware that more tools for answering this will be introduced in Section 3.6 below.]

Solution to Exercise 3.30. □

Exercise 23.130. 3.31 Find the decomposition of $\sigma_{2,1}^2$ in $G(3,6)$. This is the smallest example of a product of two Schubert classes where another Schubert class appears with multiplicity > 1.

Solution to Exercise 3.31. □

Exercise 23.131. 3.32 Give an alternative proof of Pieri's formula for intersections with σ_1 using the method of specialization.

Solution to Exercise 3.32. □

Exercise 23.132. 3.33 More generally, use Pieri to identify the degree of $\sigma_1^{k(n-k)}$ with the number of standard tableaux: that is, ways of filling in $k \times (n-k)$ matrix with the integers $1,\ldots,k(n-k)$ in such a way that

every row and column is strictly increasing. Then use the "hook formula" (see for example, Fulton [1997]) to show that this number is

$$\big(k(n-k)\big)! \prod_{i=0}^{k-1} \frac{i!}{(n-k+i)!},$$

Solution to Exercise 3.33. □

Exercise 23.133. 3.34 Using Pieri's formula, determine all products of Schubert classes in the Chow ring of the Grassmannian $\mathbb{G}(1,4)$, and compare this with the result of Proposition 3.5.

Solution to Exercise 3.34. □

Exercise 23.134. 3.35 Using Pieri's formula, determine all products of Schubert classes in the Chow ring of the Grassmannian $\mathbb{G}(2,5)$. [Note: an extra step will be required, specifically to find the square $(\sigma_{2,1})^2$.]

Solution to Exercise 3.35. □

Exercise 23.135. 3.36 Give a statement of Pieri's formula in terms of Young diagrams.

Solution to Exercise 3.36. □

23.6 Chapter 6

Exercise 23.136. 6.13 Let E be an elliptic curve, $p \in E$ any point, and $\pi : E \times E \to E$ projection on the first factor. Let $\Delta \subset E \times E$ be the diagonal, and $D = E \times \{p\}$; let $\mathcal{F}$ be the invertible sheaf

$$\mathcal{F} = \mathcal{O}_{E\times E}(\Delta - D).$$

Show that $\pi_*\mathcal{F} \otimes \kappa_p = 0$, even though $H^0(\mathcal{F}|_{\pi^{-1}(p)}) \neq 0$.

Solution to Exercise 6.13. □

Exercise 23.137. 6.14 Let $L \in \mathbb{P}^2$ be a line, and let $q, r \in \mathbb{P}^2$ be two points not lying on L. Let $X = L \times \mathbb{P}^2$, and $\pi : X \to L$ and $\alpha : X \to \mathbb{P}^2$ the two projections; let $\Gamma = \Gamma_1 \cup \Gamma_2 \subset X$ where

$$\Gamma_1 = L \times \{q, r\} \quad \text{and} \quad \Gamma_2 = \{(p,p) : p \in L\}$$

Let $\mathcal{F} = \alpha^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{I}_\Gamma$. For which $p \in L$ is it the case that $\pi_*\mathcal{F} \otimes \kappa_p \neq H^0(\mathcal{F}|_{\pi^{-1}(p)}) \neq 0$?

Solution to Exercise 6.14. □

23.7 Chapter 8

Exercise 23.138. 8.27 Let $\mathbb{P}^N$ be the space of all surfaces of degree d in $\mathbb{P}^3$. Show that the set of surfaces of degree d containing two or more lines is a subvariety of codimension $2d-6$ in $\mathbb{P}^N$. (Thus a general surface $X \subset \mathbb{P}^3$ of degree $d \geq 4$ containing a line contains only one line, and no surface in a general pencil of such surfaces will contain more than one line.)

Solution to Exercise 8.27. If L_1 and L_2 are lines that are disjoint, show that containing $L_1 \cup L_2$ imposes $h^0(\mathcal{O}_{L_1 \cup L_2}(d)) = 2d+2$ independent linear conditions on surfaces of degree $d \geq 2$, whereas if L_1, L_2 meet in point then $L_1 \cup L_2$ imposes $2d+1$ independent linear conditions.) □

Exercise 23.139. 8.28 Show that the expected number of lines on a hypersurface of degree $2n-3$ in $\mathbb{P}^n$ is always positive, and deduce that *every hypersurface of degree* $2n-3$ *in* $\mathbb{P}^n$ *must contain a line.* (This is just a special case of Corollary 8.12; the idea here is to do it without a tangent space calculation.)

Solution to Exercise 8.28. □

Exercise 23.140. 8.29 We expect that a general quartic threefold $X \subset \mathbb{P}^4$ will contain a one-parameter family of lines. (In fact, *every* smooth quartic does, though it requires a little deformation theory to see this.) Find the class of the Fano scheme $F_1(X)$, and the degree of the surface $Y \subset \mathbb{P}^4$ swept out by these lines.

Solution to Exercise 8.29. □

Exercise 23.141. 8.30 Find the class of the scheme $F_2(Q) \subset \mathbb{G}(2,5)$ of 2-planes on a quadric $Q \subset \mathbb{P}^5$. (Do the problem first, then ompare your answer to the result in Proposition 3.7.)

Solution to Exercise 8.30. □

Exercise 23.142. 8.31 Find the expected number of 2-planes on a general quartic hypersurface $X \subset \mathbb{P}^7$, that is, the degree of $c_{15}(\mathrm{Sym}_4 S^*) \in A(\mathbb{G}(2,7))$.

Solution to Exercise 8.31. □

Exercise 23.143. 8.32 We can also use the calculation carried out in this Chapter to count lines on complete intersections $X = Z_1 \cap \cdots \cap Z_k \subset \mathbb{P}^n$, simply by finding the classes of the schemes $F_1(Z_i)$ of lines on the hypersurfaces Z_i and multiplying them in $A(\mathbb{G}(1,n))$. Do this to find the number

of lines on the intersection $X = Y_1 \cap Y_2 \subset \mathbb{P}^4$ of two cubic hypersurfaces in $\mathbb{P}^5$

Solution to Exercise 8.32. □

Exercise 23.144. 8.33 Find the Chern class $c_3(\mathrm{Sym}_3 S^*) \in A^3(\mathbb{G}(1,3))$. Why is this the degree of the curve of lines on the cubic surfaces in a pencil? Note that this computation does not use the incidence correspondence Φ.

Solution to Exercise 8.33. □

Exercise 23.145. 8.34 Let $\{X_t = V(t_0F + t_1G + t_2H)\}$ be a general net of cubic surfaces in $\mathbb{P}^3$.

(a) Let $p \in \mathbb{P}^3$ be a general point. How many lines lying on some member X_t of the net pass through p?

(b) Let $H \subset \mathbb{P}^3$ be a general plane. How many lines lying on some member X_t of the net lie in H?

Compare your answer to the second half of this question to the calculation in Chapter 3 of the degree of the locus of reducible plane cubics!

Solution to Exercise 8.34. □

Exercise 23.146. 8.35 Let $X \subset \mathbb{P}^3$ be a surface of degree $d \geq 3$. Show that if $F_1(X)$ is positive-dimensional, then either X is a cone or X has a positive-dimensional singular locus.

Solution to Exercise 8.35. □

Exercise 23.147. 8.36 Prove Theorem 8.3 using the methods of Section 8.6, that is, by writing the local equations of $F_k(X) \subset \mathbb{G}(k,n)$

Solution to Exercise 8.36. □

Exercise 23.148. 8.37 Using the methods of Section 8.6, show that there exists a pair (X, Λ) with $X \subset \mathbb{P}^7$ a quartic hypersurface and $\Lambda \subset X$ a 2-plane such that Λ is an isolated, reduced point of $F_2(X)$.

Solution to Exercise 8.37. □

Exercise 23.149. 8.38 Using the result of Exercise 8.37, show that the number of 2-planes on a general quartic hypersurface $X \subset \mathbb{P}^7$ is the number calculated in Exercise 8.31.

Solution to Exercise 8.38. □

The following two exercises use a theorem of Hochster, which says that

****add statement of theorem and reference****

Exercise 23.150. 8.39 Use Hochster's theorem above to prove the analog of Corollary 8.11 for k-planes in general—that is, that if $\binom{k+d}{d} \leq (k+1)(n-k)$, then there exists a pair (X, Λ) with $X \subset \mathbb{P}^n$ a smooth hypersurface of degree d and $\Lambda \subset X$ a k-plane such that $F_k(X)$ is smooth of dimension $(k+1)(n-k) - \binom{k+d}{d}$ in a neighborhood of Λ—using methods similar to those of Exercise 8.37.

Solution to Exercise 8.39. □

Exercise 23.151. 8.40 Use the result of the preceding exercise (and the simple estimate on the dimension of the incidence correspondence Φ introduced in Section 8.1) to deduce Theorem 8.13

Solution to Exercise 8.40. □

Exercise 23.152. 8.41 To complete the proof of Proposition 8.21, let $X \subset \mathbb{P}^3$ is a cubic surface with one ordinary double point $p = (1, 0, 0, 0)$, given as the zero locus of the cubic

$$F(Z_0, Z_1, Z_2, Z_3) = Z_0 A(Z_1, Z_2, Z_3) + B(Z_1, Z_2, Z_3)$$

where A is homogeneous of degree 2 and B homogeneous of degree 3. If we write a line $L \subset X$ through p as the span $L = \overline{p, q}$ with $q = (0, Z_1, Z_2, Z_3)$, show that X is smooth along $L \setminus \{p\}$ if and only if the zero loci of A and B intersect transversely at (Z_1, Z_2, Z_3).

Solution to Exercise 8.41. □

Exercise 23.153. 8.42 Extending the results of Section 8.6, suppose that X is a general cubic surface having two ordinary double points $p, q \in X$. Describe the scheme structure of $F_1(X)$ at the point corresponding to the line $L = \overline{p, q}$, and in particular determine the multiplicity of $F_1(X)$ at L.

Solution to Exercise 8.42. □

Exercise 23.154. 8.43 Now let $X \subset \mathbb{P}^3$ be a cubic surface and $P, q \in X$ isolated singular points of X; let $L = \overline{p, q}$. Show that L is an isolated point of $F_1(X)$, and that the multiplicity

$$\operatorname{mult}_L F_1(X) \geq 4$$

Solution to Exercise 8.43. □

Exercise 23.155. 8.44 Let $X \subset \mathbb{P}^3$ be a cubic surface and $p_1, \ldots, p_\delta$ isolated singular points of X. Show that no three of the points p_i are colinear.

Solution to Exercise 8.44. □

Exercise 23.156. 8.45 Use the result of the preceding two exercises to deduce the statement that *a cubic surface $X \subset \mathbb{P}^3$ can have at most four isolated singular points.*

Solution to Exercise 8.45. □

Exercise 23.157. 8.46 Show that there exists a smooth quintic threefold $X \subset \mathbb{P}^4$ whose scheme $F_1(X)$ of lines contains an isolated point of multiplicity 2.

Solution to Exercise 8.46. □

Exercise 23.158. 8.47 Using the result of the preceding exercise, show that there exists a smooth quintic threefold whose scheme of lines consists of one double point and exactly 2873 simple points (that is, which contains exactly 2874 lines in all).

Solution to Exercise 8.47. □

Exercise 23.159. 8.48 Let Φ be the incidence correspondence of triples consisting of a hypersurface $X \subset \mathbb{P}^n$ of degree $d = 2n - 3$, a line $L \subset X$ and a singular point p of X lying on L: that is,

$$\Phi = \{(X, L, p) \in \mathbb{P}^N \times \mathbb{G}(1, n) \times \mathbb{P}^n \mid p \in L \subset X \text{ and } p \in X_{\text{sing}}\}.$$

Show that Φ is irreducible.

Solution to Exercise 8.48. □

Exercise 23.160. 8.49 Let Φ be as in the preceding exercise, and suppose that $(X, L, p) \in \Phi$ is a general point. Find the multiplicity of the Fano scheme $F_1(X)$ at L.

Solution to Exercise 8.49. □

Exercise 23.161. 8.50 Suppose F and G are two quartic polynomials on $\mathbb{P}^3$ and $\{X_t = V(t_0F + t_1G)\}$ the pencil of quartics they generate; let σ_F and σ_G be the sections of $\operatorname{Sym}_4 S^*$ corresponding to F and G. Let X_{t_0} be a member of the pencil containing a line $L \subset \mathbb{P}^3$.

(a) Find the condition on F and G that L is a reduced point of $V(\sigma_F \wedge \sigma_G) \subset \mathbb{G}(1, 3)$.

(b) Show that this is equivalent to the condition that the point $(t_0, L) \in \mathbb{P}^1 \times \mathbb{G}(1,3)$ is a simple zero of the map $\mu^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \nu^* \operatorname{Sym}_d S^*$.

Solution to Exercise 8.50. □

Exercise 23.162. 8.51 Let $\Sigma \subset \mathbb{P}^{34}$ be the space of quartic surfaces in $\mathbb{P}^3$. Interpret the condition of the preceding problem in terms of the geometry of the pencil $\mathcal{D}$ around the line L, and use this to answer two questions:

(a) What is the singular locus of Σ?

(b) What is the tangent hyperplane $\mathbb{T}_X\Sigma$ at a smooth point corresponding to a smooth quartic surface X containing a single line?

Solution to Exercise 8.51. □

The following two exercises give constructions of smooth hypersurfaces containing more than the expected families of lines.

Exercise 23.163. 8.52 Let $Z \subset \mathbb{P}^{n-2}$ be any smooth hypersurface. Show that the cone $\overline{p, Z} \subset \mathbb{P}^{n-1}$ over Z in $\mathbb{P}^{n-1}$ is the hyperplane section of a smooth hypersurface $X \subset \mathbb{P}^n$, and hence that for $d > n$ there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ whose Fano scheme $F_1(X)$ of lines has dimension strictly greater than $2n - 3 - d$.

Solution to Exercise 8.52. □

Exercise 23.164. 8.53 Take $n = 2m + 1$ odd, and let $\Lambda \subset \mathbb{P}^n$ be an m-plane. Show that there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ of any given degree d containing Λ, and deduce once more that for $d > n$ there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ whose Fano scheme $F_1(X)$ of lines has dimension strictly greater than $2n - 3 - d$.

Solution to Exercise 8.53. □

Note that the construction of Exercise 8.53 cannot be modified to provide counterexamples to the de Jong/Debarre conjecture, since there do not exist smooth hypersurfaces $X \subset \mathbb{P}^n$ containing linear spaces of dimension strictly greater than $(n-1)/2$. The following exercise shows that the construction of Exercise 8.52 is similarly extremal, but is harder: it requires use of the Second Fundamental Form of a hypersurface (see Harris [1992])

Exercise 23.165. 8.54 Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 2$. Show that X can have at most finitely many hyperplane sections that are cones.

Solution to Exercise 8.54. □

To see some of the kinds of odd behavior the variety of lines on a smooth hypersurface can exhibit, short of having the wrong dimension, the following series of exercises will look at the Fermat quartic $X \subset \mathbb{P}^4$, that is, the zero locus

$$X = V(Z_0^4 + Z_1^4 + Z_2^4 + Z_3^4 + Z_4^4).$$

The conclusion is that $F_1(X)$ has 40 irreducible components, each of which is everywhere non-reduced! We start with a useful more general fact:

Exercise 23.166. 8.55 Let $S = \overline{p, C} \subset \mathbb{P}^3$ be the cone with vertex p over a plane curve C of degree $d \geq 2$, and $L \subset S$ any line. Show that the tangent space $T_L F_1(S)$ has dimension at least two, and hence that $F_1(S)$ is everywhere nonreduced.

Solution to Exercise 8.55. □

Exercise 23.167. 8.56 Show that X has 40 conical hyperplane sections Y_i, each a cone over a quartic Fermat curve in $\mathbb{P}^2$.

Solution to Exercise 8.56. □

Exercise 23.168. 8.57 Show that the class of the reduced locus $F_1(Y_i)_{\text{red}}$ has class $\sigma_{3,2}$.

Solution to Exercise 8.57. □

Exercise 23.169. 8.58 Using your answer to Exercise 8.29, conclude that

$$F_1(X) = \bigcup_{i=1}^{40} F_1(Y_i);$$

in other words, $F_1(X)$ is the union of 40 double curves.

Solution to Exercise 8.58. □

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